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Teleporting a quantum controlled-Not with one target/two targets gate using two partially entangled states*

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This paper considers the teleportation of quantum controlled-Not (CNOT) gate by using partially entangled states. Different from the known probability schemes, it presents a method for teleporting a CNOT gate with unit fidelity and unit probability by using two partially entangled pairs as quantum channel. The method is applicable to any two partially entangled pairs satisfying the condition that their smaller Schmidt coefficients μ and ν are $(2\mu + 2\nu - 2\mu\nu - 1) \geq 0$. In this scheme, the sender's local generalized measurement described by a positive operator valued measurement (POVM) lies at the heart. It constructs the required POVM. It also puts forward a scheme for teleporting a CNOT with two targets gate with unit fidelity by using same quantum channel. With assistance of local operations and classical communications, three spatially separated users are able to complete the teleportation of a CNOT with two targets gate with probability of $(2\mu + 2\nu - 1)$. With a proper value of μ and ν, the probability could reach nearly 1.

Keywords: partially entangled state, teleportation, CNOT gate, positive operator valued measurement

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1. Introduction

The existence of quantum algorithms for some specific problems shows that a quantum computer can in principle provide a tremendous speedup in comparison with classical computer.[1] This discovery has motivated a lot of studies in recent years on the theoretical and practical aspects of quantum computation. It is well known that universal quantum computer can be built from only two kinds of gates: two-qubit CNOT gate and one-qubit unitary gate. The two kinds of quantum gates have been experimentally realized locally in several physical systems, such as ion-traps, high-finesse microwave cavities, polarized photons, and so on.[1] If the qubits involved in the process are separated in distance (e.g., distributed quantum computation), operations must be implemented nonlocally. The best way to achieve this is applying teleportation of quantum gate, i.e. one can teleport a collective quantum gate from acting on local qubits to acting on spatially separated qubits, without actually transporting the physical system that carries the state form one place to another. Teleportation of quantum gate is more challenging than teleportation of quantum state[2−6] and has attracted much attention. Some fundamental researches on teleportation of quantum gate have been presented.[7−11]

It has been shown that a single pair of particles in a maximally entangled state (one ebit) is sufficient as a quantum channel for teleporting a CNOT gate with unit success probability and unit fidelity, if local operations and classical communications (LOCCs) are permitted.[7,8] However, entanglement is fragile under the interaction with environment, so it is important to consider the teleportation of quantum gate when the entangled state shared between the sender and the receiver is only partially entangled. As is known, when only a single partially entangled pair is supplied, one can only achieve approximate teleportation of quantum gate deterministically (with imperfect fidelity but unit success probability) or exact teleportation probabilistically (with unit fidelity but certain probability of failure). A lot of efforts have been made to find protocols maximizing the average fidelity of approximate teleportation of a quantum gate or optimizing the probability of exact teleportation of a quantum gate.[12−15] Groisman and Reznik have given a way to teleport a controlled-rotation gate by using partially

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entangled state which can reach a higher probability of success in some parameter regions,[13] and recently some more efficient protocols are proposed by Ye *et al* [12] and Chen *et al.*[14] Finding a more efficient way to teleport quantum gate by using partially entangled state is the goal in this direction. Either the success probability or the average fidelity is closely related to the efficiency of distributed quantum computation.

What if multiple partially entangled pairs are available? In this paper we shall study the teleportation of a quantum CNOT with one target/two targets gate by using two partially entangled states. The paper is organized as follows. In Section 2, we present a explicit scheme for probabilistic and exact teleportation of a CNOT with two targets gate by using two partially entangled pairs, where the sender's local generalized measurement described by a positive operator valued measure (POVM) lies at the heart. We construct the required POVM. Then, in Section 3, we study the consequence of replacing the probabilistic and exact teleportation of a CNOT with two targets gate with the deterministic and exact teleportation of a CNOT with one target gate. Finally, we summarize the paper in Section 4.

