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Analysis of transition between chaos and hyper-chaos of an improved hyper-chaotic system*

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An improved hyper-chaotic system based on the hyper-chaos generated from Chen's system is presented, and some basic dynamical properties of the system are investigated by means of Lyapunov exponent spectrum, bifurcation diagrams and characteristic equation roots. Simulations show that the new improved system evolves into hyper-chaotic, chaotic, various quasi-periodic or periodic orbits when one parameter of the system is fixed to be a certain value while the other one is variable. Some computer simulations and bifurcation analyses are given to testify the findings.

Keywords: hyper-chaos, chaos, bifurcation diagram, Lyapunov exponents
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1. Introduction

Hyper-chaos is usually characterized as a chaotic attractor with more than one positive Lyapunov exponent,[1,2] it was first reported by Rössler in 1979.[2] Since then, some other hyper-chaotic generators have also been found.[3−5] Owing to its great potential in technological applications such as secure communication,[6] information security,[7] lasers,[8,9] and so on, the chaos synchronization and the chaos application, the design and the generation of hyper-chaos by using circuits have become a hot research topic recently.[10−16]

From the structure of hyper-chaos, which was recently reported, one can see that two kinds of methods of generating hyper-chaos are usually adopted, one is to introduce a linear or nonlinear controller into a state of the existing chaos. For example, Gao *et al*[12] presented a hyper-chaotic system by adding a controller to the second equation of the three-dimensional autonomous Chen's chaotic system. Gao *et al*[15] acquired a new hyper-chaos by introducing a simple quadratic dynamic feedback controller into the second equation of Lorenz chaotic system. Wang and Liu[14] constructed a hyper-chaos by adding a linear controller to an existing 3D chaotic system. The other is to couple the states of two identical chaotic systems.[17]

Among the hyper-chaotic systems recently proposed by linear feedback controller, one uses the simple linear feedback.[6,12,14] For example, the following hyper-chaotic system is obtained by adding a linear controller to the second equation of the three-dimensional autonomous Chen's chaotic system:[18]

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= dx - xz + cy - w, \\ \dot{z} &= xy - bz, \\ \dot{w} &= x + k, \end{aligned} \tag{1}$$

where $a = 36$, $b = 3$, $c = 28$, $d = -16$, and k is a variable constant. System (1) can display hyper-chaos, chaos or limit cycle when the parameter k varies.[12]

In the present paper, the linear feedback controller w is replaced by a general linear controller $\dot{w} = mx + k$ for system (1), and thus an improved system is obtained; some new and interesting results about the new system are presented; some analyses on the range of hyper-chaos and chaos with different values of parameters w and k are illustrated by computer simulation. The rest of the present paper is organized as follows: in Section 2 some simulation outcomes are

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given, and in Section 3 presented are some conclusions drawn from the present study.

2. Analysis based on the corresponding characteristic equation roots

Sensitivity to parameters is a main characteristic of hyper-chaos, so hyper-chaos may have different states with the parameters varying, and in general, these phenomena not only can be investigated by computer simulations and verified through bifurcation analysis,[12−16] but also can be demonstrated by electronic circuit experiments.[14,15] In fact, the change in sign of the characteristic equation roots of hyper-chaos can also affect the states of the hyper-chaos, and the point where the signs of the corresponding characteristic equation roots change may be the point where the system states may change.[19,20] In this section, we will give some analyses on an improved hyper-chaos to find out how its characteristic equation roots affect the transition between hyper-chaos and chaos for a hyper-chaotic system with the parameters w and k being variable.

The improved hyper-chaotic system is obtained by replacing the linear controller w with new general linear controller w, which is described in the following form:

$$\begin{aligned}\dot{x} &= a(y-x),\\ \dot{y} &= dx - xz + cy - w,\\ \dot{z} &= xy - bz,\\ \dot{w} &= mx + k.\end{aligned} \tag{2}$$

Obviously, here the linear dynamical controller w is modified into a general form; system (1) is only a special case of system (2) with the parameter $m = 1$.

