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Second-order solutions for random interfacial waves in N-layer density-stratified fluid with steady uniform currents*

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In the present paper, the random interfacial waves in N-layer density-stratified fluids moving at different steady uniform speeds are researched by using an expansion technique, and the second-order asymptotic solutions of the random displacements of the density interfaces and the associated velocity potentials in N-layer fluid are presented based on the small amplitude wave theory. The obtained results indicate that the wave–wave second-order nonlinear interactions of the wave components and the second-order nonlinear interactions between the waves and currents are described. As expected, the solutions include those derived by Chen (2006) as a special case where the steady uniform currents of the N-layer fluids are taken as zero, and the solutions also reduce to those obtained by Song (2005) for second-order solutions for random interfacial waves with steady uniform currents if $N = 2$.

Keywords: N-layer density-stratified fluid, uniform currents, random interfacial waves, second-order solutions
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1. Introduction

The random behaviour of the interfacial waves have been well recognized in many field measurements in deep and shallow water. Longuet–Higgins[1] established the second-order random wave theory of surface waves for deep water by using a perturbation method. Later, Sharma and Dean[2] extended the analysis of Longuet–Higgins[1] for deep water to water of finite depth. Recently, by using an expansion method, analogous to that of Longuet–Higgins[1] and Sharma and Dean[2] in the study of random surface waves, Song[3,4] derived the second-order solutions for random internal waves in a two-layer fluid contained between two rigid walls at the top and bottom without steady uniform currents and with steady uniform currents respectively, and Chen and Song[5] extended the study by Song[3] to the case of N-layer stratified fluid.

The wave equations describing interfacial internal waves in N-layer stratified fluid are nonlinear wave equations and finding their solutions are important for nonlinear problems. At present, there are many methods for solving nonlinear wave equations, such as the perturbation method,[3−8] the homogeneous balance principle,[9−11] expansion method about the Jacobi elliptic function,[12] various tanh method,[13] the general projective Riccati method[14] and so on.

In this paper, our previous results[4,5] are extended to a more general case of N-layer fluid with steady uniform currents, and the interactions between the interfacial waves and currents are described. The solutions obtained here would have potential applications in the study of internal waves.

2. Basic equations and boundary conditions

We consider the random interfacial waves in N-layer density-stratified fluids as steady uniform currents propagating in the interfaces, contained between two rigid wall at the top and bottom as shown in

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Fig.1. The Cartesian coordinates $(\boldsymbol{x}, z) = (x, y, z)$ are introduced with the origin at the upper boundary, where x and y are the horizontal coordinates, and z_i $(i = 1, 2, ..., N)$ is the coordinate of depth of the undisturbed ith layer, in which $z_0 = 0$ is the upper boundary and z_N is the bottom. $\rho^{(i)}$ is the density of the ith layer fluid, and $h_i = z_{i-1} - z_i$ $(i = 1, 2, ..., N)$ is the thickness of the undisturbed ith layer fluid, and $\eta^{(i)}(x, y, t)$ $(i = 1, 2, ..., N-1)$ is the vertical displacement of the ith density interface.

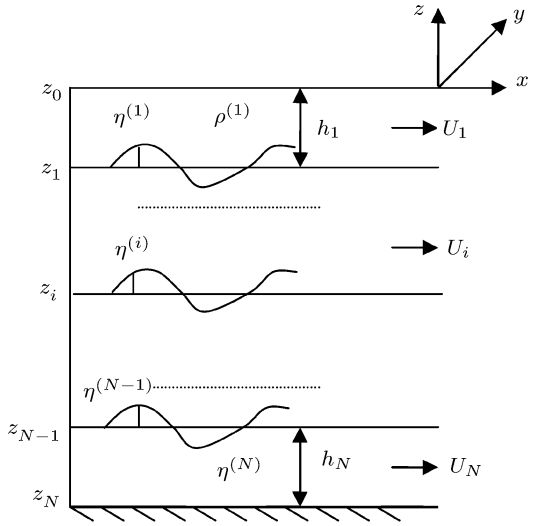


Fig.1. Definition sketch.

