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Hawking radiation from the dilaton–(anti) de Sitter black hole via covariant anomaly*

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Adopting the anomaly cancellation method, initiated by Robinson and Wilczek recently, this paper discusses Hawking radiation from the dilaton–(anti) de Sitter black hole. To save the underlying gauge and general covariance, it introduces covariant fluxes of gauge and energy–momentum tensor to cancel the gauge and gravitational anomalies. The result shows that the introduced compensating fluxes are equivalent to those of a 2-dimensional blackbody radiation at Hawking temperature with appropriate chemical potential.

Keywords: Hawking radiation, covariant anomaly, dilaton–(anti) de Sitter black hole
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1. Introduction

In 1975, Stephen Hawking made a striking discovery that black holes are not black completely but can radiate particles in the form of a precise thermal spectrum.[1] After this work, many approaches on Hawking radiation have been proposed in the past three decades.[2−14] Recently, Robinson and Wilczek proposed a new derivation of Hawking radiation from the Schwarzschild-type black hole from the anomaly point of view.[15] The key point of this method is that an arbitrary dimensional space–time can be reduced to a 2-dimensional theory effectively at the event horizon (EH). Then in the 2-dimensional reduction, they introduced a flux to eliminate the anomaly that stems from the ignorance of the classically insignificant ingoing modes. This compensating flux eventually was showed precisely to equal to that of a 2-dimensional blackbody at the Hawking temperature. Along this line, many extensions have since been done successfully.[16−19]

In this case, we further extend their work to the dilaton–(anti) de Sitter black hole. Since the dilaton is a kind of scalar field occurring in the low energy limit of the string theory, the presence of dilaton has important consequences on the causal structure and the thermodynamic properties of the black hole, thus the dilaton space–time has been discussed extensively recently.[20,21] On the other hand, there has also been some renewed interest in the study of black hole theories with the cosmological constant, because the recent observations, together with the theory of inflation, suggest that our universe approaches to de Sitter space–time geometrically in both the far past and far future only with different values of cosmological constant, which has shed light on the quantum gravitational mysteries. In addition, the anti-de-Sitter space/conformal field theory correspondence[22] also attracts attention to the de Sitter space. Thus to further understand the basic properties of dilaton black hole in de Sitter and anti-de Sitter space–time, as an intermediate, we study Hawking radiation from the dilaton–(anti) de Sitter black hole from the viewpoint of anomaly. We also ignore the back-reaction of the emission in this paper. As far as the anomaly cancellation conditions is concerned in this paper, if we along the initial approach of Robinson and Wilczek,[15] the consistent gauge and gravitational anomalies must be introduced. This implies that one must input two anomalies to restore the gauge or general coordinate covariance. More recently, it was suggested that the same work could be done by only employing the covariant gangue and gravitational anomalies.[23] This approach is believed conceptually to be clean and economical. Hence we thereafter adopt this simplified approach of cancelling the covariant anomaly to determine the near-horizon quantum anomaly.

The organization of this paper is outlined as follows. In Section 2, we establish the effective theory by

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reducing the dilaton–(anti) de Sitter space–time to an effective 2-dimensional theory. Then in Section 3, we explore the relation between the compensating fluxes and those of Hawking radiation. Section 4 is devoted to our conclusions.

2. The two-dimensional theory

The action describing the dilaton field coupled to the Maxwell field with coupling constant $\alpha = 1$ in the four dimensions is[24]

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}[R - \partial_\mu\phi\partial^\mu\phi - V(\phi) - \mathrm{e}^{-2\phi}F^2], \quad (1)$$

where R is the scalar curvature, ϕ is the dilaton field, and $F = F_{\mu\nu}F^{\mu\nu}$ is the usual Maxwell contribution. From the action, the metric of the 4-dimensional spherically dilaton black hole is obtained as