It can be seen that it is invariant under the transformation $(x, y, z, w, m, k) \to (-x, -y, z, -w, m, -k)$ for system (2). Therefore, if (x, y, z, w) is a solution of system (2) on condition that the values of k and m are definite, then $(-x, -y, z, -w)$ is also a solution when k is set to be a negative value and m keeps unchanged. For system (2), some new and interesting phenomena happen, as described below.

2.1. Choose $a = 36$, $b = 3$, $c = 28$, $d = -16$, m to be a certain value, and the value of to be variable

The equilibrium point of system (2) can be found by solving the following equations:

$$\begin{aligned}&a(y-x) = 0,\\ &dx - xz + cy - w = 0,\\ &xy - bz = 0,\\ &mx + k = 0.\end{aligned} \tag{3}$$

It can be seen that there is only one equilibrium point, $S(x_0, y_0, z_0, w_0)$, and its coordinates can be written as

$$x_0 = -\frac{k}{m}, \quad y_0 = x_0, \quad z_0 = \frac{x_0^2}{b},$$

and

$$w_0 = -k(d+c) + \frac{k^3}{b}. \tag{4}$$

By linearizing system (2) at the equilibrium point S, the Jacobean matrix is obtained as

$$\boldsymbol{J} = \begin{bmatrix} -a & a & 0 & 0 \\ d - \dfrac{x_0^2}{b} & c & k & -1 \\ x_0 & y_0 & -b & 0 \\ m & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

The corresponding characteristic equation can be written as

$$f(\lambda) = \lambda^4 + p\lambda^3 + q\lambda^2 + r\lambda + s = 0, \tag{6}$$

where

$$p = a + b - c, q = ab - ac - bc - ad + \frac{ak^2}{b},$$

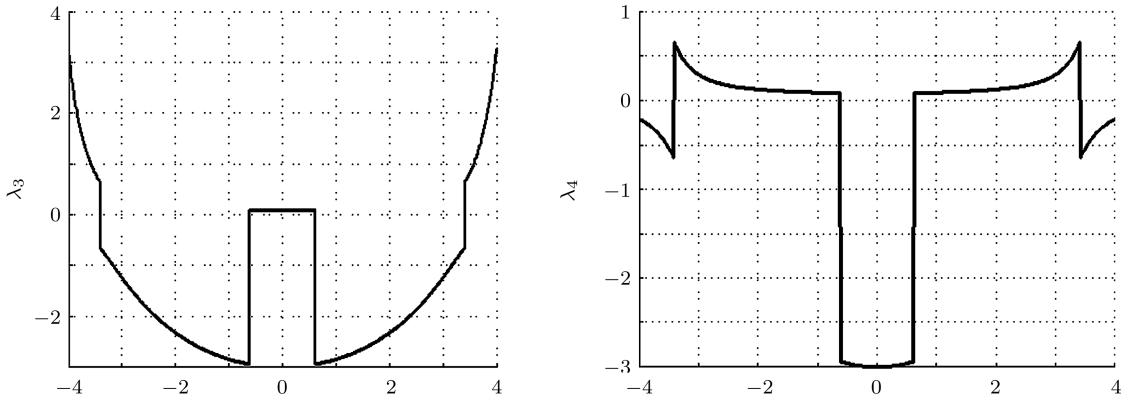
$$r = 2ak^2 - abd - abc + a, \quad \text{and} \quad s = ab.$$

Now, the following properties are observed.

(1) The change in characteristic root accords with the transition of system states when the value of parameter m is between 0 and 1.6, i.e. $0 \le m < 1.6$.

When $a = 36$, $b = 3$, $c = 28$, $d = -16$ and $0 \le m < 1.6$, for different values of m, the ranges of hyper-chaos and chaos increase with the value of m increasing. For example, taking $m = 1.2$, one can see that the range of hyper-chaos becomes $-0.9 < k < 0.9$, and the corresponding scopes of chaos also change into $-4.1 < k \le -0.9$ and $0.9 \le k < 4.1$.

