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Synchronization of noise-perturbed generalized Lorenz system by sliding mode control*

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Synchronization of a noise-perturbed generalized Lorenz system by using sliding mode control method is investigated in this paper. Two sliding mode control methods are proposed to synchronize the noise-perturbed generalized Lorenz system. Numerical simulations are also provided for the illustration and verification of the methods.

Keywords: sliding mode control, synchronization, generalized Lorenz system
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1. Introduction

Since Pecora and Carroll introduced the idea of synchronization into chaotic systems in 1990,[1] chaos synchronization has increasingly attracted the attention of researchers from different fields. Many powerful methods have been proposed to synchronize chaotic systems, including the feedback method, the adaptive control approach, the backing-stepping design method and so on.[2−4]

Usually, the aforementioned methods hold true only for unperturbed chaotic systems. However, in practical circumstances, chaotic systems are easily influenced by external and internal perturbations. Therefore, it is very significant to study the synchronization of perturbed chaotic systems. The sliding mode control method is a powerful method to synchronize perturbed chaotic systems. In the present paper we investigate the synchronization of a noise-perturbed generalized Lorenz system by using the sliding mode control method, thereby improving and further extending the work in Ref.[5]. Specifically, we first use an extended system to synchronize a generalized Lorenz system, which requires the perturbations $d_i(i = 1, 2, 3)$ to be derivable and bounded, and their derivatives to be bounded. Then, we choose a linear sliding mode surface to synchronize the generalized Lorenz system, which requires the perturbations $d_i(i = 1, 2, 3)$ to be bounded only.

The rest of the present paper is organized as follows. The generalized Lorenz system is described in Section 2. The analyses of synchronization using two different sliding mode control methods are detailed in Section 3. The numerical simulations for verifying the theoretical result are given in Section 4. And finally some concluding remarks are presented in Section 5.

2. The generalized Lorenz system

Since Lorenz[6] first found the Lorenz chaotic system in 1963, Lorenz family has aroused more and more interest. In 1999, Chen and Ueta[7] proposed the Chen chaotic system, and then Lü and Chen[8] found the Lü chaotic system and the unified system[9] in 2002. In the same year, Celikovsky and Chen [10] proposed the generalized Lorenz system which is described by

$$\dot{X} = \boldsymbol{A}X + f(X), \tag{1}$$

where $\boldsymbol{A}$ is a constant matrix, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$, $\boldsymbol{X} = (x_1, x_2, x_3)^{\mathrm{T}}$, and

$$f(\boldsymbol{X}) = x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \boldsymbol{X}. \tag{2}$$

According to the classification given in Ref.[10], the Lorenz system satisfies the condition $a_{12}a_{21} > 0$, $a_{11} < 0$, and $a_{33} < 0$. The Chen system satisfies $a_{12}a_{21} < 0$, $a_{11} < 0$, and $a_{33} < 0$, and the Lü system

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satisfies $a_{12}a_{21} = 0$, $a_{11} < 0$, and $a_{33} < 0$. In this paper, we propose the sliding mode control method to synchronize the noise-perturbed generalized Lorenz system, where the generalized Lorenz system satisfies the condition $a_{11} < 0$ and $a_{33} < 0$. Therefore, the results of this paper extends many existing results, such as the results in Ref.[5].

3. Synchronizing the generalized Lorenz system

Let the generalized Lorenz system

$$\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + f(\boldsymbol{X}), \tag{3}$$

be a master system and the controlled generalized Lorenz system with noise perturbation

$$\dot{Y} = \boldsymbol{A}Y + f(Y) + \boldsymbol{d} + \boldsymbol{U} \tag{4}$$

be a slave system, where $\boldsymbol{A}$ is a constant matrix the same as that given in the above section, $\boldsymbol{d} = (d_1, d_2, d_3)^{\mathrm{T}}$ with d_1 and d_2 being mismatched perturbations and d_3 being a matched perturbation, and $\boldsymbol{U} = (0, u, 0)^{\mathrm{T}}$. We will design a controller $\boldsymbol{U}$ such that noise-perturbed controlled generalized Lorenz system (4) is synchronous with master generalized Lorenz system (3). In this section, we will use two different sliding mode control methods to synchronize master generalized Lorenz system (3) and controlled generalized Lorenz system (4).