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r(r-2D)\mathrm{d}\Omega_2^2, \quad (2)$$

where

$$f = \left[1 - \frac{2M}{r} - r(r-2D)H^2\right], \quad (3)$$

in which M and D are the mass and dilaton charge of the black hole, H is the Hubble constant that satisfies $H^2 = \frac{\lambda}{3}$, and λ is the cosmological constant which can be positive or negative corresponding to, respectively, the de Sitter space and anti-de Sitter space. The corresponding dilaton field, dilaton charge and dilaton potential are given by

$$\mathrm{e}^{2\phi} = \mathrm{e}^{2\phi_0}\left(1 - \frac{2D}{D + \sqrt{r^2 + D^2}}\right), \quad (4)$$

$$D = \frac{Q^2\mathrm{e}^{2\phi_0}}{2M}, \quad (5)$$

$$A_t(\phi) = \frac{4}{3}\lambda + \frac{1}{3}\lambda\left[\mathrm{e}^{2(\phi-\phi_0)} + \mathrm{e}^{-2(\phi-\phi_0)}\right], \quad (6)$$

respectively, in which, ϕ_0 is the asymptotic constant value of the dilaton.

According to the null hyper-surface equation, $f(r) = 0$ will yield two positive roots in the de Sitter space denoted by r_+ and r_c respectively, and only one positive root in the anti de Sitter space, represented by r_h. For Hawking radiation in de Sitter space, as related in Ref.[18], the discussion at de Sitter cosmological horizon is similar to that at the black hole horizon other than some changes of the sign. So in this paper we focus on the physics that near the black hole EH. For the line element (2), the surface gravity at the EH takes the form as

$$\kappa_+ = \frac{f'(r)}{2}\Big|_{r_+}. \quad (7)$$

Before deriving Hawking radiation via anomaly cancellation conditions at the horizon, we first show that the scalar field on this metric can be reduced to the 2-dimensional theory effectively. The action of the scalar field on the 4-dimensional space–time is

$$\begin{aligned} S[\varphi] &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\varphi\nabla^2\varphi \\ &= \frac{1}{2}\int \mathrm{d}t\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi r^2\sin\theta\sqrt{-g}\phi \\ &\quad\times\left\{-\frac{1}{f}\partial_t^2 + f\partial_r^2 + \frac{1}{r(r-2D)}\Delta_\Omega\right\}\phi. \end{aligned} \quad (8)$$

After carrying out the partial wave decomposition in terms of the spherical harmonics $\varphi = \sum_n \varphi_n(t,r)Y_n(\theta,\varphi)$ and performing the tortoise transformation defined by $r_* = \int \mathrm{d}r/f$, when the subordinate terms are ignored, the action becomes

$$S[\varphi] = \frac{1}{2}\sum_{lm}\int \mathrm{d}t\mathrm{d}r\,\Psi\phi_{lm}\left[-\frac{1}{f}\partial_t^2 + f\partial_r^2\right]\phi_{lm}. \quad (9)$$

It is straightforwardly shown that the physics near the horizon can be effectively described by an infinite collection of massless 2-dimensional scalar fields on the metric

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2, \quad (10)$$

where the dilaton background $\Psi = r^2(r-2D)^2$ is often ignored in the static space–time.

3. Hawking radiation and covariant anomaly

Now we concentrate on exploring the relation between the covariant anomaly and Hawking radiation. In the 2-dimensional space–time, each partial wave mode has both gauge and general coordinate symmetries. Once the classically insignificant ingoing modes are ignored to formulate the effective field outside the EH, it becomes chiral and suffers from gauge and gravitational anomalies with respect to the gauge and general coordinate symmetries respectively. At the quantum level however, there is no anomaly, thus one should introduce a compensating flux to cancel it. We first discuss the gauge anomaly of charge flow. For convenience, we split the region outside the horizon

into $[r_+, r_+ + \varepsilon]$ and $[r_+ + \varepsilon, +\infty]$. The covariant form of the 2-dimensional Abelian anomaly[22,23] is given by

$$\nabla_\mu \tilde{J}^\mu = \pm \frac{e^2}{4\pi\sqrt{-g}} \varepsilon^{\mu\nu} F_{\mu\nu}. \tag{11}$$

Where $+(-)$ corresponds to the left (right) handed field while $\varepsilon^{\mu\nu}$ is the anti-symmetric tensor with non-vanishing component $\varepsilon^{10} = \varepsilon^{01} = -1$. Due to the horizon is a null hyper-surface, the ingoing modes can be discarded classically to construct the effective field. At the quantum level however, they can not be ignored, anomaly thus takes place in the near-horizon region $[r_+, r_+ + \varepsilon]$, that is

$$\nabla_r \tilde{J}^r_{(\mathrm{H})} = \frac{e^2}{4\pi} F_{\mathrm{rt}}. \tag{12}$$