In the meantime, it is observed that the change in characteristic root of system (2) with parameter varying accords with the transition of system state. For example, when $m = 1$ and $k \in [-4, 4]$, Fig.1 depicts the changes in characteristic root, indicating that the third and fourth characteristic roots λ_3 and λ_4 have some interesting changes. For example, near $k = 0.7$ and $k = -0.7$, the root λ_3 suddenly changes its sign from '+' to '−', and the sign remains unchanged until the value of k increases up to near 3.5 and −3.5, i.e. $k = 3.5$ and $k = -3.5$ respectively; at the same time, the root λ_3 again suddenly changes its sign from '−' to '+' near $k = 3.5$ and $k = -3.5$. However for the fourth root λ_4, the root spectrum changes from '−' to '+' to '−' in the intervals $k \in [-0.7, 0.7]$, $k \in [-0.3.5, -0.7) \cup (0.7, 3.5]$ and $k \in [-4, -3.5) \cup (3.5, 4]$, respectively. Here we conclude that for the interval of control parameter $k \in [-0.3.5, -0.7)$, system (1) has a chaotic behaviour, on the other hand, it has the hyper-chaotic feature when $k \in [-0.7, 0.7]$, and again it displays chaotic behaviour with parameter $k \in (0.7, 3.5]$. These conclusions accord with the results obtained in Ref.[8].



(a) Characteristic root λ_3 (b) Characteristic root λ_4

Fig.1. The characteristic roots with $m = 1$ and k being variable.

It is well known that hyper-chaotic systems are characterized by at least two positive Lyapunov exponents for typical trajectories in an arbitrarily high phase space. In a continuous four-dimensional dissipative system there are four possible types of strange attractors: their Lyapunov spectra are $(+, +, 0, -)$, $(+, -, 0, -)$, $(0, 0, -, -)$ and $(0, -, -, -)$ separately, and the corresponding orbits are hyper-chaos, chaos, quasi-periodic and periodic orbits, respectively. Therefore, the following investigation relies on a combination of mathematical analysis and computer simulations. Without the loss of generality, here we assume the Lyapunov exponents of system (2) to be λ_1, λ_2, λ_3, and λ_4 separately. When $m = 1.2$, Fig.2 shows the corresponding Lyapunov exponents, and Fig.3 displays the corresponding bifurcation diagram of state x with respect to k, while Fig.4 depicts the change in of characteristic root of system (2) with the value of k being variable.

Some typical changes in state of system (2) and Lyapunov exponents with respect to parameter k for some fixed values of parameter m are summarized in Table 1. It can be seen from the table that the ranges of hyper-chaos and chaos with respect to k increase as the value of m increases in a range $0 - 1.6$, i.e. $0 \le m < 1.6$.

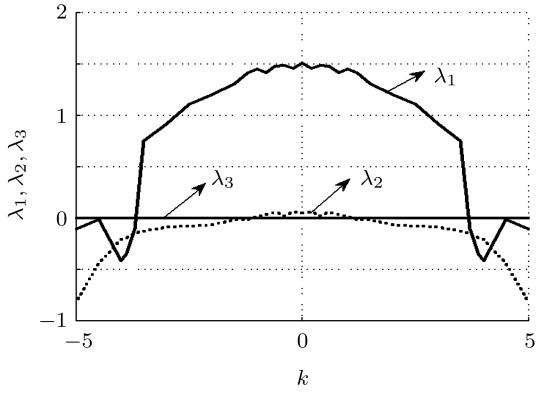


Fig.2. The Lyapunov exponent spectrum of system (2) versus k when $m = 1.2$.

