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The quantum spectral analysis of the two-dimensional annular billiard system*

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Based on the extended closed-orbit theory together with spectral analysis, this paper studies the correspondence between quantum mechanics and the classical counterpart in a two-dimensional annular billiard. The results demonstrate that the Fourier-transformed quantum spectra are in very good accordance with the lengths of the classical ballistic trajectories, whereas spectral strength is intimately associated with the shapes of possible open orbits connecting arbitrary two points in the annular cavity. This approach facilitates an intuitive understanding of basic quantum features such as quantum interference, locations of the wavefunctions, and allows quantitative calculations in the range of high energies, where full quantum calculations may become impractical in general. This treatment provides a thread to explore the properties of microjunction transport and even quantum chaos under the much more general system.

Keywords: closed-orbit theory, Fourier-transformed spectra, annular billiard, geometrical effect
PACC: 0365S

1. Introduction

Recently, nano-technology is more and more widely used in the field of micro-junction and consequently the related transport[1−3] property has become popular topic in both theoretical and experimental investigations. Many modal systems[4−6] have been analysed by using Gutzwiller periodic orbital theory. Their dynamic behaviours range from integrable to chaotic, particularly for optical billiards and chaotic systems, but it begins to deal with the mesoscopic quantum billiard in the semiclassical method.[7−9] As a typical theoretical model we study the correspondence of quantum behaviours and classical trajectories in an annular billiard based on an extended closed-orbited theory.[10] We calculate quantum spectra and compare the locations of the spectral peaks with the lengths of classical trajectories. The results show that there is a good correspondence between quantum spectra and classical orbits. This approach provides an intuitive understanding of basic quantum features such as quantum interference and locations of the wavefunctions, and allows quantitative calculations in the range of high energies, where full quantum calculations may become impractical in general.

2. Quantum spectral function and the Fourier transform

The system is made of a particle with mass μ moving freely in an annular region that is bounded by a large circle R and a confocal small circle $R_{\rm in} = fR, (0 < f < 1)$. Inside the annular region, the trajectories of the particle are straight forward until specular reflection at hard walls. Figure 1 is the schema of the annular billiard.

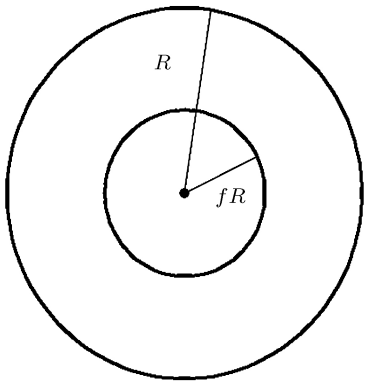


Fig.1. Geometry of the annular billiard.

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The Hamiltonian of the system is

$$H = \frac{P^2}{2\mu} + V(r), \tag{1}$$

where the potential $V(r)$ may be described by

$$V(r,\theta) = V(r) = \begin{cases} 0, & R_{\rm in} = fR < r < R, \\ \infty, & \text{others.} \end{cases} \tag{2}$$

The Schrödinger equation for the particle in the infinite potential well is

$$H\psi_n(r) = \left[\frac{P^2}{2\mu} + V(r)\right]\psi_n(r) = E_n\psi_n(r). \tag{3}$$

The E_n and ψ_n stand for the nth energy eigenvalues and eigenfunctions respectively. Suppose that A and B are two arbitrary points inside the annular region. The quantum spectral function is defined as[11]

$$\rho_{AB} = \sum_n \psi_n^*(A)\psi_n(B)\delta(E - E_n). \tag{4}$$

Similar to closed-orbit theory,[10] the semi-classical expression of $\rho_{AB}(E)$ can be written as a combination of a smooth background term and a sum of sinusoidal oscillating terms

$$\rho_{AB}(E) = \rho^0_{AB}(E) + \sum_i C_i \sin(S_i + \varphi_i). \tag{5}$$

The sum runs over all possible classical orbits from A to B. The amplitude C_i is related to the orbital stability, S_i is the action along the ith classical orbit such that $S_i = \int_A^B p{\rm d}q$, φ_i is a phase including Maslov phase correction. For billiard systems, we have the expression $S = KL$, $K = \sqrt{2\ E}$, and L is the length of classical orbit. The Fourier transform of quantum spectral function is

$$\tilde{\rho}_{AB}(L) = \int_{E_{\rm min}}^{E_{\rm max}} \rho_{AB}\exp({\rm i}kL){\rm d}E, \tag{6}$$

where $E_{\rm max}$ and $E_{\rm min}$ are the chosen lowest and highest energies respectively, both of them must meet the calculated precise demand. Substituting Eq.(4) into Eq.(6), we obtain the Fourier transform formula

$$\tilde{\rho}_{AB}(L) = \sum_m \psi_m^*(x_1,y_1)\psi_m(x_2,y_2)\exp({\rm i}K_mL). \tag{7}$$