While in $[r_+ + \varepsilon, +\infty]$, there is not anomaly thus the covariant charge flow obeys

$$\nabla_r \tilde{J}^r_{(0)} = 0. \tag{13}$$

The covariant current outside the horizon can be solved in each region as

$$\tilde{J}^r_{(\mathrm{H})} = e_{(\mathrm{H})} + \frac{e^2}{2\pi} \left(A_t - A_t \left(r_+ \right) \right), \tag{14}$$

$$\tilde{J}^r_{(\mathrm{O})} = e_0, \tag{15}$$

where e_{H} and e_0 are integration constants, which stand respectively for the values of the covariant fluxes of charge flow at the EH and infinity. Then the Ward identity is

$$\begin{aligned} \nabla_\mu \tilde{J}^\mu &= \partial_r \tilde{J}^r \\ &= \partial_r \left(\frac{e^2}{2\pi} A_t H \right) + \left[\tilde{J}^r_{(0)} - \tilde{J}^r_{(\mathrm{H})} + \frac{e^2 A_t(r)}{2\pi} \right] \\ &\quad \times \delta \left(r - r_+ - \varepsilon \right). \end{aligned} \tag{16}$$

Where, we have written the covariant flux as the sum of two regions, namely $\tilde{J}^\mu_{(2)} = \tilde{J}^u_{(0)} \Theta_+(r) + \tilde{J}^\mu_{(\mathrm{H})} H(r)$, by the scalar step function $\Theta_+ = \Theta \left(r - r_+ - \varepsilon \right)$ and scalar top hat function $H = 1 - \Theta_+$. The first term would be cancelled by the quantum effects of the classically irrelevant ingoing modes in the right of Eq.(16). This is the Wess–Zumino term induced by the ingoing modes, which contributes to the total gauge flux $-e^2 A_t H/2\pi$ at the horizon.[23] Therefore to save the underlying general covariance, the coefficients of the delta function should satisfy

$$e_0 = e_{\mathrm{H}} - \frac{e^2}{2\pi} A_1(r_+), \tag{17}$$

in which, the integration constant e_{H} is determined by imposing the covariant gauge flux to vanish at the EH. This is the so-called regular boundary condition.[13] Thus the total covariant compensating flux of electric charge, which is required to cancel the gravitational anomaly at the EH to save the gauge covariance at the quantum level, can be written as

$$e_0 = \frac{e^2}{2\pi} A_t(r_+). \tag{18}$$

Where $A_t(r_+)$ is the value of A_t at the horizon, this current will be showed to precisely equal to that of the Hawking radiation flux with an appropriate chemical potential.

Next, we turn to study gravitational anomaly. When the classically irrelevant ingoing modes are excluded to formulate an effective field at the EH, due to the pileup of outgoing high frequency modes, the energy–momentum tensor is divergent, which makes the effective field becoming chiral and exhibiting gravitational anomaly. For the right-handed field, the covariant anomaly takes the form as[25,26]

$$\nabla_\mu \tilde{T}^\mu_\nu = -\frac{1}{96\pi} \varepsilon_{\mu\nu} \partial^\mu R = \tilde{A}_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \tilde{N}^\mu_\nu. \tag{19}$$

For the 2-dimensional space–time, the covariant anomaly is purely time-like ($\tilde{A}_r = 0$) while

$$\tilde{A}_t = \partial_r \tilde{N}^r_t, \qquad \tilde{N}^r_t = \frac{[2ff'' - (f')^2]}{192\pi}. \tag{20}$$

Since the existence of electric field, the covariant energy–momentum tensor in the region $[r_+ + \varepsilon, \infty]$ is not conserved even classically but keeps to the Lorenz force law

$$\nabla_\mu \tilde{T}^\mu_{\nu(0)} = F_{\mu\nu} \tilde{J}^\mu_{(0)}. \tag{21}$$

While in the region $[r_+, r_+ + \varepsilon]$, by adding the covariant gravitational anomaly that stems from the divergent energy–momentum tensor, the Ward identity becomes

$$\nabla_\mu \tilde{T}^\mu_{\nu(\mathrm{H})} = F_{\mu\nu} \tilde{J}^\mu_{(\mathrm{H})} + \tilde{A}_\nu. \tag{22}$$