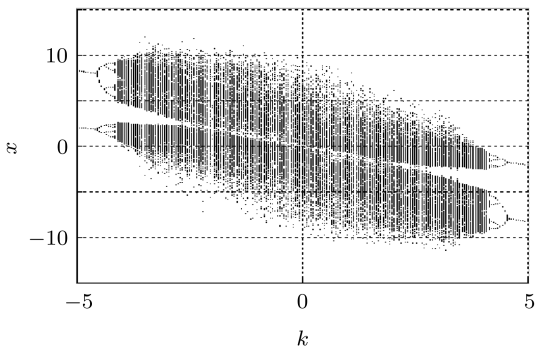
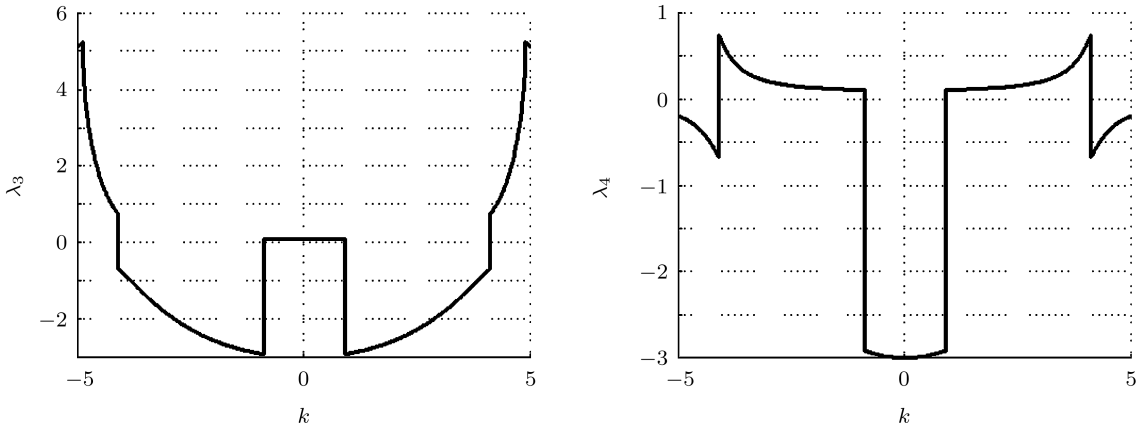


Fig.3. Bifurcation diagram with k increasing for $m = 1.2$.

Some typical phase portraits are also given in Fig.5.

(2) When $1.6 \leq m < 12$, the range of hyper-chaos decreases as the value of m increases; when $5 \leq m \leq 9.5$, for any constant k, there exists no hyper-chaotic state; when $m > 12$ for any value of k the system will enter into quasi-periodic or periodic orbit.

When the value of parameter m varies from 1.6 to 5, we find that the scope of hyper-chaos decreases as the value of m increases: for example, the range of hyper-chaos is $-0.3 \leq k \leq -0.3$ when $m = 3$, and $-0.2 \leq k \leq -0.2$ when $m = 4$. When $5 \leq m \leq 9.5$,



(a) Characteristic root λ_3

(b) Characteristic root λ_4

Fig.4. The characteristic roots with $m = 1.2$ and k being variable.

Table 1. Ranges of typical parameters m and k leading to hyper-chaos and chaos.

m	Range of hyper-chaos	Range of chaos	Lyapunov exponents for typical k
0.5	$-0.6 < k < 0.6$	$-1.6 < k \leq -0.7,\ 0.7 \leq k < 1.6$	$k = 0.5$:(hyper-chaos) $\lambda_1 = 1.5582,\ \lambda_2 = 0.0111,\ \lambda_3 = 0,\ \lambda_4 = -12.5627$ $k = 1$(chaos) $\lambda_1 = 1.2121,\ \lambda_2 = -0.03179,\ \lambda_3 = 0,\ \lambda_4 = -12.1789$
1.2	$-0.9 < k < 0.9$	$-4.1 < k \leq -0.9,\ 0.9 \leq k < 4.1$	$k = 0.8$:(hyper-chaos) $\lambda_1 = 1.4107,\ \lambda_2 = 0.0520,\ \lambda_3 = 0,\ \lambda_4 = -12.4592$ $k = 3$(chaos) $\lambda_1 = 0.9132, \lambda_2 = -0.0904,\ \lambda_3 = 0,\ \lambda_4 = -11.8274$
1.3	$-1.0 < k < 1.0$	$-4.5 < k \leq -1.0,\ 1.0 \leq k < 4.5$	$k = 0.9$(hyper-chaos) $\lambda_1 = 1.4745,\ \lambda_2 = 0.0353,\ \lambda_3 = 0,\ \lambda_4 = -12.5040$ $k = 4$:(chaos) $\lambda_1 = 0.5354, \lambda_2 = -0.0763,\ \lambda_3 = 0,\ \lambda_4 = -11.4608$
1.5	$-1.2 < k < 1.2$	$-5.2 < k \leq -1.2,\ 1.2 \leq k < 5.2$	$k = 1.1$(hyper-chaos) $\lambda_1 = 1.3467,\ \lambda_2 = 0.0256,\ \lambda_3 = 0,\ \lambda_4 = -12.3664$ $k = 5$:(chaos) $\lambda_1 = 0.3711,\ \lambda_2 = -0.1172,\ \lambda_3 = 0,\ \lambda_4 = -12.2527$