Assuming that the fluids are homogeneous, inviscid, incompressible and irrotational, and move as steady uniform currents parallel to the $x - y$ plane. The total potential $\varphi^{(i)}$ can be expressed as[4]

$$\varphi^{(i)}(x, y, z, t) = \boldsymbol{U}_i \cdot \boldsymbol{x} + \Phi^{(i)}(x, y, z, t), \quad i = 1, 2, ..., N, \tag{1}$$

where t is time; $\boldsymbol{U}_i$ $(i = 1, 2, ..., N) = (U_{ix}, U_{iy})$ are the steady uniform horizontal current vectors; $\boldsymbol{x} = (x, y)$; $\Phi^{(i)}$ $(i = 1, 2, ..., N)$ is the associated unsteady wave potentials.

The velocity potential $\Phi^{(i)}$ $(i = 1, 2, ..., N)$ satisfies the Laplace equations:

$$\begin{aligned} &\Delta \Phi^{(i)} = 0, \\ &z_i + \eta^{(i)}(x, y, t) \le z \le z_{i-1} + \eta^{(i-1)}(x, y, t), \\ &i = 1, 2, ..., N, \end{aligned} \tag{2}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$, $\eta^{(0)} = 0$ and $\eta^{(N)} = 0$.

At the density interfaces, the kinematic boundary conditions are

$$\begin{aligned} &\frac{\partial \Phi^{(i)}}{\partial z} = \frac{\partial \eta^{(i)}}{\partial t} + \nabla_{\mathrm{h}} \eta^{(i)} \cdot \nabla_{\mathrm{h}} \Phi^{(i)} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \eta^{(i)}, \\ &z = z_i + \eta^{(i)}(x, y, t), \quad i = 1, 2, ..., N-1, \end{aligned} \tag{3}$$

$$\begin{aligned} &\frac{\partial \Phi^{(i)}}{\partial z} = \frac{\partial \eta^{(i-1)}}{\partial t} + \nabla_{\mathrm{h}} \eta^{(i-1)} \cdot \nabla_{\mathrm{h}} \Phi^{(i)} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \eta^{(i-1)}, \\ &z = z_{i-1} + \eta^{(i-1)}(x, y, t), \quad i = 2, 3, ..., N, \end{aligned} \tag{4}$$

and the dynamic boundary conditions are

$$\begin{aligned} &\rho^{(i)} \left[g\eta^{(i)} + \frac{\partial \Phi^{(i)}}{\partial t} + \frac{1}{2} \nabla \Phi^{(i)} \cdot \nabla \Phi^{(i)} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \Phi^{(i)} \right] \\ &= \rho^{(i+1)} \left[g\eta^{(i)} + \frac{\partial \Phi^{(i+1)}}{\partial t} + \frac{1}{2} \nabla \Phi^{(i+1)} \cdot \nabla \Phi^{(i+1)} + \boldsymbol{U}_{i+1} \cdot \nabla_{\mathrm{h}} \Phi^{(i+1)} \right], \\ &z = z_i + \eta^{(i)}(x, y, t), \quad i = 1, 2, ..., N-1, \end{aligned} \tag{5}$$

where g is the gravitational acceleration , $\nabla_{\mathrm{h}} = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right)$ and $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right)$.

The top and bottom boundary conditions are

$$\frac{\partial \Phi^{(1)}}{\partial z} = 0, \quad z = z_0 = 0, \tag{6}$$

$$\frac{\partial \Phi^{(N)}}{\partial z} = 0, \quad z = z_N. \tag{7}$$

3. First-order random interfacial wave solutions

As Longuet–Higgins[1] and Song,[3,4] we used the perturbation method to solve the above basic equation and boundary conditions. The velocity potential and vertical displacements of the density interfaces can be

expressed as

$$\Phi^{(i)} = \varepsilon \Phi_{\mathrm{I}}^{(i)} + \varepsilon^2 \Phi_{\mathrm{II}}^{(i)} + O(\varepsilon^3), \quad i = 1, 2, ..., N, \quad (8)$$