In the region $[r_+ + \varepsilon, \infty]$, with the aid of $\tilde{J}^r_{(0)} = e_0$, the solution of the Ward identity with $\nu = t$ component is

$$\tilde{T}^r_{t(0)} = a_0 + e_0 A_t. \tag{23}$$

In addition, inserting $\tilde{J}^r_{(\mathrm{H})}$ in Eq.(14) into Eq.(22), we obtain

$$\tilde{T}^r_{t(\mathrm{H})} = a_{(\mathrm{H})} + \int_{r_+}^{r} \mathrm{d}r \partial_r \left[e_0 A_t + \frac{e^2}{4\pi} A_t^2 + \tilde{N}^r_t \right]. \tag{24}$$

Then we find that the Ward identity outside the EH is

$$\begin{aligned}&\nabla_\mu \tilde{T}^\mu_t \\ &= \partial_r \tilde{T}^r_t = e_0 \partial_r A_t(r) + \left[\partial_r\left(\frac{e^2}{4\pi}A_t^2(r) + \tilde{N}^r_t\right)H\right] \\ &+\tilde{T}^r_{t(0)} - \tilde{T}^\mu_{t(\mathrm{H})} + \frac{e^2}{4\pi}A_t^2 + \tilde{N}^r_t]\delta\left(r - r_+ - \varepsilon\right).\end{aligned} \tag{25}$$

The first term is the classical effect that arises from the Lorenz force, the second term has to be cancelled by the quantum effects of the classically irrelevant ingoing modes. Therefore, to restore the underlying diffeomorphism invariance at the EH, the properties of the delta function would force

$$a_0 = a_{\mathrm{H}} + \frac{e^2}{4\pi}A_t^2(r_+) - \tilde{N}^r_t(r_+). \tag{26}$$

As before, we impose the vanishing condition for the flux of covariant energy–momentum tensor at the EH, which yields $a_{\mathrm{H}} = 0$. The compensating flux of energy–momentum tensor then can be written as

$$\begin{aligned}a_0 &= \frac{e^2}{4\pi}A_t^2(r_+) - \tilde{N}^r_t(r_+) \\ &= \frac{e^2}{4\pi}A_t^2(r_+) + \frac{\pi T_+^2}{12}.\end{aligned} \tag{27}$$

Where $T_+ = \kappa/2\pi$ is the Hawking temperature at the horizon of dilaton–(anti) de Sitter black hole.

At the EH, the Planck distribution of fermions with a chemical takes the form as

$$N_{\pm e}\left(\omega\right) = 1/\left(\exp\left(\frac{\omega \pm eA_t\left(r_+\right)}{T_+}\right) + 1\right). \tag{28}$$

Integrating it, we find

$$\begin{aligned}&\int_0^\infty \frac{e}{2\pi}[N_e\left(\omega\right) - N_{-e}\left(\omega\right)]\mathrm{d}\omega \\ &= -\frac{e^2}{2\pi}A_t\left(r_+\right),\end{aligned} \tag{29}$$

$$\begin{aligned}&\int_0^\infty \frac{\omega}{2\pi}[N_e\left(\omega\right) + N_{-e}\left(\omega\right)]\mathrm{d}\omega \\ &= \frac{e^2}{4\pi}A_t^2(r_+) + \frac{\pi T_+^2}{12}.\end{aligned} \tag{30}$$

Comparing Eqs.(18) and (27) with Eqs.(29) and (30), we find the fluxes of gauge and energy–momentum tensor, required cancelling the gauge and gravitational anomalies at the EH to restore the gauge and general invariance at the quantum level, precisely equal to those of a 2-dimensional blackbody at the Hawking temperature. That is, one can determine the Hawking radiation by the compensating fluxes of gauge and gravitational anomalies as the underlying gauge, and general coordinate covariance are saved.

4. Conclusions

In this paper, we investigated Hawking radiation from dilaton– (anti) de Sitter black hole, which is obtained from the string theory, from the viewpoint of anomaly. We introduced fluxes of electric current and energy–momentum to cancel the gauge anomaly and gravitational anomaly at the quantum level. The result showed that the compensating fluxes of the gauge and energy–momentum equal those of Hawking radiation, which supports the viewpoint of Robinson and Wilczek.

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