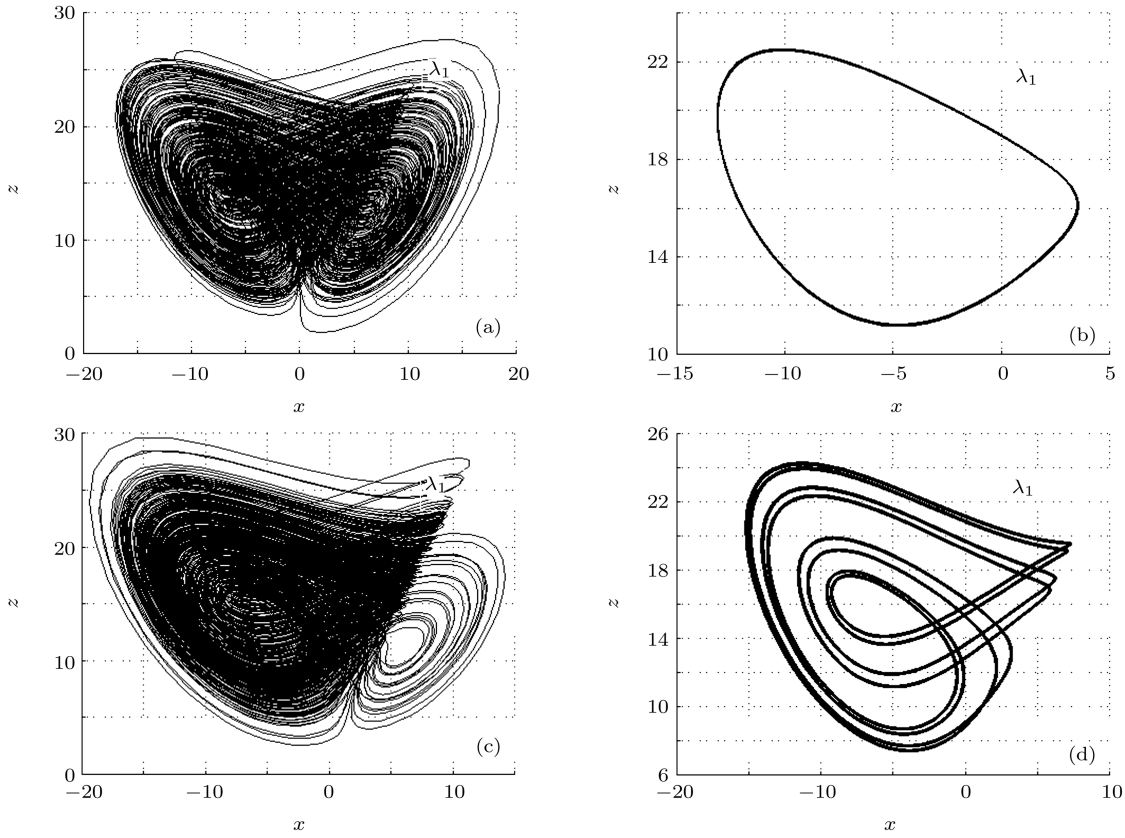


Fig.5. Phase portraits of system (2) for different values of k and m, showing hyper-chaos with $m = 0.5$ and $k = 0.2$ (a), quasi-periodic orbits with $m = 0.5$ and $k = 2$ (b), chaos with $m = 1.2$ and $k = 3$ (c), and quasi-periodic orbits with $m = 1.3$ and $k = 4.5$ (d).

