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Subleading terms of thermodynamic quantities around static spherical black holes

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The thermodynamic quantities for the gases of massless particles with spin $s = 1/2, 1, 3/2, 2$ around static spherical black holes are investigated by using the brick-wall method. The appearance of the spin-dependent subleading terms is demonstrated and the terms are shown to contain not only the linear and quadratic terms of the spins but also a zero-power term of the spins.

Keywords: thermodynamic quantity, brick-wall method, Boulware vacuum state, static spherical black holes

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1. Introduction

Logarithmic corrections to Bekenstein–Hawking entropy have been extensively studied for the last several years and the existing computations of logarithmic corrections have very different physical starting points, such as Carlip's work[1] based on the Cardy formula, and thermal fluctuations.[2] In particular, logarithmic corrections to Bekenstein–Hawking entropy due to spin fields propagating in black hole background were obtained in Refs.[3–17] by using the brick-wall method[18] and were found to be spin-dependent.

One generally assumes that the thermodynamic quantities in curved spacetime take the same forms as that in flat space–time. Obviously, this assumption is inconsistent with the results mentioned above. In fact, Li [19,20] applied the quantization procedure, referred to as Boulware vacuum state[21] and Killing time t to study the thermodynamic quantities for a perfect relativistic gas around the Schwarzschild black hole and the Reissner–Nordström black hole, and found that, although different fields obey different statistics, the corrected expressions of these quantities include additional subleading terms with spin-dependence, which include a quadratic term and a linear term of the spins. In this paper, we attempt to extend this method to the general static spherical black holes and show that the subleading terms of the thermodynamic quantities for the perfect relativistic gases with spin $s = 1/2, 1, 3/2, 2$ usually include the quadratic term of the spins, the linear term of the spins and a zero-power term of the spins.

2. Space–time and spin fields

In the Boyer–Lindquist coordinates, the metrics of the general static spherical black holes can be written as

$$\begin{aligned} \mathrm{d}s^2 \;=\; & B(r)\mathrm{d}t^2 - B^{-1}(r)\mathrm{d}r^2 \\ & -C(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2). \end{aligned} \tag{1}$$

The horizons are determined by $B = 0$. If r_{H} denotes the event horizon position, the Hawking temperature is $\beta^{-1} = B'(r_{\mathrm{H}})/(4\pi)$. We establish the following null tetrad vectors:

$$\begin{aligned} l^\mu &= B^{-1}\delta^\mu_0 + \delta^\mu_1; \\ n^\mu &= \frac{1}{2}\delta^\mu_0 - \frac{B}{2}\delta^\mu_1; \\ m^\mu &= \frac{1}{\sqrt{2C}}\delta^\mu_2 + \frac{\mathrm{i}}{\sqrt{2C}\sin\theta}\delta^\mu_3. \end{aligned} \tag{2}$$

From Eqs.(1) and (2), we can obtain the nonvanishing spin coefficients and component of the Weyl tensor by using the Newman–Penrose formula[22]

$$\alpha \;=\; -\beta = -\frac{\mathrm{ctg}\theta}{2\sqrt{2C}}, \qquad \rho = -\frac{C'}{2C},$$

$$\mu = -\frac{BC'}{4C}, \quad \gamma = \frac{B'}{4},$$
$$\Psi_2 = \frac{5}{12}B'' + \frac{1}{12}\frac{B'C'}{C} + \frac{1}{12}\frac{B'C''}{C} - \frac{1}{12}\frac{B'C'^2}{C^2} + \frac{1}{6C}. \tag{3}$$

Equations (3) show that the metric (1) is of Petrov type-D. Using the result of Teukolsky and Press,[23] the field equations of the Weyl neutrino ($s = 1/2$), electromagnetic ($s = 1$), massless Rarita–Schwinger ($s = 3/2$) and gravitational ($s = 2$) fields for the course free case can be combined into[3]

$$\begin{aligned}&\{[D-(2s+1)\rho](\Delta - 2s\gamma + \mu) - [\delta + (2s-2)\alpha]\\ &\times(\bar{\delta} - 2s\alpha) - (2s-1)(s-1)\,\Psi_2\}\,\Phi_{+s} = 0;\\ &\{[\Delta + (2s-2)\gamma + (2s+1)\mu](D-\rho)\\ &-[\bar{\delta} + (2s-2)\alpha](\delta - 2s\alpha) - (2s-1)(s-1)\,\Psi_2]\\ &\times\Phi_{-s} = 0,\end{aligned} \tag{4}$$

where $D = l^\mu\partial_\mu, \Delta = n^\mu\partial_\mu, \delta = m^\mu\partial_\mu$ are the directional derivatives. In Eq.(4) the first equation is for spin states $p = s$ and the second one for $p = -s$.