We adopt the rotating-wave approximation and consider that the C_iK changes slowly in the integral range, and the Fourier transform of Eq.(5) is approximately given by

$$\tilde{\rho}_{AB}(L) \approx \tilde{\rho}^0_{AB}(L) + \sum_i \bar{k}_i c_i \left[-\frac{\exp(-{\rm i}\varphi_i)}{2{\rm i}}\right]\Delta(L - L_i), \tag{8}$$

where $\Delta(x) \equiv \int_{k_{\rm min}}^{k_{\rm max}} \exp({\rm i}xk){\rm d}k$ is a function with a peak at $x = 0$, and $\tilde{\rho}^0_{AB}(L) = \int_{E_{\rm min}}^{E_{\rm max}} \tilde{\rho}^0_{AB}(E)\exp({\rm i}\sqrt{2\ E}L){\rm d}E$, $\bar{k}_i$ is an average value of k in the interval $[k_i, k_{i+1}]$.

3. Quantum spectra of the two-dimensional annular billiard

In the two-dimensional annular billiard, the Schrödinger equation of the particle can be written in polar coordinates as

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi(r,\theta) = E\psi(r,\theta). \tag{9}$$

We assume that the solution for separating variables is $\psi(r,\theta) = R(r)\Theta(\theta)$, and obtain readily the angular equation and radial equation, respectively

$$\frac{{\rm d}^2\Theta_m(\theta)}{{\rm d}\theta^2} = -m^2\Theta_m(\theta), \tag{10}$$

$$\frac{{\rm d}^2R(r)}{{\rm d}r^2} + \frac{1}{r}\frac{{\rm d}R(r)}{{\rm d}r} - \frac{m^2}{r^2}R(r) = -k^2R(r), \tag{11}$$

where $k = \sqrt{2\mu E/\hbar^2}$, m^2 is the constant to be determined.

The angular equation has normalized solutions of the form

$$\Theta_m(\theta) = \frac{1}{\sqrt{2\pi}}{\rm e}^{{\rm i}m\theta},\ m = 0, \pm1, \pm2, \pm3, ... \tag{12}$$

Let $z = kr$, the radial equation becomes

$$\frac{{\rm d}^2R(z)}{{\rm d}z^2} + \frac{1}{z}\frac{{\rm d}R(z)}{{\rm d}z} + \left(1 - \frac{m^2}{z^2}\right)R(z) = 0. \tag{13}$$

The general solution of Eq.(13) is

$$\begin{aligned} R(z) &= \alpha J_m(z) + \beta N_m(z) \\ &= \alpha J_m(kr) + \beta N_m(kr), \end{aligned} \tag{14}$$

where $J_m(z)$ is the mth order cylindrical Bessel function and $N_m(z)$ is the mth order Neumann function; both of them satisfy Dirichlet boundary conditions:

$$\psi(\boldsymbol{r})\,|_{r=fR} = 0; \psi(\boldsymbol{r})\,|_{r=R} = 0. \tag{15}$$

So the eigen equations[12] in the annular well satisfy

$$\begin{aligned} &J_m(k_{m,n_r}fR)N_m(k_{m,n_r}R) \\ &-J_m(k_{m,n_r}R)N_m(k_{m,n_r}fR) = 0. \end{aligned} \tag{16}$$

And the corresponding eigenvalues are given by

$$E_{(m,n_r)} = \frac{\hbar^2 k_{m,n_r}^2}{2\mu}, \tag{17}$$

where $k_{m,n_r} = x_{n_r}^{(m)}$ is the n_rth zero of the eigen equation (16), the normalized radial wavefunction is $R(k_{m,n_r}r) = A_{m,n_r}\left[J_m(k_{m,n_r}r) - \frac{J_m(k_{m,n_r}fR)}{N_m(k_{m,n_r}fR)}N_m(k_{m,n_r}r)\right]$ and the normalized coefficients A_{m,n_r} satisfy the following equation: $|A_{m,n_r}|^2 \int_a^b \left|J_m(k_{m,n_r}r) - \frac{J_m(k_{m,n_r}a)}{N_m(k_{m,n_r}a)}N_m(k_{m,n_r})\right|^2 r\mathrm{d}r = 1$.