First, we use an extended system to synchronize the generalized Lorenz system perturbed by noise. In this method, we assume that these perturbations are bounded in C^1, i.e. $|d_i|_1 \leq \delta < 1 (i = 1, 2, 3)$, where δ is a constant and $|\cdot|_1$ is a usual norm in C^1, i.e. $|u(t)|_1 = \sup_{t\in R}(|u(t)| + |u'(t)|)$.

Set the error variable $\boldsymbol{e} = (e_1, e_2, e_3)^{\mathrm{T}} = (y_1 - x_1, y_2 - x_2, y_3 - x_3)^{\mathrm{T}}$, then we will have the error state dynamics equations

$$\begin{aligned} \dot{e}_1 &= a_{11}e_1 + a_{12}e_2 + d_1, \\ \dot{e}_2 &= a_{21}e_1 + a_{22}e_2 + d_2 + u_1, \\ \dot{e}_3 &= a_{33}e_3 + x_1e_2 + x_2e_1 + e_1e_2 + d_3, \end{aligned} \tag{5}$$

where $u_1 = u - y_1y_3 + x_1x_3$.

According to the concept of extended system,[11] the extended system is written as

$$\begin{aligned} \dot{e}_1 &= a_{11}e_1 + a_{12}e_2 + d_1, \\ \dot{e}_2 &= e_4, \\ \dot{e}_3 &= a_{33}e_3 + x_1e_2 + x_2e_1 + e_1e_2 + d_3, \\ \dot{e}_4 &= a_{11}a_{21}e_1 + a_{12}a_{21}e_2 + a_{22}e_4 \\ &\quad + a_{21}d_1 + \dot{d}_2 + \dot{u}_1. \end{aligned} \tag{6}$$

We choose an appropriate sliding mode surface as $S_1 = 0$, where

$$S_1 = e_4(t) + \int_0^t (c_1e_1(s) + c_2e_2(s) + c_4e_4(s))\mathrm{d}s \tag{7}$$

with c_1, c_2 and c_4 being constants to be designed. Then we have the following lemma.

Lemma 1 If the controller is selected as

$$\begin{aligned} u_1 = \int_0^t &(-a_{11}a_{21}e_1 - a_{12}a_{21}e_2 - a_{22}e_4 - c_1e_1 \\ &-c_2e_2 - c_4e_4 - k\mathrm{sgn}(S_1))\mathrm{d}s, \end{aligned} \tag{8}$$

where c_1, c_2 and c_4 are constants and $k > (|a_{21}| + 1)\delta + 1$, then error system (6) will approach the sliding mode surface $S_1 = 0$ in a finite time.

Proof We choose $V = S_1^2$ as the Lyapunov function of the system (6), then we will have

$$\begin{aligned} \dot{V} &= 2S_1\dot{S}_1 \\ &= 2S_1(\dot{e}_4 + c_1e_1 + c_2e_2 + c_4e_4) \\ &= 2S_1(a_{11}a_{21}e_1 + a_{12}a_{21}e_1 + a_{22}e_4 + a_{21}a_{21}d_1 \\ &\quad + \dot{d}_2 + \dot{u}_1 + c_1e_1 + c_2e_2 + c_4e_4). \end{aligned} \tag{9}$$

From the condition of the lemma we obtain

$$\begin{aligned} \dot{V} &= 2S_1[a_{21}d_1 + \dot{d}_2 - k\mathrm{sgn}(S_1)] \\ &\leq 2|S_1|[(|a_{21}| + 1)\delta - k] \\ &\leq -2|S_1| \\ &= -2V^{\frac{1}{2}}(t), \end{aligned} \tag{10}$$

which implies $V^{\frac{1}{2}}(t) \leq V^{\frac{1}{2}}(0) - t, t \in [0, t_s)$. Thus, $V(t) = 0$ when $t = t_s = V^{\frac{1}{2}}(0)$, i.e. error system (6) approaches the sliding mode surface $S_1 = 0$ in a finite time. The proof is completed here.

