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On dynamics of elastic rod based on exact Cosserat model*

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The analysis of kinematics and dynamics of an elastic rod with circular cross section is studied on the basis of exact Cosserat model under consideration of the tension and shear deformation of the rod. The dynamical equations of a rod with arbitrary initial shape are established in general form. The dynamics of a straight rod under axial tension and torsion is discussed as an example. In discussion of static stability in the space domain the Greenhill criteria of stability and the Euler load are corrected by the influence of tension and shear strain. In analysis of dynamical stability in the time domain it is shown that the Lyapunov and Euler stability conditions of the rod in space domain are the necessary conditions of Lyapunov's stability in the time domain. The longitudinal, torsional and lateral vibrations of a straight rod based on exact model are discussed, and an exact formula of free frequency of lateral vibration is obtained. The free frequency formulas of various simplified models, such as the Rayleigh beam, the Kirchhoff rod, and the Timoshenko beam, can be seen as special cases of the exact formula under different conditions of simplification.

Keywords: dynamics of elastic rod, exact Cosserat model, lateral vibration of rod
PACC: 0320, 0340D

1. Introduction

The analysis of deformation and its stability of a thin elastic rod have a background in engineering and play an important role in the study of DNA. The problems of equilibrium of the rod have been discussed extensively on the basis of the Kirchhoff's kinetic analogy.[1−8] More recently, consideration has been given to the dynamics of elastic rod, and it is necessary to treat the rod as a system with double independent variables s and t.[9−13] The Kirchhoff model of the rod is based on the hypothesis that the rod is inextensible and the cross section is unshearable. Obviously it is not true for DNA and other soft materials. The Cosserat model of an elastic rod is a more general model including the Kirchhoff model as a special case. The idea of the Cosserat model is to describe the configuration of the rod by the position of each point on the centreline and the attitude of the cross section about its centre.[14,15] The exact Cosserat model abandons the hypothesis of inextensible and unshearable characters, and is closer to a real elastic rod.[16,17] In the present paper the kinematics of an elastic rod with circular cross section is studied on the basis of the exact Cosserat model, considering the tension of the centreline and the shear deformation of the cross section. The exact dynamical equations of a rod with circular cross section are established in general form, which can be applied to calculate the motion of a rod with arbitrary initial shape by numerical computation. A straight rod under axial tension and torsion or under axial compression is discussed as examples. In discussion of static stability in the space domain the Greenhill criteria of stability and the Euler's load are corrected by considering the tension and shear strain. In analysis of dynamical stability in the time domain it is shown that the conclusions for a Kirchhoff rod are effective too for the exact Cosserat model. Namely, the Lyapunov and Euler stability conditions of the Cosserat rod in the space domain are the necessary conditions of Lyapunov's stability in the time domain. The longitudinal, torsional and lateral vibrations of a straight rod are discussed, and the exact formulas of free frequencies are derived. The free frequency formulas of lateral vibration based on various simplified models, such as the Rayleigh beam, the Kirchhoff rod, and the Timoshenko beam, can be seen as special cases of the exact formula under different conditions of simplification.

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2. Kinematic analysis of a rod with circular section

Consider the motion of a thin elastic rod with circular cross section. Taking the fixed end point O as origin, we establish the inertial reference frame $(O-\xi\eta\zeta)$ and define the arc-coordinate s along the unstretched centreline of the rod. Assume that the rod is homogeneous and isotropic, and the cross section of the rod is rigid. At arbitrary time t, consider an arbitrary infinitesimal segment P_0Q_0 of the rod before local elastic deformation. The positions of end points P_0 and Q_0 are determined by vectors $OP_0=\boldsymbol{r}_0$ and $OQ_0=\boldsymbol{r}_0+\Delta\boldsymbol{r}_0$ in space, and by arc-coordinates s and $s+\Delta s$ on the centreline. The vector $P_0Q_0=\Delta\boldsymbol{r}_0$ is along the tangent of centreline at time t when $\Delta s\to 0$. Letting the direction of P_0Q_0 be the z_0-axis we define a reference frame $(P-x_0y_0z_0)$ with basis-vectors $\boldsymbol{e}_{i0}$ $(i=1,2,3)$ as the principal coordinates of cross section. Assume that points P_0, Q_0 move to P, Q at time $t+\Delta t$, the position of which are determined by $OP=\boldsymbol{r}$ and $OQ=\boldsymbol{r}+\Delta\boldsymbol{r}$. Resolve the procedure of displacement to three stages. At first let P_0Q_0 translate to PQ_1 (see Fig.1), then it follows that

$$PQ_1=P_0Q_0=\Delta\boldsymbol{r}_0=\boldsymbol{e}_{30}\Delta s. \tag{1}$$

Then let PQ_1 rotate about point P to PQ_2 after an infinitesimal rigid rotation ϕ of the section. Denote the new position of $(P-x_0y_0z_0)$ by $(P-xyz)$, where z-axis is normal to the cross section at time $t+\Delta t$ and is along PQ_2 when $\Delta s\to 0$. As the principal coordinates of cross section at time $t+\Delta t$, the basis-vectors of $(P-xyz)$ are denoted by $\boldsymbol{e}_i(i=1,2,3)$. Then the vector PQ_2 can be written as

$$PQ_2=(\boldsymbol{e}_{30}+\boldsymbol{\phi}\times\boldsymbol{e}_{30})\,\Delta s=\boldsymbol{e}_3\Delta s. \tag{2}$$

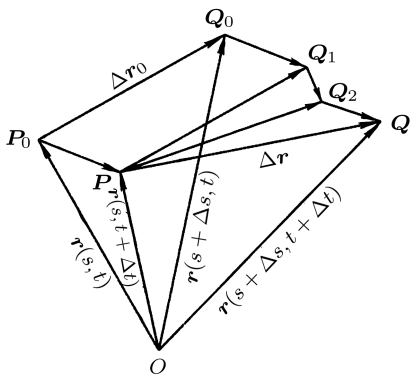


Fig.1. Displacement of infinitesimal segment of a rod.