2. Probabilistic and exact teleportation of a quantum CNOT with two targets gate by using two partially entangled states

It is well known that taking $|A\rangle = \alpha|0\rangle + \beta|1\rangle$ as the control qubit and $|B_1\rangle = \gamma_1|0\rangle + \delta_1|1\rangle$, $|B_2\rangle = \gamma_2|0\rangle + \delta_2|1\rangle$, ..., $|B_N\rangle = \gamma_N|0\rangle + \delta_N|1\rangle$ as the target qubits, and performing a CNOT with N targets gate $U_{\rm CNOT}(A; B_1, \ldots, B_N)$ on $|A\rangle|B_1\rangle|B_2\rangle \ldots |B_N\rangle$ we have

$$|A\rangle_A|B_1\rangle|B_2\rangle \ldots |B_N\rangle \xrightarrow{U_{\rm CNOT}(A;B_1B_2\ldots B_N)} |A\rangle|B_1 \oplus A\rangle|B_2 \oplus A\rangle \ldots |B_N \oplus A\rangle. \quad (1)$$

The operation $U_{\rm CNOT}(A; B_1, \ldots, B_N)$ may be constructed by concatenating N CNOT with one target gates

$$U_{\rm CNOT}(A; B_1, \ldots, B_N) = U_{\rm CNOT}(A; B_1)U_{\rm CNOT}(A; B_2) \ldots U_{\rm CNOT}(A; B_N). \quad (2)$$

The CNOT with N targets gate is convenient to use, for example, in constructing classical functions such as permutations, or in encoders and decoders for quantum error-correction computing such as the 7-qubit Calderbank–Shor–Steane code.[1]

In the case of nonlocal operation, because $|A\rangle$, $|B_1\rangle$, $|B_2\rangle$, ..., and $|B_N\rangle$ are spatially separated (by convention, held by Alice, $\rm Bob_1$, $\rm Bob_2$, ..., and $\rm Bob_N$, respectively), entangled states have to be prepared and shared by Alice, $\rm Bob_1$, $\rm Bob_2$, ..., and $\rm Bob_N$. For the sake of simplicity, we describe a scheme in the case of three parties (N=2), which can be directly generalized to the $N > 2$ case.

Suppose that Alice, $\rm Bob_1$, and $\rm Bob_2$ share two partially entangled pairs, the states of which can be written in Schmidt decomposition as

$$|\varphi\rangle_{a_1b_1} = (\sqrt{\mu}|00\rangle + \sqrt{1-\mu}|11\rangle)_{a_1b_1},$$
$$|\varphi\rangle_{a_2b_2} = (\sqrt{\nu}|00\rangle + \sqrt{1-\nu}|11\rangle)_{a_2b_2}, \quad (3)$$

where the Schmidt coefficients μ and ν are positive real numbers and should be known in advance. Without loss of generality, we can set $1/2 > \mu \geq \nu > 0$. The qubits (a_1, a_2), qubit b_1, and qubit b_2 belong to Alice, $\rm Bob_1$, and $\rm Bob_2$, respectively. Alice has a qubit A in the unknown state $|A\rangle_A = (\alpha|0\rangle + \beta|1\rangle)_A$, $\rm Bob_1$ has $|B_1\rangle_{B_1} = (\gamma_1|0\rangle + \delta_1|1\rangle)_{B_1}$, and $\rm Bob_2$ has $|B_2\rangle_{B_2} = (\gamma_2|0\rangle + \delta_2|1\rangle)_{B_2}$. The initial state of the whole system is $|\psi\rangle_0 = |\varphi\rangle_{a_1b_1}|\varphi\rangle_{a_2b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}$.

First, Alice performs a POVM on her qubits a_1 and a_2. Two of the authors of the present paper have presented a protocol[16] for constructing the required POVM following the line of Jensen and Schack[17] and Gu *et al.*[18] In that letter, we present the deduction of the doubly stochastic matrix connecting the initial and the final states in detail, and give the general solution and the simplest special solution. We decompose the simplest doubly stochastic matrix into permutation matrices and construct the

required POVM. In the lexicographically ordered basis $(|00\rangle_{a_1a_2},\ |01\rangle_{a_1a_2},\ |10\rangle_{a_1a_2},\ |11\rangle_{a_1a_2})$, the measurement operators $A^j_{a_1a_2}$ (j=1, 2, 3, 4) are represented by four 4×4 matrices, each has only two non-zero elements