From this lemma we have the following theorem.

Theorem 1 If the controller is selected as $u = u_1 + y_1y_3 - x_1x_3$, where

$$\begin{aligned} u_1 = \int_0^t &(-a_{11}a_{21}e_1 - a_{12}a_{21}e_1 - a_{22}e_4 - c_1e_1 \\ &-c_2e_2 - c_4e_4 - k\mathrm{sgn}(S_1))\mathrm{d}s \end{aligned} \tag{11}$$

with c_1, c_2 and c_4 being constants and $k > (|a_{21}| + 1)\delta + 1$, then there exists a constant c such that controlled generalized Lorenz system (4) is synchronous with master generalized Lorenz system (3) with an ultimate error bound $c\delta$, i.e. $\overline{\lim}_{t\to\infty} |y_1 - x_1| \le c\delta$, $\overline{\lim}_{t\to\infty} |y_2 - x_2| \le c\delta$, and $\overline{\lim}_{t\to\infty} |y_3 - x_3| \le c\delta$.

Proof From Lemma 1, we only need to analyse the error system on the sliding mode surface. On the sliding mode surface $S_1 = 0$, the error system reads

$$\begin{aligned}
\dot{e}_1 &= a_{11}e_1 + a_{12}e_2 + d_1,\\
\dot{e}_2 &= e_4,\\
\dot{e}_3 &= a_{33}e_3 + x_1e_2 + x_2e_1 + e_1e_2 + d_3,\\
\dot{e}_4 &= -c_1e_1 - c_2e_2 - c_4e_4 + a_{21}d_1 + \dot{d}_2.
\end{aligned} \tag{12}$$

Thus,

$$\begin{pmatrix} e_1(t)\\ e_2(t)\\ e_3(t)\end{pmatrix} = \mathrm{e}^{At}\begin{pmatrix} e_1(0)\\ e_2(0)\\ e_3(0)\end{pmatrix} + \int_0^{\mathrm{T}} \mathrm{e}^{A(t-s)}\begin{pmatrix} d_1\\ 0\\ a_{21}d_1 + \dot{d}_2\end{pmatrix}\mathrm{d}s, \tag{13}$$

where

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & 0\\ 0 & 0 & -1\\ -c_1 & -c_2 & -c_4\end{pmatrix}.$$

Its characteristic polynomial is

$$\begin{aligned}
f(\lambda) &= \lambda^3 + (c_4 - a_{11})\lambda^2 - (a_{11}c_4 - c_2)\lambda\\
&\quad + a_{12}c_1 - a_{11}c_2\\
&= 0.
\end{aligned}$$

According to the Routh–Hurwitz theorem we know that the real parts of its all characteristic roots are negative if and only if

$$\begin{aligned}
\Delta_1 &= c_4 - a_{11} > 0,\\
\Delta_2 &= (c_4 - a_{11})(c_2 - a_{11}c_4) - (a_{12}c_1 - a_{11}c_2) > 0,\\
\Delta_3 &= a_{12}c_1 - a_{11}c_2 > 0.
\end{aligned} \tag{14}$$

Obviously, there exist constants c_1, c_2 and c_4 satisfying the above conditions (For example, we can choose $c_1 = 0, c_2 > 0, c_4 > 0$ and c_4 to be big enough). Therefore, there exist positive constants $a > 0$ and $b > 0$ such that $|\mathrm{e}^{At}x| \le a\mathrm{e}^{-bt}|x|$ for each $x \in R^3$ and $t \ge 0$. Thus

$$|e_i| \le a\mathrm{e}^{-bt}\max_{i=1,2,4}|e_i(0)| + \frac{a\delta}{b}(|a_{21}| + 1), \quad i = 1, 2, 4,$$

which implies

$$\overline{\lim_{t\to\infty}}\,|e_i| \le \frac{a\delta}{b}(|a_{21}| + 1), \quad i = 1, 2, 4.$$