The shear deformation occurs only in the planes (y,z) and (x,z), because of the hypothesis of rigid cross section. The shear strains along (y,z) and (x,z) are denoted by γ_{23} and γ_{13}, and cause an infinitesimal rotation of the vector PQ_2. The normal strain ε_3 leads to an axial tension of the centreline. Therefore the displacement Q_2Q caused by the shear strain and axial tension can be written as

$$Q_2Q=(\gamma\times\boldsymbol{e}_3+\varepsilon_3\boldsymbol{e}_3)\,\Delta s=\boldsymbol{\varepsilon}\Delta s, \tag{3}$$

where $\gamma=\gamma_{23}\boldsymbol{e}_1+\gamma_{13}\boldsymbol{e}_2$, and $\boldsymbol{\varepsilon}$ is defined as the strain vector. The vector $\boldsymbol{\phi}$ and $\boldsymbol{\varepsilon}$ have components ϕ_i and $\varepsilon_i\,(i=1,2,3)$ in $(P-xyz)$ with $\varepsilon_1=\gamma_{13},\ \ \varepsilon_2=-\gamma_{23}$. When the terms of more than second order of ϕ_i and $\varepsilon_i\ \ (i=1,2)$ are neglected, components of $\boldsymbol{\phi}$ and $\boldsymbol{\varepsilon}$ in $(P-x_0y_0z_0)$ are the same as ϕ_i and $\varepsilon_i\,(i=1,2,3)$. The vector $PQ=PQ_2+Q_2Q$ can be obtained from Eqs.(2) and (3) as (Fig.1):

$$\begin{aligned} PQ &= \Delta\boldsymbol{r}\\ &= (\boldsymbol{e}_3+\boldsymbol{\varepsilon})\,\Delta s\\ &= [\boldsymbol{e}_{30}+(1+\varepsilon_3)\,\boldsymbol{\phi}\times\boldsymbol{e}_{30}+\boldsymbol{\varepsilon}]\,\Delta s. \end{aligned} \tag{4}$$

The position of $(P-xyz)$ after shear deformation γ is denoted by $(P-x^*y^*z^*)$ with PQ as z^*-axis, which is along the tangent of centreline at time $t+\Delta t$ when $\Delta s\to 0$. Dividing $\Delta\boldsymbol{r}$ by Δs and letting $\Delta s\to 0$ we obtain the derivative $\partial\boldsymbol{r}/\partial s$, the project matrices of which in $(P-xyz)$ and $(P-x_0y_0z_0)$ can be written as

$$(\partial\boldsymbol{r}/\partial s)_{(xyz)}=(\varepsilon_1\ \ \varepsilon_2\ \ 1+\varepsilon_3)^{\mathrm{T}},$$

$$\begin{aligned} &(\partial\boldsymbol{r}/\partial s)_{(x_0y_0z_0)}\\ &=(\varepsilon_1+(1+\varepsilon_3)\phi_2\quad \varepsilon_2-(1+\varepsilon_3)\phi_1\ \ 1+\varepsilon_3)^{\mathrm{T}}. \end{aligned} \tag{5}$$

The modulus of $\Delta\boldsymbol{r}$ is the length of the infinitesimal segment and is denoted by $\Delta\bar{s}$. Neglecting the terms of more than second order of $\phi_i,\ \varepsilon_i\,(i=1,2)$, we obtain from Eq.(5)

$$\begin{aligned} \Delta\bar{s} &= |\Delta\boldsymbol{r}|\\ &= \Delta s\,(1+2\varepsilon_3+\ldots)^{1/2}\\ &= \Delta s\,(1+\varepsilon_3+\ldots), \end{aligned} \tag{6}$$

where the ellipsis denotes all terms of more than second order of ε_3. It follows

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}s}=\lim_{\Delta s\to 0}\left(\frac{\Delta\bar{s}}{\Delta s}\right)=1+\varepsilon_3+\ldots. \tag{7}$$

Denote the Cartesian coordinates of point P in $(O-\xi\eta\zeta)$ by ξ,η,ζ, and the projects of vector $\Delta\boldsymbol{r}$ by $\Delta\xi,\ \Delta\eta,\ \Delta\zeta$, then the vector $\partial\boldsymbol{r}/\partial s$ can be expressed as

$$(\partial\boldsymbol{r}/\partial s)_{(\xi\eta\zeta)}=(\xi'\ \eta'\ \zeta')^{\mathrm{T}}, \tag{8}$$

where we use the dash or dot as symbols of partial derivative with respect to s or t, respectively. Moving the origin of $(O-\xi\eta\zeta)$ to the point P, we transform $\partial\boldsymbol{r}/\partial s$ from $(P-\xi\eta\zeta)$ to $(P-x_0y_0z_0)$ using the rotation matrix $\boldsymbol{C}^{(0)}=\left(c_{ij}^{(0)}\right)$ between both frames, and obtain

$$(\partial\boldsymbol{r}/\partial s)_{(x_0y_0z_0)}=\begin{pmatrix}\xi'c_{11}^{(0)}+\eta'c_{12}^{(0)}+\zeta'c_{13}^{(0)}\\ \xi'c_{21}^{(0)}+\eta'c_{22}^{(0)}+\zeta'c_{23}^{(0)}\\ \xi'c_{31}^{(0)}+\eta'c_{32}^{(0)}+\zeta'c_{33}^{(0)}\end{pmatrix}. \tag{9}$$