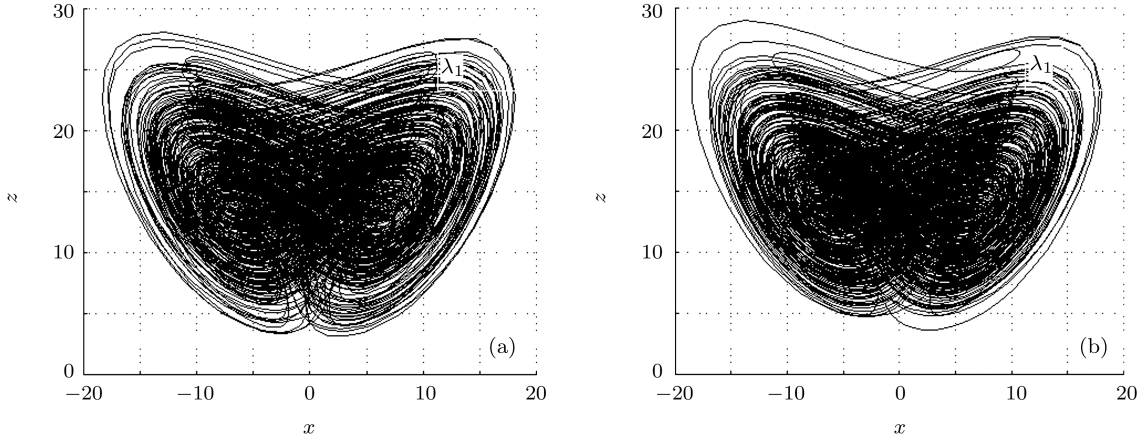
some simulation results show that system (2) displays only chaotic dynamical behaviours, while when $m > 9.5$, the system enters quasi-periodic or periodic orbit for any value of k. Some typical Lyapunov exponents and the phase portraits with different values of m and k are summarized as follows:

1) when $m = 4$ and $k = 0.2$, then $\lambda_1 = 1.1753$, $\lambda_2 = 0.0615$, $\lambda_3 = 0$ and $\lambda_4 = -12.2176$, and system (2) is hyper-chaotic (Fig.6(a));

2) when $m = 6$ and $k = 0$, then $\lambda_1 = 0.7398$, $\lambda_2 = -0.0655$, $\lambda_3 = 0$ and $\lambda_4 = -11.6859$, and system (2) is chaotic (Fig.6(b));

3) when $m = 8$ and $k = 2$, then $\lambda_1 = 0.5369$, $\lambda_2 = -0.3490$, $\lambda_3 = 0$ and $\lambda_4 = -11.1860$, and system (2) is chaotic and the chaotic attractor is shown in Fig.6(c);

4) when $m = 13$ and $k = 4$, then $\lambda_1 = -0.4840$, $\lambda_2 = -1.3664$, $\lambda_3 = 0$ and $\lambda_4 = -9.1518$, and system (2) is periodic-orbits, which is shown in Fig.6(d).