From the equation $\dot{e}_3(t) = a_{33}e_3 + x_1e_2 + x_2e_1 + e_1e_2 + d_3$, we have

$$\begin{aligned}
e_3(t) &= \mathrm{e}^{a_{33}t}\Bigg[e_3(0)\\
&\quad + \int_0^t \mathrm{e}^{-a_{33}s}(x_1e_2 + x_2e_1 + e_1e_2 + d_3)\mathrm{d}s\Bigg].
\end{aligned}$$

Since a chaotic system has bounded trajectories, then there exists a positive constant M, such that $|x_1| \le M$, $|x_2| \le M$, thus,

$$\begin{aligned}
|e_3| &\le \mathrm{e}^{a_{33}t}\Bigg[|e_3(0)| + \int_0^t \mathrm{e}^{-a_{33}s}(|x_1||e_2|\\
&\quad + |x_2|e_1| + |e_1||e_2| + |d_3|)\mathrm{d}s\Bigg],
\end{aligned}$$

i.e.

$$\begin{aligned}
|e_3| &\le \mathrm{e}^{a_{33}t}|e_3(0)| + \left[\frac{2aM}{b}(|a_{21}| + 1)\right.\\
&\quad \left. + \frac{a^2}{b^2}(|a_{21}| + 1)^2\delta + 1\right]\frac{(1 - \mathrm{e}^{a_{33}t})}{-a_{33}}\delta.
\end{aligned}$$

Because $a_{33} < 0$, we have

$$\begin{aligned}
\overline{\lim_{t\to\infty}}\,|e_3| &\le -\frac{\delta}{a_{33}}\left[\frac{2aM}{b}(|a_{21}| + 1)\right.\\
&\quad \left. + \frac{a^2}{b^2}(|a_{21}| + 1)^2\delta + 1\right].
\end{aligned}$$

Letting

$$\begin{aligned}
c &= \max\left\{\frac{a\delta}{b}(|a_{21}| + 1), -\frac{\delta}{a_{33}}\left[\frac{2aM}{b}(|a_{21}| + 1)\right.\right.\\
&\quad \left.\left. + \frac{a^2}{b^2}(|a_{21}| + 1)^2\delta + 1\right]\right\},
\end{aligned}$$

we prove the conclusion of the theorem.

Corollary 1 Under the conditions of Theorem 1, the unperturbed controlled generalized Lorenz system is synchronous with the generalized Lorenz system, i.e.

$$\begin{aligned}
\lim_{t\to\infty}|y_1 - x_1| &= 0,\\
\lim_{t\to\infty}|y_2 - x_2| &= 0,\\
\lim_{t\to\infty}|y_3 - x_3| &= 0.
\end{aligned}$$

According to the above analysis, we can find that the method is complicated and requires the perturbations $d_i (i = 1, 2, 3)$ to be derivable and bounded,

and their derivatives to be bounded. Actually, the majority of perturbations are not derivable. Even if they are derivable, their derivatives are not necessarily bounded. Hence, we need to find another sliding mode control method.

In the following, we choose a linear siding mode surface to synchronize the generalized Lorenz system, where the perturbations $d_i(i = 1, 2, 3)$ are required to be bounded only.

Assume that these perturbations $d_i(i = 1, 2, 3)$ are bounded, i.e. $||d_i|| < \delta < 1(i = 1, 2, 3)$, where δ is a constant and $||\cdot||$ is a usual norm. We directly choose a sliding mode surface of the error system (4) as $S_2 = 0$, where

$$S_2 = e_2 + \rho e_1 \tag{15}$$

and ρ is a constant that can be determined. Then we have the following lemma.

Lemma 2 If the controller is selected as

$$\begin{aligned} u_1 = & -a_{21}e_1 - a_{22}e_2 - \rho a_{11}e_1 \\ & -\rho a_{12}e_2 - k\mathrm{sgn}(S_2), \end{aligned} \tag{16}$$

where ρ is a constant and $k > (\rho + 1)\delta + 1$, then error system (5) will approach the sliding mode surface $S_2 = 0$ in a finite time.

We omit its proof because it is similar to the proof of Lemma 1. Thus, we have the following theorem.