3. Thermodynamic quantities

In this section, we calculate the thermodynamic quantities by WKB approximation. In the quantization procedure, referred to as Boulware vacuum state and Killing time t, the mode functions Φ_p of these spin fields around the static spherical black hole are $\Phi_p = \exp[\mathrm{i}S(r,p,l,E,)] \cdot {}_pY_l^m(\theta,\varphi) \cdot \mathrm{e}^{-\mathrm{i}Et}$, where ${}_pY_l^m(\theta,\varphi)$ is a spin-weighted spherical harmonic and l, m are integers satisfying the inequalities $l \geq s$ and $-l \leq m \leq l$. After separating the variables of Eq.(4), we obtain the radial wave number by using the formula $k_p \equiv \partial_r S$,

$$k_p^2 = \frac{E^2}{B^2} + \frac{1}{BC}[A_p - l(l+1)], \tag{5}$$

where

$$\begin{aligned}A_{+s} =\ & sB''C + \frac{2s^2+s+1}{2}B'C' + \frac{1}{2}BC'' \\ &+\frac{2s-1}{4}\frac{BC'^2}{C} + 2(2s-1)(s-1)C\,\Psi_2 \\ &+s^2 + s, \\ A_{-s} =\ & \frac{1-s}{2}B'C' + \frac{1}{2}BC'' + \frac{2s-1}{4}\frac{BC'^2}{C} \\ &+2(2s-1)(s-1)C\,\Psi_2 + s^2 - s.\end{aligned} \tag{6}$$

According to the semi-classical quantization rule, the wave number is quantized as

$$\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L} k_p \mathrm{d}r = n\pi, \quad n \in N, \tag{7}$$

under the brick-wall boundary conditions[18] $\Phi_p = 0$ at $r = r_\mathrm{H} + \varepsilon$ and $r = L$. Here the quantum number $n > 0$ is integer and ε is a small positive constant denoting the distance of the brick wall from the horizon. Then, the number of modes with energy smaller than E reads

$$\begin{aligned}g(E) =\ & \int_s^{l_{\max}} (2l+1)\mathrm{d}l \cdot \frac{1}{\pi}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L} (k_{+s} + k_{-s})\mathrm{d}r \\ =\ & \frac{2}{3\pi}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L} \frac{C}{B^2}\left[\left(E^2 + \frac{B}{C}(A_{+s} - s^2 - s)\right)^{3/2}\right. \\ &\left.+\left(E^2 + \frac{B}{C}(A_{-s} - s^2 - s)\right)^{3/2}\right]\mathrm{d}r,\end{aligned} \tag{8}$$

where the upper limit $l_{\max}$ is determined by Eq.(5).

The free energy of the system is given by

$$-\beta F = \pm\sum_\alpha \ln(1 \pm \mathrm{e}^{-\beta E_\alpha}), \tag{9}$$

where the plus sign corresponds to the Fermi case and the minus sign to the Bose case. Using Eq.(8) to determine the state density, we obtain the free energy

$$\begin{aligned}F =\ & \mp\frac{1}{\beta}\int_0^\infty \mathrm{d}E\frac{\mathrm{d}g(E)}{\mathrm{d}E}\ln(1 \pm \mathrm{e}^{-\beta E}) \\ =\ & -\int_0^\infty \frac{g(E)}{\mathrm{e}^{\beta E} \pm 1}\mathrm{d}E \\ =\ & -\frac{15 + (-1)^{2s}}{16}\frac{2\omega\pi^3}{45\beta^4}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L}\frac{C}{B^2}\mathrm{d}r - \frac{3 + (-1)^{2s}}{4} \\ &\times\frac{\pi}{6\beta^2}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L}\frac{A_{+s} + A_{-s} - 2s^2 - 2s}{B}\mathrm{d}r,\end{aligned} \tag{10}$$

where $\omega = 2$ is the spin degeneracy of the particles of the fields considered here. Using formulas $S = \beta^2\partial F/\partial\beta$ and $U = \partial(\beta F)/\partial\beta$, we can obtain the entropy and energy

$$\begin{aligned}S =\ & \frac{15 + (-1)^{2s}}{16}\frac{8\omega\pi^3}{45\beta^3}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L}\frac{C}{B^2}\mathrm{d}r + \frac{3 + (-1)^{2s}}{4} \\ &\times\frac{\pi}{3\beta}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L}\frac{A_{+s} + A_{-s} - 2s^2 - 2s}{B}\mathrm{d}r, \\ U =\ & \frac{15 + (-1)^{2s}}{16}\frac{2\omega\pi^3}{15\beta^4}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L}\frac{C}{B^2}\mathrm{d}r + \frac{3 + (-1)^{2s}}{4} \\ &\times\frac{\pi}{6\beta^2}\int_{r_\mathrm{H}+\varepsilon}^{r_\mathrm{H}+L}\frac{A_{+s} + A_{-s} - 2s^2 - 2s}{B}\mathrm{d}r.\end{aligned} \tag{11}$$