Equating each elements of matrix (9) to matrix (5), we obtain the components of $\boldsymbol{\varepsilon}$ in $(P-xyz)$ as

$$\begin{aligned}\varepsilon_1&=\xi'c_{11}^{(0)}+\eta'c_{12}^{(0)}+\zeta'c_{13}^{(0)}-(1+\varepsilon_3)\,\phi_2,\\ \varepsilon_2&=\xi'c_{21}^{(0)}+\eta'c_{22}^{(0)}+\zeta'c_{23}^{(0)}+(1+\varepsilon_3)\,\phi_1,\\ \varepsilon_3&=\xi'c_{31}^{(0)}+\eta'c_{32}^{(0)}+\zeta'c_{33}^{(0)}-1.\end{aligned} \tag{10}$$

Dividing $\Delta\boldsymbol{r}$ by Δt, and letting $\Delta t\to 0$, we obtain the velocity $\boldsymbol{v}=\dot{\boldsymbol{r}}$ of point P. Let $\boldsymbol{C}=(c_{ij})$ be the rotation matrix of $(P-xyz)$ relative to $(P-\xi\eta\zeta)$, and $\boldsymbol{C}=\tilde{\boldsymbol{\phi}}\boldsymbol{C}^{(0)}$, where the matrix $\tilde{\boldsymbol{\phi}}$ is defined as

$$\tilde{\boldsymbol{\phi}}=\begin{pmatrix}1&-\phi_3&\phi_2\\ \phi_3&1&-\phi_1\\ -\phi_2&\phi_1&1\end{pmatrix}. \tag{11}$$

Then the components of $\boldsymbol{v}$ in $(P-xyz)$ can be obtained as

$$\begin{aligned}v_1&=\dot{\xi}c_{11}+\dot{\eta}c_{12}+\dot{\zeta}c_{13},\\ v_2&=\dot{\xi}c_{21}+\dot{\eta}c_{22}+\dot{\zeta}c_{23},\\ v_3&=\dot{\xi}c_{31}+\dot{\eta}c_{32}+\dot{\zeta}c_{33}.\end{aligned} \tag{12}$$

Denote the arc-derivative of rotational displacement of $(P-xyz)$ by ω,

$$\boldsymbol{\omega}=\boldsymbol{\phi}'=\phi_1'\boldsymbol{e}_1+\phi_2'\boldsymbol{e}_2+\phi_3'\boldsymbol{e}_3. \tag{13}$$

Assume that $(P-xyz)$ becomes a body-fixed frame $(P-x_{\mathrm{S}}y_{\mathrm{S}}z_{\mathrm{S}})$ after a rotation φ about z-axis, where z_{S} is the normal axis of the cross section. Denote the derivatives of rotational displacements of $(P-xyz)$ and $(P-x_{\mathrm{S}}y_{\mathrm{S}}z_{\mathrm{S}})$ with respect to t by $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_{\mathrm{S}}$, respectively. It follows

$$\begin{aligned}&\boldsymbol{\Omega}_1=\boldsymbol{\Omega}_{\mathrm{S}1}=\dot{\phi}_1,\ \ \boldsymbol{\Omega}_2=\boldsymbol{\Omega}_{\mathrm{S}2}=\dot{\phi}_2,\\ &\boldsymbol{\Omega}_3=\dot{\phi}_3,\ \boldsymbol{\Omega}_{\mathrm{S}3}=\dot{\phi}_3+\dot{\varphi},\end{aligned} \tag{14}$$

where $\boldsymbol{\Omega}_{\mathrm{S}}$ is the angular velocity of rigid rotation of the cross section.

Assume that a rotation φ^* about the z^*-axis makes $(P-x^*y^*z^*)$ to be fixed on the cross section and is denoted by $(P-x_{\mathrm{S}}^*y_{\mathrm{S}}^*z_{\mathrm{S}}^*)$, where z_{S}^* is the tangent axis of the centreline. The angle φ^* is equal to φ when only the terms of first order of ϕ_i and $\varepsilon_i\ \ (i=1,2)$ are considered. The derivative of rotational displacement of $(P-x_{\mathrm{S}}^*y_{\mathrm{S}}^*z_{\mathrm{S}}^*)$ with respect to the deformed arc-coordinate $\bar{s}$ is defined as the curvature-twisting vector and is denoted by ω_{S},

$$\boldsymbol{\omega}_{\mathrm{S}}=(\boldsymbol{\phi}'+\gamma'+\varphi'\boldsymbol{e}_3)\,(1+\varepsilon_3)^{-1}. \tag{15}$$

Considering Eq.(7) we can write the components of $\boldsymbol{\omega}_{\mathrm{S}}$ in $(P-xyz)$ as

$$\begin{aligned}\omega_{\mathrm{S}1}&=(1+\varepsilon_3)^{-1}\,(\phi_1'-\varepsilon_2'),\\ \omega_{\mathrm{S}2}&=(1+\varepsilon_3)^{-1}\,(\phi_2'+\varepsilon_1'),\\ \omega_{\mathrm{S}3}&=(1+\varepsilon_3)^{-1}\,(\phi_3'+\varphi'),\end{aligned} \tag{16}$$

where $\omega_{\mathrm{S}1}$, $\omega_{\mathrm{S}2}$ are curvatures of the centreline and $\omega_{\mathrm{S}3}$ is the twisting of the rod.