Therefore, the eigenfunctions of the billiard are

$$\psi_{m,n_r}(r,\theta) = A_{m,n_r}\left[J_m(k_{m,n_r}r) - \frac{J_m(k_{m,n_r}fR)}{N_m(k_{m,n_r}fR)}N_m(k_{m,n_r}r)\right] \times \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\theta}. \tag{18}$$

Due to the energy conservation, we have

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2}. \tag{19}$$

When $\dot{r} = 0$ in the inner turning point[6] (the distance of closest approach to the origin), we have

$$E = \frac{L^2}{2\mu R_{\min}^2} \quad \text{or} \quad \frac{R_{\min}}{R} = \sqrt{\frac{L^2}{2\mu E R^2}}. \tag{20}$$

Combining Eqs.(17) and (20) we obtain

$$1 \geq \frac{R_{\min}}{R} = \sqrt{\frac{L^2}{2\mu E R^2}} = \sqrt{\frac{m^2\hbar^2}{\hbar^2(x)^2}} = \frac{|m|}{x},$$

where $x \equiv x_{n_r}^m$ and $\frac{R_{\min}}{R} \geq \frac{R_{\text{in}}}{R}$.

4. Calculated results and discussion

We choose the parameters as $R = 1$, $R_{\text{in}} = fR = 0.45$ and $E_{\max} = 180000$ to yield a suitable resolution of spectra (in natural units).

We have researched two different trajectories according to the relative position of the connecting line of AB and the inner circle. First, we take $A\left(0.9, \frac{2}{3}\pi\right)$, $B\left(0.75, \frac{1}{3}\pi\right)$ as the two points inside the annular billiard, whose connecting line is apart away from the inner circle, the importance of this kind of orbit lies on including the 'whispering gallery'. The ballistic trajectories travelling from A to B obey the law of specular reflection. Because of the special symmetry of the annular geometry, we can obtain several classical orbits that have near identical length but different eject angles and therefore different paths which contribute to the same quantum spectral peak (this resembles the degeneration in energy structure). The results are shown in Fig.2, where we have evaluated Fourier transform $|\tilde{\rho}_{AB}(L)|^2$ for the 500 lowest energy values $x_{n_r}^m$, and displayed the correspondence between the resonant peak positions and the lengths of classical orbits.

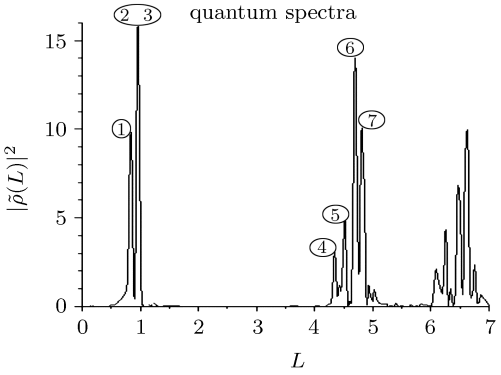


Fig.2. Correspondence of quantum spectra and classical ballistic trajectories.

Some corresponding shorter classical trajectories are delineated in Fig.3.

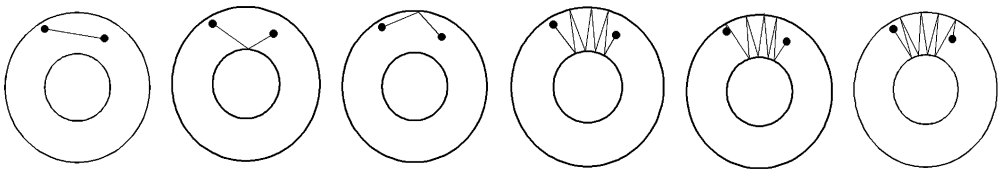


Fig.3. Some classical orbits connecting A and B in annular billiard.

It is found that the intensity of every peak of the quantum spectra is different from each other and the same peak may correspond to the numbers of classical orbits as indicated in columns 2 and 3 of the Table 1.

Table 1. The position of the peaks of the Fourier-transformed quantum spectra and the lengths of classical orbits going from A to B.

number	1	2	3	4	5	6
the location of peak	0.836	0.962	0.962	4.331	4.520	4.685
the length of orbit	0.835	0.973	1.008	4.318	4.509	4.623

In the second situation we take the two points to be $A\left(0.9, \frac{2}{3}\pi\right)$, $B(0.9, 0)$ inside the annular billiard, whose connecting line is tangential to the inner circle. Figure 4 shows the calculated results of parts of the correspondence between the peak positions and the lengths of classical orbits and the specific data are in Table 2.