Theorem 2 If the controller is selected as $u = u_1 + y_1y_3 - x_1x_3$, where

$$\begin{aligned} u_1 = & -a_{21}e_1 - a_{22}e_2 - \rho a_{11}e_1 \\ & -\rho a_{12}e_2 - k\mathrm{sgn}(S_2) \end{aligned} \tag{17}$$

with ρ being a constant and $k > (\rho + 1)\delta + 1$, then there exists a constant c_1 such that the controlled generalized Lorenz system is synchronous with the master generalized Lorenz system with an ultimate error bound $c_1\delta$, i.e.

$$\begin{aligned} \overline{\lim_{t\to\infty}}\,|y_1 - x_1| \le c_1\delta, \\ \overline{\lim_{t\to\infty}}\,|y_2 - x_2| \le c_1\delta, \\ \overline{\lim_{t\to\infty}}\,|y_3 - x_3| \le c_1\delta. \end{aligned}$$

Proof Because of Lemma 2, also, we only need to analyse the error system on the sliding mode surface.

On the sliding mode surface $S_2 = 0$, the error system reads

$$\begin{aligned} \dot{e}_1 &= (a_{11} - \rho a_{12})e_1 + d_1, \\ \dot{e}_2 &= a_{21}e_1 + a_{22}e_2 + d_2 + u_1, \\ \dot{e}_3 &= a_{33}e_3 + x_1e_2 + x_2e_1 + e_1e_2 + d_3. \end{aligned} \tag{18}$$

From the first equation of Eq.(18) we have

$$\begin{aligned} e_1(t) = & \,\mathrm{e}^{(a_{11}-\rho a_{12})t}e_1(0) \\ & + \int_0^t \mathrm{e}^{(a_{11}-\rho a_{12})(t-s)}d_1\mathrm{d}s, \end{aligned}$$

therefore,

$$\begin{aligned} |e_1(t)| = & \Big|\mathrm{e}^{(a_11-\rho a_12)t}e_1(0) \\ & + \int_0^t \mathrm{e}^{(a_{11}-\rho a_{12})(t-s)}d_1\mathrm{d}s\Big| \\ \le & \Big|\mathrm{e}^{(a_11-\rho a_12)t}e_1(0)\Big| \\ & + \Big|\int_0^t \mathrm{e}^{(a_{11}-\rho a_{12})(t-s)}d_1\mathrm{d}s \\ \le & \,|\mathrm{e}^{(a_11-\rho a_12)t}e_1(0)| \\ & + d_1\Big|\int_0^t \mathrm{e}^{(a_{11}-\rho a_{12})(t-s)}\mathrm{d}s\Big|. \end{aligned} \tag{19}$$

We choose ρ such that $a_{11} - \rho a_{12} < 0$. Thus,

$$|e_1(t)| \le \mathrm{e}^{(a_{11}-\rho a_{12})t}|e_1(0)| + \delta\frac{1 - \mathrm{e}^{(a_{11}-\rho a_{12})t}}{\rho a_{12} - a_{11}},$$

So

$$\overline{\lim_{t\to\infty}}\,|e_1| \le \gamma_1 = \frac{\delta}{\rho a_{12} - a_{11}}.$$

Because $e_2 = -\rho e_1$, thus

$$\overline{\lim_{t\to\infty}}\,|e_2| \le \gamma_2 = \frac{|\rho|\delta}{\rho a_{12} - a_{11}}.$$

From the third equation of Eq.(18) we have

$$\begin{aligned} e_3(t) = & \,\mathrm{e}^{a_{33}t}\Bigg[e_3(0) \\ & + \int_0^t (d_3 + x_1e_2 + x_2e_1 + e_1e_2)\mathrm{e}^{-a_{33}s}\mathrm{d}s\Bigg]. \end{aligned}$$

Since a chaotic system has bounded trajectories, there exists a positive constant M such that $|x_1| \le M$ and $|x_2| \le M$. Thus

$$\begin{aligned} |e_3(t)| \le & \,\mathrm{e}^{a_{33}t}|e_3(0)| \\ & + \frac{\delta}{-a_{33}}\Bigg[1 + \frac{M|\rho|}{\rho a_{12} - a_{11}} + \frac{M}{\rho a_{12} - a_{11}} \\ & + \frac{|\rho|\delta}{(\rho a_{12} - a_{11})^2}\Bigg](1 - \mathrm{e}^{a_{33}t}). \end{aligned}$$