3. Dynamical equations of a rod with circular section

Assume that the constitutive relation of the rod is linear, and the resultant internal force $\boldsymbol{F}$ and torque $\boldsymbol{M}$ on the cross section of point P are given as

$$\boldsymbol{F}=\boldsymbol{K}\cdot\boldsymbol{\varepsilon},\quad \boldsymbol{M}=\boldsymbol{A}\cdot\boldsymbol{\omega}_{\mathrm{S}}, \tag{17}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}_{\mathrm{S}}$ are strain and curvature-twisting vectors, the intrinsic values of which in unstressed state are not considered, and $\boldsymbol{K}$, $\boldsymbol{A}$ are the stiffness tensors of the rod. The projective matrices of $\boldsymbol{K}$, $\boldsymbol{A}$ in the principal coordinates of the cross section $(P-xyz)$ are diagonal as

$$\boldsymbol{K}=\mathrm{diag}\,(K\ K\ K_0)\,,\ \boldsymbol{A}=\mathrm{diag}\,(A\ A\ C)\,, \tag{18}$$

where $K=GS$ and $K_0=ES$ are the shear and tension stiffness of the cross section, $A=EI$ and $C=GI_0$ are the bending and twisting stiffness about x- and z-axes, respectively, and E and G are the Young's and shear modulus. For a homogeneous and isotropic material, $G=E/2\,(1+\nu)$ and ν are the Poisson ratio. S, I and I_0 are the area, inertial and polar inertial moment of the cross section, respectively.

In the case of circular cross section with radius r, we have $I_0 = 2I$, $S = \pi r^2$, $I = \pi r^4/4$.

Considering that the segment PQ has the same mass and inertia tensor as P_0Q_0, we can write the linear momentum $\Delta\boldsymbol{Q}$ and angular momentum $\Delta\boldsymbol{H}$ of the element as

$$\Delta\boldsymbol{Q} = \rho S\Delta\boldsymbol{v}\Delta s, \quad \Delta\boldsymbol{H} = \boldsymbol{J}\cdot\Delta\boldsymbol{\Omega}_{\mathrm{S}}\Delta s, \tag{19}$$

where ρ, $\boldsymbol{J}$ are the density and the inertia tensor of the rod with unit length, respectively. The projective matrix of $\boldsymbol{J}$ in $(P-xyz)$ is diagonal as

$$\boldsymbol{J} = \mathrm{diag}\left(\rho I\ \rho I\ \rho I_0\right). \tag{20}$$

According to the theorems of linear and angular momenta the dynamical equations of the rod can be derived as

$$\Delta\boldsymbol{Q} = \Delta\boldsymbol{F}\Delta t, \quad \Delta\boldsymbol{H} = \left(\Delta\boldsymbol{M} + \Delta\boldsymbol{r}\times\boldsymbol{F}\right)\Delta t. \tag{21}$$

Substituting Eqs.(19) into Eqs.(21), dividing each terms by Δs and Δt, and letting $\Delta s\to 0$, $\Delta t\to 0$, we obtain

$$\begin{aligned}&\frac{\partial\boldsymbol{F}}{\partial s} - \rho S\frac{\partial\boldsymbol{v}}{\partial t} = 0,\\ &\frac{\partial\boldsymbol{M}}{\partial s} + \frac{\partial\boldsymbol{r}}{\partial s}\times\boldsymbol{F} - \frac{\partial(\boldsymbol{J}\cdot\boldsymbol{\Omega}_{\mathrm{S}})}{\partial t} = 0.\end{aligned} \tag{22}$$

Taking $(P-xyz)$ as a moving reference frame in derivation of vectors $\boldsymbol{F}$, $\boldsymbol{M}$,$\boldsymbol{v}$, $\boldsymbol{r}$, $(\boldsymbol{J}\cdot\boldsymbol{\Omega}_{\mathrm{S}})$ with respect to t or s, we obtain

$$\begin{aligned}&\frac{\tilde\partial\boldsymbol{F}}{\partial s} + \boldsymbol{\omega}\times\boldsymbol{F} - \rho S\left(\frac{\tilde\partial\boldsymbol{v}}{\partial t} + \boldsymbol{\Omega}\times\boldsymbol{v}\right) = 0,\\ &\frac{\tilde\partial\boldsymbol{M}}{\partial s} + \boldsymbol{\omega}\times\boldsymbol{M} + \frac{\partial\boldsymbol{r}}{\partial s}\times\boldsymbol{F} - \frac{\tilde\partial(\boldsymbol{J}\cdot\boldsymbol{\Omega}_{\mathrm{S}})}{\partial t}\\ &-\boldsymbol{\Omega}\times(\boldsymbol{J}\cdot\boldsymbol{\Omega}_{\mathrm{S}}) = 0,\end{aligned} \tag{23}$$

where the symbol $\tilde\partial$ denotes the local derivative relative to $(P-xyz)$. Let the components in $(P-xyz)$ of all vectors be denoted by the index 1, 2, 3, and introduce following complex variables

$$\begin{aligned}&F = F_1 + \mathrm{i}F_2, \quad M = M_1 + \mathrm{i}M_2, \quad v = v_1 + \mathrm{i}v_2,\\ &\phi = \phi_1 + \mathrm{i}\phi_2, \quad \varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2.\end{aligned} \tag{24}$$