On the other hand, the total entropy and the total energy are given by

$$\begin{aligned} S &= \int_{r_{\rm H}+\varepsilon}^{r_{\rm H}+L} \delta(r)\frac{A(r){\rm d}r}{\sqrt{B}} = \int_{r_{\rm H}+\varepsilon}^{r_{\rm H}+L} \sigma(r)\frac{4\pi C{\rm d}r}{\sqrt{B}}, \\ U &= \int_{r_{\rm H}+\varepsilon}^{r_{\rm H}+L} \rho(r)A(r){\rm d}r = L\int_{r_{\rm H}+\varepsilon}^{r_{\rm H}+\varepsilon+h} \rho(r)4\pi C{\rm d}r, \end{aligned} \tag{12}$$

where we have taken the spherical shell as the volume element and $A(r) = \int \sqrt{g_{\theta\theta}g_{\varphi\varphi}}{\rm d}\theta{\rm d}\varphi$ is the area of the curved surface at random point r outside the horizon in the space–time (1). The factor $1/\sqrt{B}$ does not appear in the integral for the total energy of the thermal excitations.[24] Comparing Eq.(12) with Eq.(11), we can obtain the entropy density and energy density

$$\begin{aligned} \sigma(r) &= \frac{15+(-1)^{2s}}{16}\frac{2\omega\pi^2}{45}T^3(r) \\ &\quad +\frac{3+(-1)^{2s}}{4}\frac{1}{12}\eta(r)T(r), \end{aligned} \tag{13}$$

$$\begin{aligned} \rho(r) &= \frac{15+(-1)^{2s}}{16}\frac{\omega\pi^2}{30}T^4(r) \\ &\quad +\frac{3+(-1)^{2s}}{4}\frac{1}{24}\eta(r)T^2(r), \end{aligned} \tag{14}$$

where $T(r) = (\beta\sqrt{B})^{-1}$ is the local temperature.[25]

Using the Unruh–Wald relation $P(r) = \sigma(r)T(r) - \rho(r)$,[26] we can obtain the pressure

$$\begin{aligned} P(r) &= \frac{15+(-1)^{2s}}{16}\frac{\omega\pi^2}{90}T^4(r) \\ &\quad +\frac{3+(-1)^{2s}}{4}\frac{1}{24}\eta(r)T^2(r). \end{aligned} \tag{15}$$

The equation of state is

$$\rho(r) - 3P(r) = -\frac{3+(-1)^{2s}}{4}\frac{1}{12}\eta(r)T^2(r), \tag{16}$$

which is not zero. The function $\eta(r)$ in Eqs.(13)–(16) is given by

$$\begin{aligned} \eta(r) &= \frac{A_{+s}+A_{-s}-2s^2-2s}{C} \\ &= a_2 s^2 + a_1 s + a_0, \end{aligned} \tag{17}$$

where

$$\begin{aligned} a_2 &= \frac{B'C'}{C} + 8\,\Psi_2 \\ &= \frac{10}{3}B'' + \frac{5}{3}\frac{B'C'}{C} + \frac{2}{3}\frac{BC''}{C} \\ &\quad -\frac{2}{3}\frac{BC'^2}{C^2} + \frac{4}{3}\frac{1}{C}; \\ a_1 &= B'' + \frac{BC'^2}{C^2} - \frac{2}{C} - 12\,\Psi_2 \\ &= -4B'' - \frac{B'C'}{C} - \frac{BC''}{C} \\ &\quad +2\frac{BC'^2}{C^2} - 4\frac{1}{C}; \\ a_0 &= \frac{B'C'}{C} + \frac{BC''}{C} - \frac{1}{2}\frac{BC'^2}{C^2} + 4\,\Psi_2 \\ &= \frac{5}{3}B'' + \frac{4}{3}\frac{B'C'}{C} + \frac{4}{3}\frac{BC''}{C} \\ &\quad -\frac{5}{6}\frac{BC'^2}{C^2} + \frac{2}{3}\frac{1}{C}. \end{aligned} \tag{18}$$

4. Discussion

Our results (13)–(16) show that the usual results, i.e., the entropy density, energy density, pressure and equation of state in curved space–time take the same forms as those in flat one; this holds only for the leading terms in powers of the temperature. The additional sub-leading term in Eqs.(13)–(16) explicitly depends on the spins of the fields and usually contains not only a quadratic term and a linear term of the spins but also a zero-power term of them. The existence of the term is compatible with the conclusions that the black hole entropy is not exactly proportional to the horizon area, and it is important for a further discussion of black hole properties.

We can easily show that the zero-power term in Eq.(17) vanishes, i.e., $a_0 = 0$ for the Schwarzschild black holes and Reissner–Nordström black holes, so the subleading terms in Eqs.(13)–(16) just depend on the quadratic term and the linear term of the spins. This result is the same as that obtained in Refs.[19] and [20].

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