The corresponding classical trajectories are exhibited in Fig.5.

It is clear that the intensity of the first peak in Fig.4 is low in contrast to other peaks, but the corresponding classical orbit L_1 does not violate the optical Fermat principle. Due to the standing of the inner annulus on the travelling path, such that the ballistic trajectory was obstructed, then the propagating probability was reduced. From the view of the classical orbit theory this result can be attributed to the geometrical effect. Therefore the intensity of the first peak is not as strong as generally regarded.

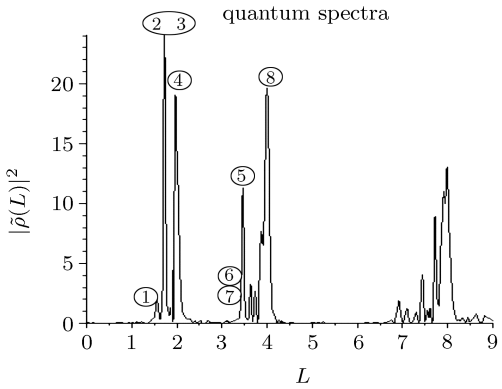


Fig.4. Correspondence of quantum spectra and classical orbit numbers in annular billiard.

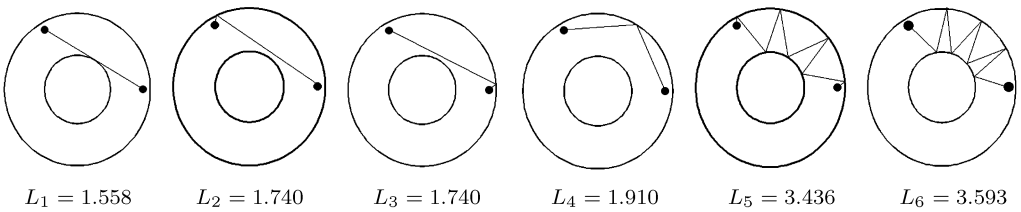


Fig.5. Some classical orbits in annular billiard in second situation.

Table 2. The position of the peaks of the Fourier-transformed quantum spectra and the lengths of classical orbits going from A to B.

number	1	2	3	4	5	6
the location of peak	1.560	1.724	1.841	1.936	3.458	3.560
the length of orbit	1.558	1.740	1.851	1.910	3.436	3.593

5. Conclusion

Our calculation demonstrates that the peak positions of quantum spectra match with the lengths of the classical orbits very well no matter the connecting line of AB is apart away from the inner annulus or tangential with the inner annulus and finds the 'geometrical effect' of the inner annular barrier.

One may notice that the diffraction may appear when the inner radius is of the order of the de Broglie wavelength of the particle,[13] but not very small. Generally, in most practical applications of the quantum transport the electron's orbit is not closed or periodic but open, so the semi-classical method has ac-

tual significance. In the analysis we have developed the closed-orbit theory to open-orbit case. We will explore the dynamics of the eccentric annular billiard in the next step to reveal the quantum manifestations of the classical chaos.

References

[1] Stöckmann H J 2002 *J. Mod. Opt.* **49** 2045

[2] Ludger W, Christöph S, Stefan R and Joachim B 2003 *Phys. Rev.* E **67** 1

[3] Dembowski C and Dgraf H 2000 *Phys. Rev. Lett.* **84** 867

[4] Gutzwiller M C 1971 *J. Math. Phys.* **12** 343

[5] Robinett R W 1997 *Am. J. Phys.* **65** 1167

[6] Robinett R W 1999 *Am. J. Phys.* **67** 67

[7] Lin S L, Gao F, Hong Z P and Du M L 2005 *Chin. Phys. Lett.* **22** 9

[8] Zhang Y H, Zhang L Q, Xu X Y, Ge M H, Lin S L and Du M L 2006 *Chin. Phys.* **15** 502

[9] Xue X Y, Zhang Y H, Huang F Z, Lin S L and Du M L 2005 *Acta Phys. Sin.* **54** 4538 (in Chinese)

[10] Du M L and Delos J B 1988 *Phys. Rev.* A **38** 1896

[11] Lu J and Du M L 2004 *Acta Phys. Sin.* **53** 2450 (in Chinese)

[12] Bohigas O, Boosé D, Egydio de Carvalho R and Marvulle V 1993 *Nucl. Phys.* A **560** 197

[13] Reimann S M, Brack M and Magner A G 1996 *Surf. Rev. Lett.* **3** 19