Substituting Eqs.(5), (13), (14), (16), (17) into Eq.(23), and projecting it to $(P-xyz)$ lead to the dynamical equations of a rod with circular cross section:

$$F' + \mathrm{i}\left(\phi_3'F - \phi'F_3\right) - \rho S\left[\dot v + \mathrm{i}\left(\phi_3'v - \phi' v_3\right)\right] = 0, \tag{25a}$$

$$\begin{aligned}&M' + \mathrm{i}\left(\phi_3'M - \phi' M_3\right) - \mathrm{i}F_3\varepsilon + \mathrm{i}F\left(1+\varepsilon_3\right)\\ &-\rho I\left(\ddot\phi - \dot\phi_3\dot\phi\right) + \mathrm{i}\rho I_0\dot\phi\dot\varphi = 0,\end{aligned} \tag{25b}$$

$$F_3' + \phi_1'F_2 - \phi_2'F_1 - \rho S\left(\dot v_3 + \phi_1'v_2 - \phi_2'v_1\right) = 0, \tag{25c}$$

$$\begin{aligned}&M_3' + \phi_1'M_2 - \phi_2'M_1 + F_2\varepsilon_1 - F_1\varepsilon_2\\ &-\rho I_0\left(\ddot\phi_3 + \ddot\varphi\right) = 0.\end{aligned} \tag{25d}$$

4. Straight equilibrium and its perturbed equations

Equations (25) permit the following special solutions:

$$\begin{aligned}&\phi = 0, \quad \phi_3 = 0, \quad \varphi = \omega_0 s, \quad \varepsilon = 0,\\ &\varepsilon_3 = \varepsilon_0, \quad v = 0, \quad v_3 = 0,\\ &F = 0, \quad F_3 = F_0 = K_0\varepsilon_0,\\ &M = 0, \quad M_3 = M_0 = C\omega_0.\end{aligned} \tag{26}$$

The solutions (26) correspond to a straight equilibrium of a rod with centreline along the ζ-axis, and without intrinsic strains, curvature and twisting. The straight rod has constant tension strain ε_0 and twisting ω_0 caused by the external axial force F_0 and torque M_0. Since $(P-x_0y_0z_0)$ is coincident with $(P-\xi\eta\zeta)$, the elements $c_{ij}^{(0)}$ are equal to the Kronecker delta $\delta_{ij}(i,j=1,2,3)$. Introduce a complex variable $z=\xi+\mathrm{i}\eta$ as the deflection of the perturbed rod from straight equilibrium, then the relationships between strains, velocities and coordinates after perturbation can be derived from Eqs.(10) and (12)

$$\begin{aligned}&\varepsilon = z' + \mathrm{i}\left(1+\varepsilon_3\right)\phi,\\ &\varepsilon_3 = \zeta' - 1, \ v = \dot z, \quad v_3 = \dot\zeta.\end{aligned} \tag{27}$$

In perturbed state when $\phi_3 = 0$, and φ is defined as the absolute twisting angle, the perturbed force and torque on the cross section can be written as

$$\begin{aligned}&F = K\varepsilon, \qquad F_3 = K_0\varepsilon_3,\\ &M = A\left(1+\varepsilon_3\right)^{-1}\left(\mathrm{i}z'' - \varepsilon_3\phi'\right),\\ &M_3 = C\varphi'\left(1+\varepsilon_3\right)^{-1}.\end{aligned} \tag{28}$$

When the lateral force F is treated as a unknown variable, the dynamical equations (23a), (23b) can be lin-

earized as

$$F' - \mathrm{i}F_0\phi' - \rho S\ddot{z} = 0, \tag{29a}$$

$$A\left(1+\varepsilon_0\right)^{-1}\left(z''' + \mathrm{i}\varepsilon_0\phi''\right) - M_0\phi' - F_0\left[z' + \mathrm{i}\left(1+\varepsilon_0\right)\phi\right] + \left(1+\varepsilon_0\right)F + \mathrm{i}\rho I\ddot{\phi} = 0. \tag{29b}$$

Substitution of $F = K\varepsilon$ and $F_0 = K_0\varepsilon_0$ into Eqs.(29) leads to a closed system of differential equations with z, ϕ as unknown variables

$$Kz'' + \mathrm{i}\left[K\left(1+\varepsilon_0\right) - K_0\varepsilon_0\right]\phi' - \rho S\ddot{z} = 0, \tag{30a}$$

$$A\left(1+\varepsilon_0\right)^{-1}\left(z''' + \mathrm{i}\varepsilon_0\phi''\right) - M_0\phi' + \left[K\left(1+\varepsilon_0\right) - K_0\varepsilon_0\right]\left[z' + \mathrm{i}\left(1+\varepsilon_0\right)\phi\right] + \mathrm{i}\rho I\ddot{\phi} = 0. \tag{30b}$$

Equations (25c) and (25d) are decoupled from Eqs.(30):

$$K_0\zeta'' - \rho S\ddot{\zeta} = 0, \tag{31a}$$

$$C\varphi''\left(1+\varepsilon_0\right)^{-1} - \rho I_0\ddot{\varphi} = 0. \tag{31b}$$

Therefore the longitudinal and torsional motions of the rod can be determined independently.