$$A^1_{a_1a_2} = (2\mu+2\nu-2\mu\nu-1)\mathrm{diag}\left(0, \frac{1/2}{(1-\mu)\nu}, \frac{1/2}{\mu(1-\nu)}, 0\right), \tag{4a}$$

$$A^2_{a_1a_2} = (2\mu\nu)\mathrm{diag}\left(\frac{1/2}{\mu\nu}, 0, 0, \frac{1/2}{(1-\mu)(1-\nu)}\right), \tag{4b}$$

$$A^3_{a_1a_2} = (1-2\mu)\mathrm{diag}\left(0, \frac{1/2}{(1-\mu)\nu}, 0, \frac{1/2}{(1-\mu)(1-\nu)}\right), \tag{4c}$$

$$A^4_{a_1a_2} = (1-2\nu)\mathrm{diag}\left(0, 0, \frac{1/2}{\mu(1-\nu)}, \frac{1/2}{(1-\mu)(1-\nu)}\right), \tag{4d}$$

where we suppose $(2\mu+2\nu-2\mu\nu-1)\geq 0$. It is clear that each $A^j_{a_1a_2}$ is a positive matrix and that the sum of $A^j_{a_1a_2}$ is the unit matrix, namely $\sum\limits_j A^j_{a_1a_2}=1$. Thus $\{A^j_{a_1a_2}\}$ defines a POVM. After Alice performing a POVM described by $A^j_{a_1a_2}$ in Eq.(4) on her qubits a_1, a_2, the post-measurement states corresponding to the measurement results j are

$$(1/\sqrt{2})(|1100\rangle+|0011\rangle)_{a_1b_1a_2b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}, \quad j=1, \tag{5a}$$

$$(1/\sqrt{2})(|0000\rangle+|1111\rangle)_{a_1b_1a_2b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}, \quad j=2, \tag{5b}$$

$$|11\rangle_{a_1b_1}(1/\sqrt{2})(|00\rangle+|11\rangle)_{a_2b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}, \quad j=3, \tag{5c}$$

$$(1/\sqrt{2})(|00\rangle+|11\rangle)_{a_1b_1}|11\rangle_{a_2b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}, \quad j=4 \tag{5d}$$

with probabilities

$$p_1 = 2\mu+2\nu-2\mu\nu-1, \tag{6a}$$

$$p_2 = 2\mu\nu, \tag{6b}$$

$$p_3 = 1-2\mu, \tag{6c}$$

$$p_4 = 1-2\nu. \tag{6d}$$

Now Alice must do two things. First, she chooses the target qubits/qubit from qubits B_1 and B_2. If Alice's measurement result is $j=1$ or $j=2$, she chooses both of them as the target qubits (when j=1, Alice and Bob_1 (or Alice and Bob_2) must first perform respectively an additional operation σ_x on her qubit a_1 and his qubit b_1(or on a_2 and b_2).); if $j=3$, she chooses qubit B_2; and if $j=4$, she chooses qubit B_1. Second, she sends the result of her measurement to the selected target receivers/receiver, and at the same time she (he) discards one of her (his) qubits a_1, a_2 (b_1, b_2) according to j. Now the states of Eq.(5) become

$$(1/\sqrt{2})(|000\rangle+|111\rangle)_{a_1b_1b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}, \quad j=1,2 \tag{7a}$$

$$(1/\sqrt{2})(|00\rangle+|11\rangle)_{a_2b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}, \quad j=3, \tag{7b}$$

$$(1/\sqrt{2})(|00\rangle+|11\rangle)_{a_1b_1}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}, \quad j=4. \tag{7c}$$