Since $a_{33} < 0$, we have

$$\begin{aligned} & \overline{\lim_{t\to\infty}}\,|e_3| \le \gamma_3 \\ = & \,\frac{\delta}{-a_{33}}\Bigg[1 + \frac{M|\rho|}{\rho a_{12} - a_{11}} \\ & + \frac{M}{\rho a_{12} - a_{11}} + \frac{|\rho|\delta}{(\rho a_{12} - a_{11})^2}\Bigg]. \end{aligned}$$

Letting

$$c_1 = \max\left\{\frac{\delta}{\rho a_{12}-a_{11}}, \frac{|\rho|\delta}{\rho a_{12}-a_{11}}, \frac{\delta}{-a_{33}}\left[1+\frac{M|\rho|}{\rho a_{12}-a_{11}}+\frac{M}{\rho a_{12}-a_{11}} + \frac{|\rho|\delta}{(\rho a_{12}-a_{11})^2}\right]\right\},$$

we finish the proof of the theorem.

Corollary 2 Under the conditions of Theorem 2, the noise-unperturbed controlled generalized Lorenz system is synchronous with the generalized Lorenz system, i.e.

$$\lim_{t\to\infty}|y_1-x_1|=0,$$
$$\lim_{t\to\infty}|y_2-x_2|=0,$$
$$\lim_{t\to\infty}|y_3-x_3|=0.$$

4. Improvement on improving the sliding mode control

The controllers of the above two methods contain the discontinuous nonlinear sgn(S), as both the theoretical and the practical problems come up from discontinuous controllers. The theoretical problems, such as the existence and uniqueness of solutions and the validity of Lyapunov analysis, will have to be examined in a framework that does not require the state equation to have locally Lipschitz right-hand side functions. There is also a practical problem about chattering due to imperfections in switching devices and delays. Chattering can cause low control accuracy, high heat loss in electrical power circuits, and high wear of moving mechanical parts. It may also induce unmodelled high frequency dynamics, which degrades the performance of the system and may even lead to instability.

One approach to eliminating the chattering is to use a continuous approximation of the discontinuous sliding mode controller. By using the continuous approximation, we can also evade the theoretical difficulties associated with discontinuous controllers. We use a saturation nonlinearity with high slop to approximate the signum nonlinearity. A saturation function is defined as

$$\mathrm{sat}(y)=\begin{cases} y & \text{if } |y|\le 1,\\ \mathrm{sgn}(y) & \text{if } |y|>1. \end{cases} \tag{20}$$

Then in the first method we choose the controller as

$$\widetilde{u}_1 = \int_0^t \Big(-a_{11}a_{21}e_1 - a_{12}a_{21}e_2 - a_{22}e_4 - c_1e_1 - c_2e_2 - c_4e_4 - k\mathrm{sat}\left(\frac{S_1}{\varepsilon}\right)\Big)\mathrm{d}s, \tag{21}$$

where ε is a positive constant. Let $\varepsilon\to 0$, then the saturation nonlinearity $\mathrm{sat}\left(\frac{S_1}{\varepsilon}\right)$ will approach the function $\mathrm{sat}(S_1)$. Therefore, we have to examine the approaching control by using the Lyapunov function $V=S_1^2$. Let $V=S_1^2$ be the Lyapunov function, the derivative of V along the solution of system (5) satisfies the equality

$$\dot{V} = 2S_1\left[a_{21}d_1+\dot{d}_2-k\mathrm{sat}\left(\frac{S_1}{\varepsilon}\right)\right]. \tag{22}$$