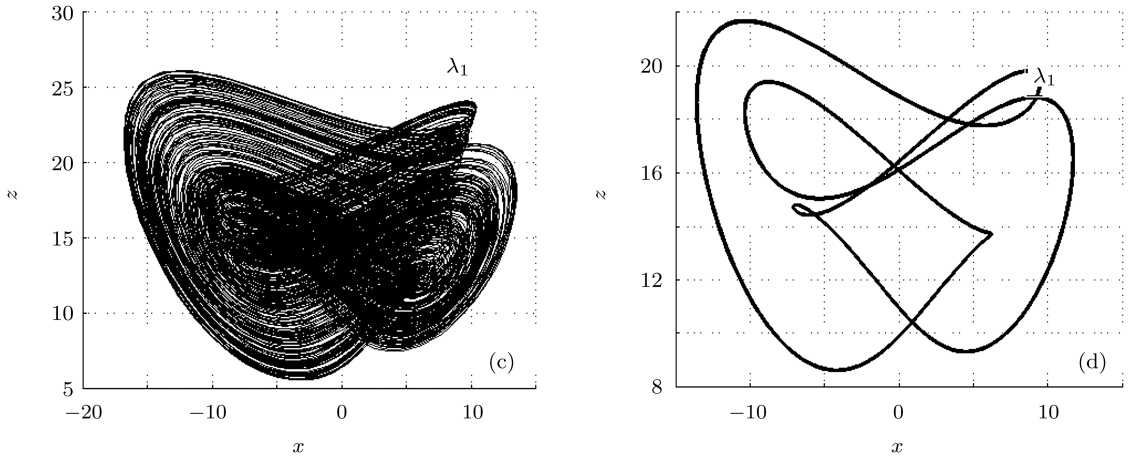


Fig.6. Phase portraits of system (2) for different values of k and m, showing hyper-chaos with $m = 4$ and $k = 0.2$ (a), chaos with $m = 6$ and $k = 0$ (b), chaos with $m = 8$ and $k = 2$ (c), and periodic orbits with $m = 13$ and $k = 4$ (c).

2.2. Choose $a = 36$, $b = 3$, $c = 28$, $d = -16$, k to be a certain value, and the value of m to be variable

For system (2), from the above conclusions it follows that when $a = 36$, $b = 3$, $c = 28$ and $d = -16$, k is fixed to be a certain value, and the value of m is variable, the dynamical characteristics can be summarized in the following:

(1) when $-1.2 \leq k \leq 1.2$, system (2) undergoes hyper-chaos, chaos and periodic orbits with the value of m varying, and the scope of hyper-chaos gradually decreases as the absolute value of k increases. For example, when $k = 0.5$, the dynamical characteristic of system (2) is described in the following:

1) when $0 < m \leq 0.14$, the system displays quasi-periodic or periodic orbits (Fig.7(a));

2) when $0.14 < m \leq 0.4$, the system is chaotic (Fig.7(b));

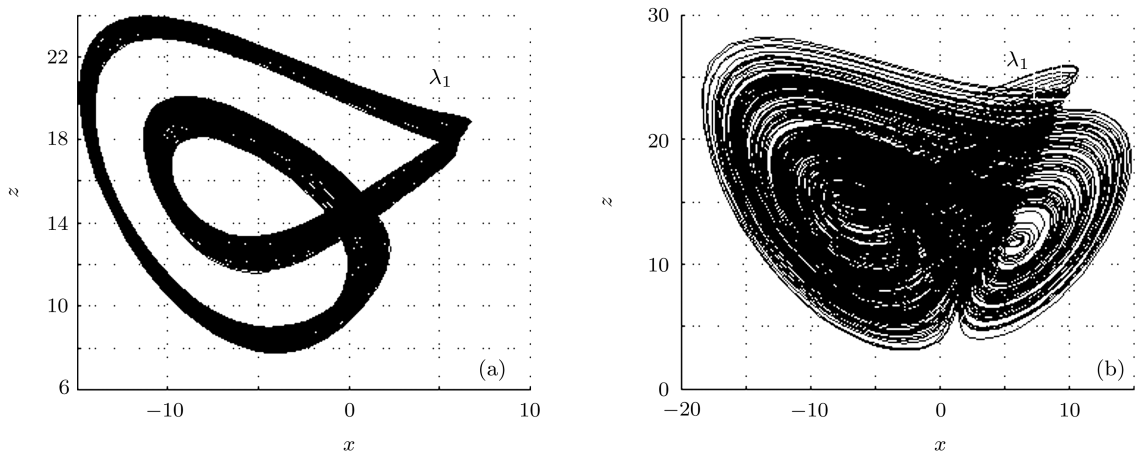
3) when $0.4 < m \leq 1.8$, the system enters into the hyper-chaotic state (Fig.7(c));

4) when $1.8 < m \leq 8.5$, the system displays chaos again;

5) when $8.5 < m \leq 10$, the system exhibits quasi-periodic or periodic orbits again (Fig.7(d)).