Suppose Alice's measurement result is $j=1$ in step 1. Now Alice wants to perform a local quantum CNOT gate $U_{\mathrm{CNOT}}(A;\ a_1)$ on her qubits $(A,\ a_1)$, and then teleports exactly this gate from qubits $(A,\ a_1)$ to the qubits $(A,\ B_1, B_2)$ through gate teleportation $[U_{\mathrm{CNOT}}(A;\ a_1)\rightarrow U_{\mathrm{CNOT}}(A;\ B_1,\ B_2)]$. The gate teleportation is possible because of the following identity:

$$
\begin{aligned}
&U_{\rm CNOT}(b_2;B_2)U_{\rm CNOT}(b_1;B_1)U_{\rm CNOT}(A;a_1)[(1/\sqrt{2})(|000\rangle+|111\rangle)_{a_1b_1b_2}|A\rangle_A|B_1\rangle_{B_1}|B_2\rangle_{B_2}]\\
=&(1/\sqrt{2})^3\{|+\rangle_{b_2}|+\rangle_{b_1}|0\rangle_{a_1}\otimes[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\\
&+|-\rangle_{b_2}|+\rangle_{b_1}|0\rangle_{a_1}\otimes(\sigma_z)_A[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\\
&+|+\rangle_{b_2}|-\rangle_{b_1}|0\rangle_{a_1}\otimes(\sigma_z)_A[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\\
&+|-\rangle_{b_2}|-\rangle_{b_1}|0\rangle_{a_1}\otimes[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\\
&+|+\rangle_{b_2}|+\rangle_{b_1}|1\rangle_{a_1}\otimes(\sigma_x)_{b_2}(\sigma_x)_{b_1}[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\\
&+|-\rangle_{b_2}|+\rangle_{b_1}|1\rangle_{a_1}\otimes(\sigma_z)_A(\sigma_x)_{b_2}(\sigma_x)_{b_1}[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\\
&+|+\rangle_{b_2}|-\rangle_{b_1}|1\rangle_{a_1}\otimes(\sigma_z)_A(\sigma_x)_{b_2}(\sigma_x)_{b_1}[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\\
&+|-\rangle_{b_2}|-\rangle_{b_1}|1\rangle_{a_1}\otimes(\sigma_x)_{b_2}(\sigma_x)_{b_1}[U_{\rm CNOT}(A;B_1,B_2)|A\rangle_A|B_1\rangle_{B_1}|B\rangle_{B_2}]\}. \qquad (8)
\end{aligned}
$$

This identity shows that the teleportation of the CNOT with two targets gate is achieved as follows: first, we apply respectively CNOT gates $U_{\rm CNOT}(b_1;B_1)$ and $U_{\rm CNOT}(b_2;B_2)$ on the local qubits (b_1,B_1) and (b_2,B_2), and then measure the qubits a_1, b_1, and b_2, respectively, in the basis $\{|0\rangle_{a_1},|1\rangle_{a_1}\}$, $\{|+\rangle_{b_1},\ |-\rangle_{b_1}\}$ $(|\pm\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle))$, and $\{|+\rangle_{b_2},|-\rangle_{b_2}\}$. Conditional on the measurement outcomes, we performs one of the following single-qubit corrections $\{I,\ (\sigma_z)_A,\ (\sigma_z)_A,\ I,\ (\sigma_x)_{b_2}(\sigma_x)_{b_1},\ (\sigma_z)_A(\sigma_x)_{b_2}(\sigma_x)_{b_1},\ (\sigma_z)_A(\sigma_x)_{b_2}(\sigma_x)_{b_1},\ (\sigma_x)_{b_2}(\sigma_x)_{b_1}\}$ on the qubits $A,\ b_2$, and b_1. The probability of success teleportation is $(1/\sqrt{8})^2\times 8\times p_1 = p_1$. When Alice's measurement result is $j=2$ in first step, Alice, $\rm Bob_1$, and $\rm Bob_2$ can obtain the remote quantum gate $U_{\rm CNOT}(A;B_1,B_2)$ with probability $(1/\sqrt{8})^2\times 8\times p_2 = p_2$ following the same way. The total probability of success is

$$p_1+p_2 = 2\mu+2\nu-1. \qquad (9)$$

With a proper value of μ and ν, the probability could reach nearly 1.