In the region $|S_1|>\varepsilon$, we have

$$\dot{V}\le 2|S_1|[(|a_{21}|+1)\delta-k]\le -2|S_1| = -2V^{\frac{1}{2}}(t), \tag{23}$$

if $k>(|a_{21}|+1)\delta+1$. This implies that whenever $|S_1(0)|>\varepsilon$, $|S_1|$ will be strictly decreasing until it reaches the bounded layer $\{|S_1|\le\varepsilon\}$ in a finite time and remains inside thereafter. On the other hand, in the bounded layer, we have

$$\begin{aligned}\dot{V} &\le 2|S_1|[(|a_{21}|+1)\delta]-\frac{2k}{\varepsilon}S_1^2\\ &= 2|S_1|[(|a_{21}|+1)\delta]-\frac{2k}{\varepsilon}V,\end{aligned} \tag{24}$$

which implies that,

$$V^{\frac{1}{2}}\le V^{\frac{1}{2}}(0)\mathrm{e}^{-\frac{kt}{\varepsilon}}+\frac{\varepsilon}{k}[(|a_{21}|+1)\delta]. \tag{25}$$

Additionally, in the bounded layer the error system reads

$$\begin{aligned}\dot{e}_1 &= a_{11}e_1+a_{12}e_2+d_1,\\ \dot{e}_2 &= e_4,\\ \dot{e}_3 &= a_{33}e_3+x_1e_2+x_2e_1+e_1e_2+d_3,\\ \dot{e}_4 &= -c_1e_1-c_2e_2-c_4e_4+a_{21}d_1\\ &\quad +\dot{d}_2-\frac{k}{\varepsilon}S_1.\end{aligned} \tag{26}$$

In the same way as that of proving Theorem 1, we can have the following theorem easily.

Theorem 3 If the controller is selected as $u=\widetilde{u}_1+y_1y_3-x_1x_3$, where

$$\widetilde{u}_1 = \int_0^t(-a_{11}a_{21}e_1-a_{12}a_{21}e_1-a_{22}e_4-c_1e_1 - c_2e_2-c_4e_4-k\mathrm{sat}(S_1\varepsilon))\mathrm{d}s,$$

with c_1, c_2 and c_4 being constants and $k>|a_{21}|+1)\delta+1$, then there exists a constant c such that the

controlled generalized Lorenz system is synchronous with the mater generalized Lorenz system with an ultimate error bound $c\delta$, i.e.

$$\overline{\lim_{t\to\infty}}\,|y_1 - x_1| \le c\delta,$$
$$\overline{\lim_{t\to\infty}}\,|y_2 - x_2| \le c\delta,$$
$$\overline{\lim_{t\to\infty}}\,|y_3 - x_3| \le c\delta.$$

In the second method, we only choose $\tilde{u}_1 = -a_{21}e_1 - a_{22}e_2 - \rho e_1 - \rho a_{12}e_2 - k\mathrm{sat}(S_2)$, and we have the following theorem.

Theorem 4 If the controller is selected as $u = \widetilde{u}_1 + y_1y_3 - x_1x_3$, where

$$\tilde{u}_1 = -a_{21}e_1 - a_{22}e_2 - \rho e_1 - \rho a_{12}e_2 - k\mathrm{sat}(S_2),$$

ρ is a constant and $k > (\rho+1)\delta + 1$, then there exists a constant c' such that the controlled generalized Lorenz system is synchronous with the master generalized Lorenz system with an ultimate error bound $c'\delta$, i.e.

$$\overline{\lim_{t\to\infty}}\,|y_1 - x_1| \le c'\delta,$$
$$\overline{\lim_{t\to\infty}}\,|y_2 - x_2| \le c'\delta,$$
$$\overline{\lim_{t\to\infty}}\,|y_3 - x_3| \le c'\delta.$$

5. Numerical simulations

In order to verify the validity of the proposed method, some numerical simulations are presented in this section. In all numerical simulations we separately select the Lorenz chaotic system $\left(a_{11} = -10,\ a_{12} = 10,\ a_{21} = 28,\ a_{22} = -1 \text{ and } a_{33} = -\frac{8}{3}\right)$, the Chen system $\left(a_{11} = -35,\ a_{12} = 35, a_{21} = -7,\ a_{22} = 28, \text{ and } a_{33} = -\frac{8}{3}\right)$, and the Lü system ($a_{11} = -36$, $a_{12} = 36$, $a_{21} = 0, a_{22} = 20$, and $a_{33} = -3$). The initial states of the master system and the controlled system are $(1.0, 2.0, 3.0)^{\mathrm{T}}$ and $(4.0, 5.0, 6.0)^{\mathrm{T}}$, respectively. We choose $\rho = 1$. And the results of the numerical simulations are shown in Figs.1–3, separately. Obviously, the numerical simulations well verify the theoretical analysis.