5. Stability analysis of straight equilibrium

Substitute the following special solutions with exponential form into Eqs.(30)

$$z(s,t) = z_0\exp\left(\lambda s + wt\right),$$
$$\phi(s,t) = \phi_0\exp\left(\lambda s + wt\right). \tag{32}$$

Then a characteristic equation consisting two eigenvalues λ and w in different domains is derived as

$$a\left(\lambda\right)w^4 + b\left(\lambda\right)w^2 + c\left(\lambda\right) = 0. \tag{33}$$

When only the first term of ε_0 is considered the coefficients of Eq.(33) are defined as

$$\begin{aligned} a\left(\lambda\right) &= \rho^2 SI, \\ b\left(\lambda\right) &= \rho S\{\left(E\varepsilon_0 - G\right)I\lambda^2 + \mathrm{i}M_0\lambda \\ &\quad + GS\left[1 - \left(\sigma - 2\right)\varepsilon_0\right]\}, \\ c\left(\lambda\right) &= GS\lambda^2\{\left[1 - \left(\sigma+1\right)\varepsilon_0\right]EI\lambda^2 - \mathrm{i}M_0\lambda \\ &\quad - F_0\left[1 - \left(\sigma-1\right)\varepsilon_0\right]\}, \end{aligned} \tag{34}$$

where the parameter σ is the ratio of Young's and shear modulus, which depends on the Poisson ratio ν for a homogeneous and isotropic rod:

$$\sigma = E/G = 2\left(1+\nu\right). \tag{35}$$

In the category of statics all the perturbation variables are functions of arc-coordinate s only, and the eigenvalue w in time domain should be equal to zero. Then the characteristic equation (33) is simplified

$$c\left(\lambda\right) = 0. \tag{36}$$

Besides the zero roots, the condition of purely imaginary roots of λ in Eq.(36) requires

$$F_0 \le F_{0,\mathrm{L}}, \quad F_{0,\mathrm{L}} = \frac{M_0^2}{4EI}\left(1 + \sigma_1\varepsilon_0\right). \tag{37}$$

The inequality (37) is the necessary condition of Lyapunov's stability of static equilibrium, where the parameter σ_1 is defined as

$$\sigma_1 = 2\sigma - 1 = 3 + 4\nu. \tag{38}$$

When $\varepsilon_0 = 0$, we obtain the well-known Greenhill formula of Kirchhoff rod as a special case of the condition (37). The axial tension of the rod increases the critical value of the force ensuring the static stability.

When the stability condition (37) is satisfied let $\lambda = \pm\mathrm{i}k$, where k is the angular frequency of the perturbed elastica with respect to s. Considering that the Lyapunov's stability is a concept of stability with respect to the initial perturbation, occurring on one end of the rod, it is necessary to add a complementary condition for the perturbed elastica to satisfy the boundary condition on another end. For the rod with both hinged ends and unstretched length L, the following boundary conditions are required:

$$z(L) = z(0) = 0, \;\; \phi(L) = \pm\phi(0). \tag{39}$$

Then the parameter k can be determined as

$$k = n\pi/L \quad (n = 1, 2, \ldots). \tag{40}$$

Substituting $\lambda = \mathrm{i}n\pi/L$ into Eqs.(36), and letting $n = 1$, we obtain

$$F_{0,\mathrm{E}} = \frac{\pi M_0}{L}\left[1 + \left(\sigma - 1\right)\varepsilon_0\right] - \frac{\pi^2 EI}{L^2}\left(1 - \varepsilon_0\right). \tag{41}$$

From the viewpoint of Euler's stability concept, when the perturbation equations have nontrivial solution satisfying the boundary conditions on both ends, then the rod is unstable and buckling occurs. Therefore the special value $F_{0,\mathrm{E}}$ in Eq.(41) can be understood as the Euler's load of the rod for given external torque M_0.

In the case of a rod under axial compression only, let $M_0 = 0$, $F_0 = -\left|F_0\right|$, $\varepsilon_0 = -\left|\varepsilon_0\right|$, the Euler's load of the compressive rod can be obtained as

$$\left|F_0\right|_{\mathrm{E}} = \frac{\pi^2 EI}{L^2}\left(1 + \left|\varepsilon_0\right|\right). \tag{42}$$

The compressive deformation increases the Euler's load.

When the stability condition of the rod has been satisfied in the space domain, and the space eigenvalues have been determined by the boundary conditions as $\lambda = \mathrm{i}n\pi/L$, the time eigenvalue w becomes the sole unknown in the characteristic equation (33). As a part of the conditions of purely imaginary roots of the time eigenvalue w in Eq.(33), the condition $c(\mathrm{i}n\pi/L) > 0$ requires that $F_0 > F_{0,\mathrm{E}}$. Consequently it leads to a necessary condition of stability of the straight equilibrium in both space domain and time domain as

$$F_{0,\mathrm{L}} \geq F_0 > F_{0,\mathrm{E}}. \tag{43}$$

It means that the tension force should be restricted between $F_{0,\mathrm{L}}$ and $F_{0,\mathrm{E}}$. For the rod under axial compression only, the stability condition in time domain is

$$|F_0| < |F_0|_{\mathrm{E}}. \tag{44}$$

Therefore we come to the same conclusion in the stability analysis based on Kirchhoff model, namely, the Lyapunov's and Euler's stability conditions of the rod in the space domain are the necessary conditions of Lyapunov's stability in the time domain.[11−13]

6. Vibrations of a straight rod

The decoupled equations (31) determine the longitudinal and torsional vibrations of the rod. Define the following disturbances:

$$x_1 = \zeta - (1+\varepsilon_0)\, s, \quad x_2 = \varphi - \omega_0 s. \tag{45}$$

Substituting it into Eqs.(31) leads to the same one-dimensional wave equation:

$$\frac{\partial^2 x_i}{\partial t^2} = a^2 \frac{\partial^2 x_i}{\partial s^2} \quad (i = 1, 2), \tag{46}$$

where

$$a = \sqrt{E/\rho} \ \ (i = 1)$$

or

$$a = \sqrt{G/\rho\,(1+\varepsilon_0)} \ \ (i = 2). \tag{47}$$

Then the angular frequencies μ of free longitudinal and torsional vibrations can be derived as

$$\mu = n\pi a/L \ \ (n = 1, 2, \ldots). \tag{48}$$

In discussion of the lateral vibration of straight rod, the different results of various models can be deemed as different simplifications of the exact model.