3. Deterministic and exact teleportation of a quantum CNOT with one target gate by using two partially entangled states

If we restrict ourselves to just two parties, Alice and Bob, we have N=1. After Bob performing appropriate local operations on his qubit b_1 or b_2, the states of Eq.(7) become

$$(1/\sqrt{2})(|00\rangle+|11\rangle)_{ab}|A\rangle_A|B\rangle_B,\ \ (j=1,2,3,4). \qquad (10)$$

Alice could perform a local quantum CNOT gate $U_{\rm CNOT}(A;a)$ on her qubits $(A,\ a)$, and then teleport deterministically and exactly this gate from qubits $(A,\ a)$ to the qubits $(A,\ B)$ $[U_{\rm CNOT}(A;a)\rightarrow U_{\rm CNOT}(A;B)]$,[8,19] which could be realized as follows.

Firstly, Alice applies a local CNOT operation $U_{\rm CNOT}(A;a)$ on A and a, and performs a computation basis measurement on qubit a. Alice will broadcast the result of her measurement, so that qubits A and b can be transformed [according to the two possible results $|0\rangle_a$ and $|1\rangle_a$, the corresponding transformations are I_b and $(\sigma_X)_b$, respectively] to a common form

$$(1/\sqrt{2})(\alpha|00\rangle_{Ab}+\beta|11\rangle_{Ab})|B\rangle_B. \qquad (11)$$

Then, Bob applies a local CNOT operation $U_{\rm CNOT}(b;B)$ onto his qubits b and B, followed by a measurement on qubit b in the basis $\{|+\rangle_b,|-\rangle_b\}$. If the result is $|-\rangle_b$, Alice needs to perform $(\sigma_z)_A$ on qubit A to obtain

$$(1/2)U_{\rm CNOT}(A;B)|A\rangle_A|B\rangle_B. \qquad (12)$$

The total probability of successful teleportation is $P=(1/2)^2\times 4\times(p_1+p_2+p_3+p_4)=1$. We have thus shown how to teleport a CNOT with one target gate with unit fidelity and unit probability by using two partially entangled states. This is just the so-called deterministic and exact teleportation of a quantum CNOT with one target gate. It is especially suitable for situations requiring high reliability.

4. Conclusion

Teleportation of quantum gates has been the subject of intensive study because of the key role which

is played in quantum communication network and distributed quantum computation. Since partially entanglement corrupts the quality of teleportation of quantum gate, efforts have been made to overcome the difficulty. In this work we have investigated such problem theoretically. In particular, we have given a method to realize deterministic and exact teleportation of a quantum CNOT with one target gate by using two partially entangled states. The method is applicable to any two partially entangled pairs satisfying the condition that their smaller Schmidt coefficients μ and ν are $(2\mu+2\nu-2\mu\nu-1) \geq 0$. Different from the probabilistic scheme,[10,12−15] our scheme makes it possible for us to perform exact teleportation of a CNOT with one target gate via two partially entangled pairs without gambling, i.e., in a deterministic fashion. Therefore, it is especially suitable for situations requiring high reliability. In the scheme, the sender's local generalized measurement described by a POVM lies at the heart. We have constructed the required POVM. We have also put forward a scheme for probabilistically teleporting a CNOT with two targets gate with unit fidelity by using same quantum channel, in which we have one control qubit and two target qubits. We have found that by using two partially entangled pairs and LOCCs, three spatially separated users Alice, Bob_1, and Bob_2 are able to complete the non-local CNOT with two targets gate with probability $(2\mu + 2\nu - 1)$. With the proper values of μ and ν, the probability could reach nearly 1.

During recent years various schemes for teleportation of quantum gates were successfully demonstrated experimentally with several different physical systems. Particular attentions have been paid to optical realizations. For example, the quantum teleportation scheme for arbitrary single qubit rotations was implemented experimentally in a linear optical setup.[20] Reference [19] reported an experiment along this line, where the teleportation of a quantum CNOT gate was implemented using linear optics. The POVM on polarized photons has been realized.[21] Therefore we hope that our scheme will be realized by experiment in the near future.

Finally, we indicate that the work can also be generalized to d-dimensional quantum system (qudit, $d \geq 3$), for instance the d-dimensional cat-like states.[3] Recent studies show that qudits are better suited for certain purpose. For example, with the presence of noise, qudit schemes for quantum cryptographic are more secure than protocols using two-dimensional state.[22] The work is under study and we will publish elsewhere the study on the remote quantum operation in the high dimensional case.

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