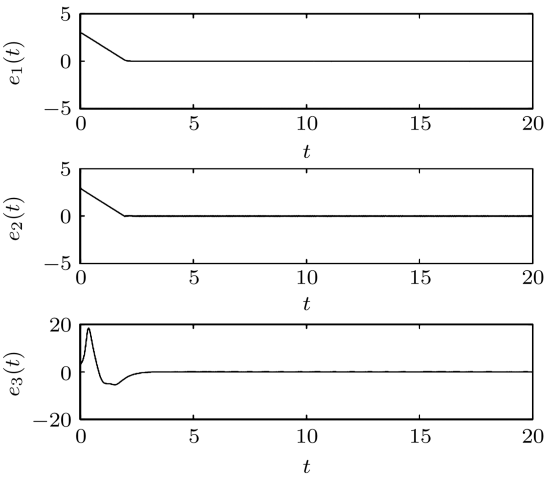


Fig.1. (Lorenz system) Errors obtained by sliding mode control between drive system and slave system with a perturbation ($d_1 = 0.05\sin(t)$, $d_2 = 0.1\cos(t)$, $d_3 = 0.25\sin(2t)$).

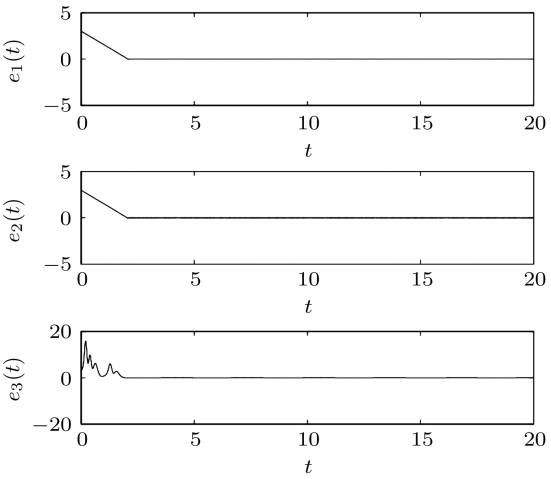


Fig.2. (Chen system) Errors obtained by sliding mode control between drive system and slave system with a perturbation ($d_1 = 0.05\sin(t)$, $d_2 = 0.1\cos(t)$, $d_3 = 0.25\sin(2t)$).

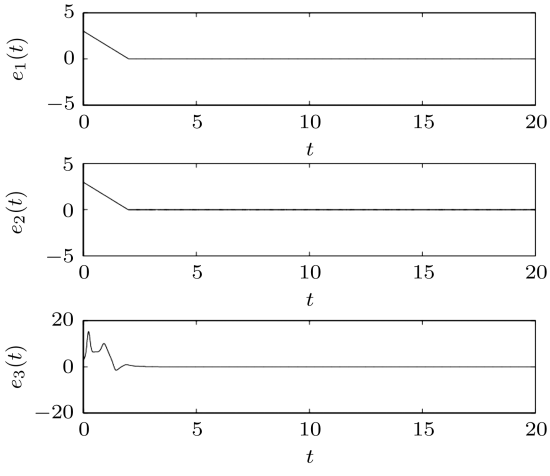


Fig.3. (Lü system) Errors obtained by sliding mode control between drive system and slave system with a perturbation ($d_1 = 0.05\sin(t)$, $d_2 = 0.1\cos(t)$, $d_3 = 0.25\sin(2t)$).

6. Conclusion

In this paper, we use two different sliding mode control methods to synchronize the perturbed generalized Lorenz system. The sufficient conditions of synchronization between these systems are established and some numerical simulations are presented to verify the methods.

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