The corresponding bifurcation diagram of state x with respect to m with the parameter $k = 0.5$ is given in Fig.8.

(2) when $k > 1.2$ or $k < -1.2$, for any value of m, there exists only chaos or periodic orbit, but no hyper-chaotic phenomenon.



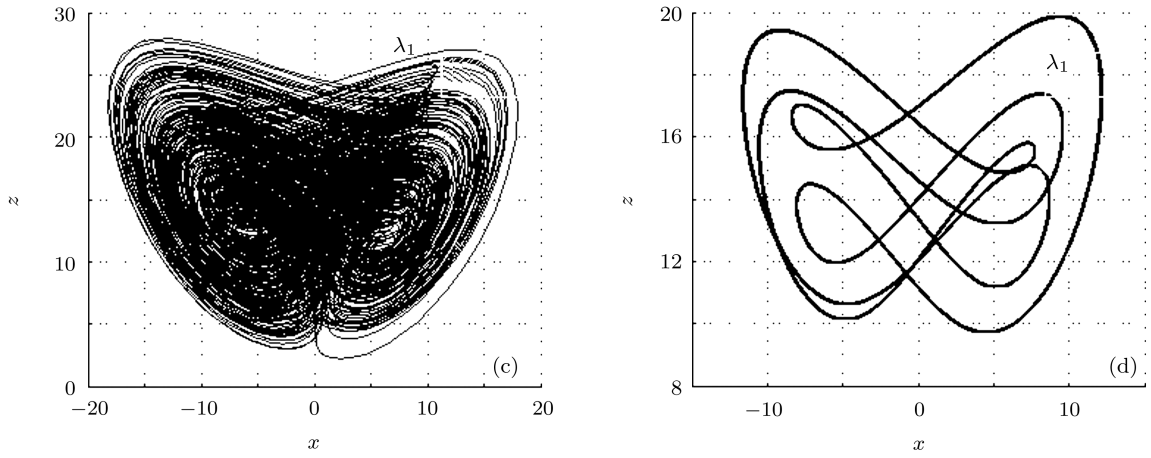


Fig.7. Phase portraits of system (2) for $k = 0.5$ and different values of m, showing quasi-periodic orbits with $m = 0.13$ (a), chaos with $m = 0.4$ (b), hyper-chaos with $m = 1$ (c), and periodic orbits with $m = 9.5$ (d).

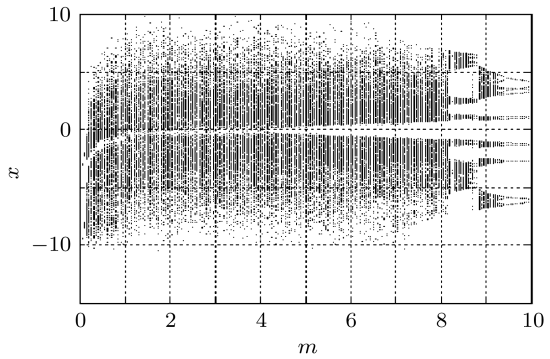


Fig.8. Bifurcation diagram with m increasing for $k = 0.5$.

3. Conclusions

An improved hyper-chaotic system based on the hyper-chaos generated from Chen's system is presented, and some basic dynamical properties of the system are investigated by means of Lyapunov exponent spectrum, bifurcation diagrams and characteristic equation roots. The simulations show that the new improved system evolves into hyper-chaotic, chaotic, various quasi-periodic or periodic orbits when one parameter of the system is fixed to be a certain value while the other one is variable. Some computer simulations and bifurcation analyses are given to testify the findings.

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