6.1. Rayleigh beam

As a simplest model of straight rod, the Rayleigh beam model neglects the extension of the rod and the rigid rotation of the cross section, and only the shear deformation is considered. Assume that the beam has no external axial force and torque, let $\varepsilon_0 = 0$, $\phi \equiv 0$, $F_0 = M_0 = 0$, Eqs.(29) are simplified as

$$F' - \rho S \ddot{z} = 0, \tag{49a}$$

$$EIz''' + F = 0. \tag{49b}$$

Eliminating F from Eqs.(49), we obtain the equation of three-dimensional lateral vibration of Rayleigh beam as

$$EI\frac{\partial^4 z}{\partial s^4} + \rho S \frac{\partial^2 z}{\partial t^2} = 0. \tag{50}$$

Letting μ be the free frequency of lateral vibration, we obtain the well-known formula for the hinged Rayleigh beam:

$$\mu = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho S}} \quad (n = 1, 2, \ldots). \tag{51}$$

6.2. Kirchhoff rod[2,10]

As another simplified model, the Kirchhoff rod neglects the extension of the rod and the shear deformation of the cross section, and only the rigid rotation is considered. Let $\varepsilon = \varepsilon_0 = 0$, $F_0 = M_0 = 0$, the condition $\varepsilon = 0$ requires $\phi = \mathrm{i}z'$, then Eqs.(29) are simplified as

$$F' - \rho S \ddot{z} = 0, \tag{52a}$$

$$EIz''' + F - \rho I \ddot{z}' = 0. \tag{52b}$$

Eliminating F from Eqs.(52), we obtain the equation of lateral vibration of Kirchhoff rod as

$$EI\frac{\partial^4 z}{\partial s^4} + \rho S \frac{\partial^2 z}{\partial t^2} - \rho I \frac{\partial^4 z}{\partial s^2 \partial t^2} = 0. \tag{53}$$

Substituting the special solution $z(s,t) = z_0 \exp\{\mathrm{i}[(n\pi/L)\, s + \mu t]\}$ into Eq.(53), and assuming that the rod is sufficient slender, that the terms higher than fourth order of r/L are neglected, we obtain the free frequency of hinged Kirchhoff rod as

$$\mu = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho S}} \left[1 - \frac{1}{2}\left(\frac{n\pi r}{2L}\right)^2\right] \quad (n = 1, 2, \ldots). \tag{54}$$

The free frequency of Kirchhoff rod, when taking rigid rotation of the cross section into consideration, is smaller than that of the Rayleigh beam with shear deformation.

6.3. Timoshenko beam[18]

When both rigid rotation and shear deformation of the cross section are considered, the Kirchhoff rod becomes the Timoshenko beam. Let $\varepsilon_0 = 0$, $F_0 = M_0 = 0$, Eqs.(30) are simplified as

$$G\left(z'' + \mathrm{i}\phi'\right) - \rho\ddot{z} = 0, \tag{55a}$$

$$EIz''' + GS\left(z' + \mathrm{i}\phi\right) + \mathrm{i}\rho I\ddot{\phi} = 0. \tag{55b}$$

Eliminating ϕ from Eqs.(55) leads to the equation of three-dimensional lateral vibration of Timoshenko beam as

$$EI\frac{\partial^4 z}{\partial s^4} + \rho S\frac{\partial^2 z}{\partial t^2} - \rho I\frac{\partial^4 z}{\partial s^2 \partial t^2} + \left(\frac{\rho^2 I}{G}\right)\frac{\partial^4 z}{\partial t^4} = 0. \tag{56}$$

Assume that the coefficient of the last term $\delta = \rho^2 I/G$ of Eq.(56) is small enough and only the first order of δ is considered. Using the same special solution as in Kirchhoff model, we obtain the free frequency of hinged Timoshenko beam

$$\mu = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho S}}\left[1 - \frac{1}{2}\left(\frac{n\pi r}{2L}\right)^2 + \frac{2E}{G}\left(\frac{n\pi r}{2L}\right)^4\right] \quad (n = 1, 2, \ldots). \tag{57}$$

The influence of shear deformation is to increase the free frequency more than that in the Kirchhoff rod.

6.4. Kirchhoff rod with axial force and extension

When the axial force and extension of the Kirchhoff rod are considered. Let $M_0 = 0$, $\varepsilon = 0$, Eqs.(29) are simplified as

$$F' + F_0 z'' - \rho S\ddot{z} = 0, \tag{58a}$$

$$EI\left(1 - 2\varepsilon_0\right)z''' + \left(1 + \varepsilon_0\right)F - \rho I\ddot{z}' = 0, \tag{58b}$$

where the axial force F_0 and axial tension ε_0 are treated as different parameters, because in the case of inextensible Kirchhoff formulation, F_0 can exist when ε_0 is equal to zero. Eliminating F from Eqs.(58), we obtain the equation of lateral vibration of Kirchhoff rod as

$$EI\frac{\partial^4 z}{\partial s^4} + \left(1 + 3\varepsilon_0\right)\left(\rho S\frac{\partial^2 z}{\partial t^2} - F_0\frac{\partial^2 z}{\partial s^2}\right) - \rho I\left(1 + 2\varepsilon_0\right)\frac{\partial^4 z}{\partial s^2 \partial t^2} = 0. \tag{59}$$

Substituting the same special solution into Eq.(58), and assuming that the product of F_0 and ε_0 can be neglected, we obtain the free frequency of hinged Kirchhoff rod

$$\mu = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho S}}\left[1 + \frac{F_0}{EI}\left(\frac{L}{n\pi}\right)^2\right]^{1/2} \times \left[1 - \frac{3\varepsilon_0}{2} - \frac{1}{2}\left(\frac{n\pi r}{2L}\right)^2\right] \quad (n = 1, 2, \ldots). \tag{60}$$

The free frequency of Kirchhoff rod is increased by the axial extension force, and decreased by the compression force, but the axial deformation weakens this effect.

6.5. Exact Cosserat model

The exact Cosserat model abandons all simplified hypotheses on deformations of centreline and cross section. When the dynamical stability conditions are satisfied, let $\lambda = \mathrm{i}n\pi/L$, $w = \pm\mathrm{i}\mu$, the characteristic equation (33) is changed to

$$a\left(\mathrm{i}n\pi/L\right)\mu^4 - b\left(\mathrm{i}n\pi/L\right)\mu^2 + c\left(\mathrm{i}n\pi/L\right) = 0, \tag{61}$$

where

$$\begin{aligned} a &= \rho^2 SI, \\ b &= \rho S[I\left(G - E\varepsilon_0\right)\left(n\pi/L\right)^2 - M_0\left(n\pi/L\right) \\ &\quad + GS\left[1 - \left(\sigma - 2\right)\varepsilon_0\right]], \\ c &= GS\left(n\pi/L\right)^2\{EI\left(1 - \sigma\varepsilon_0\right)\left(n\pi/L\right)^2 \\ &\quad - M_0\left(n\pi/L\right) + F_0\left[1 - \left(\sigma - 1\right)\varepsilon_0\right]\}. \end{aligned} \tag{62}$$

The free frequency of lateral vibration can be determined as follows:

$$\mu = \left[\left(b \pm \sqrt{b^2 - 4ac}\right)/2a\right]^{1/2}. \tag{63}$$

The aforementioned different formulas of free frequency for Rayleigh beam, Kirchhoff rod and Timoshenko beam can be deemed as different special cases of the exact Cosserat model, and can be derived directly from the exact formula (63) after different simplification.

7. Conclusions

The exact Cosserat model under consideration of tension and shear deformation of an elastic rod is a model more close to a real rod. On the basis of kinematic analysis the dynamical equations in general form are established for arbitrary initial configuration of a rod with circular cross section. The dynamics of

a straight rod under axial tension and torsion or under axial compression is discussed as example. The stability problem of the straight equilibrium in space domain and in time domain is analysed. Considering the influence of normal and shear strain of the rod, the Greenhill criteria of stability and the Euler's load are corrected. The Lyapunov's and Euler's stability conditions of the rod in space domain are the necessary conditions of Lyapunov's stability in time domain. The longitudinal, torsional and lateral vibrations of a straight rod are discussed based on the exact model, and exact formulas of free frequency are obtained. The free frequency formulas of lateral vibration of various simplified models, such as the Rayleigh beam, the Kirchhoff rod, and the Timoshenko beam, can be seen as different special cases of the exact formula under different simplified conditions. In numerical computation the dynamical equations can be utilized to calculate the motion of a rod with arbitrary initial shape.

References

[1] Love A E H 1944 *A Treatise on Mathematical Theory of Elasticity* (New York: Dover)

[2] Liu Y Z 2006 *Nonlinear Mechanics of Thin Elastic Rod* (Beijing: Tsinghua University/Springer) (in Chinese)

[3] Liu Y Z and Zu J W 2004 *Acta Mechanica* **167** 29

[4] Xue Y, Liu Y Z and Chen L Q 2004 *Chin. Phys.* **13** 794

[5] Xue Y, Chen L Q and Liu Y Z 2004 *Acta Phys. Sin.* **53** 2040 (in Chinese)

[6] Liu Y Z and Xue Y 2004 *Technische Mechanik* **24** 206

[7] Huang L, Bao G W and Liu Y Z 2005 *Acta Phys. Sin.* **54** 2457 (in Chinese)

[8] Xue Y and Liu Y Z 2006 *Acta Phys. Sin.* **55** 3845 (in Chinese)

[9] Liu Y Z, Xue Y and Chen L Q 2004 *Acta Phys. Sin.* **53** 2424 (in Chinese)

[10] Liu Y Z 2005 *Acta Phys. Sin.* **54** 4989 (in Chinese)

[11] Liu Y Z and Sheng L W 2007 *Acta Phys. Sin.* **56** 891 (in Chinese)

[12] Liu Y Z 2007 *Chin. Phys.* **16** 891

[13] Liu Y Z and Sheng L W 2007 *Acta Mechanica Sin.* **23** 215

[14] Antman S S 1995 *Nonlinear Problems of Elasticity* (New York: Springer)

[15] Tucker R W and Wang C 2003 *Meccanica* **38** 143

[16] Schuricht F 1998 *Arch. Rational Mech. Anal.* **145** 23

[17] Bishop T C, Cortez R and Zhmudsky O O 2004 *J. Comput. Phys.* **193** 642

[18] Timoshenko S 1937 *Vibration Problems in Engineering* (New York: Van Nostrand)