$$\eta^{(i)} = \varepsilon \eta_{\mathrm{I}}^{(i)} + \varepsilon^2 \eta_{\mathrm{II}}^{(i)} + O(\varepsilon^3), \quad i = 1, 2, .., N-1, \quad (9)$$

where ε is the perturbation parameter , O is the order symbol, and the subscripts I and II denote quantities corresponding to the first-order and second-order perturbation solutions, respectively. Also the velocity potential $\Phi^{(i)}(i = 1, 2, \ldots, N)$ at the density interfaces $z = z_{i-1} + \eta^{(i-1)}$ and $z = z_i + \eta^{(i)}(i = 1, 2, \ldots, N-1)$ can be expanded in Taylor series about $z = z_{i-1}$ and $z = z_i$ as

$$\begin{aligned}\Phi^{(i)}\big|_{z=z_{i-1}+\eta^{(i-1)}} =& \Phi^{(i)}\big|_{z=z_{i-1}} + \eta^{(i-1)} \frac{\partial \Phi^{(i)}}{\partial z}\Big|_{z=z_{i-1}} \\ &+ O((\eta^{(i-1)})^2), \\ &(i = 2, 3, \cdots, N), \end{aligned} \quad (10)$$

$$\begin{aligned}\Phi^{(i)}\big|_{z=z_i+\eta^{(i)}} =& \Phi^{(i)}\big|_{z=z_i} + \eta^{(i)} \frac{\partial \Phi^{(i)}}{\partial z}\big|_{z=z_i} \\ &+ O((\eta^{(i)})^2), \\ &(i = 1, 2, \ldots, N-1). \end{aligned} \quad (11)$$

So we can replace the density interface boundary conditions in Eqs.(3)–(5) by the conditions to be satisfied for $z = z_i$ $(i = 1, 2, \ldots, N-1)$ instead of $z = z_i + \eta^{(i)}$. Substituting (8)–(9) into (2)–(7) and (10)–(11) into (3)–(7), we derive the governing equations for the first and second powers as follows.

The first-order equations and the boundary conditions are

$$\begin{aligned}&\Delta \Phi_{\mathrm{I}}^{(i)} = 0, \\ &z_i + \eta^{(i)}(x, y, t) \le z \le z_{i-1} + \eta^{(i-1)}(x, y, t), \\ &i = 1, 2, ..., N, \end{aligned} \quad (12)$$

$$\frac{\partial \Phi_{\mathrm{I}}^{(1)}}{\partial z} = 0, \quad z = z_0 = 0, \quad (13)$$

$$\begin{aligned}&\frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial z} = \frac{\partial \eta_{\mathrm{I}}^{(i)}}{\partial t} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \eta_{\mathrm{I}}^{(i)}, \\ &z = z_i, \quad i = 1, 2, ..., N-1, \end{aligned} \quad (14)$$

$$\begin{aligned}&\frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial z} = \frac{\partial \eta_{\mathrm{I}}^{(i-1)}}{\partial t} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \eta_{\mathrm{I}}^{(i-1)}, \\ &z = z_{i-1}, \quad i = 2, 3, ..., N, \end{aligned} \quad (15)$$

$$\frac{\partial \Phi_{\mathrm{I}}^{(N)}}{\partial z} = 0, \quad z = z_N, \quad (16)$$

$$\begin{aligned}&\rho^{(i)} \left[g\eta_{\mathrm{I}}^{(i)} + \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial t} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \Phi_{\mathrm{I}}^{(i)} \right] \\ =&\rho^{(i+1)} \left[g\eta_{\mathrm{I}}^{(i)} + \frac{\partial \Phi_{\mathrm{I}}^{(i+1)}}{\partial t} + \boldsymbol{U}_{i+1} \cdot \nabla_{\mathrm{h}} \Phi_{\mathrm{I}}^{(i+1)} \right], \\ z =& z_i, \quad i = 1, 2, ..., N-1. \end{aligned} \quad (17)$$

The second-order equations and the boundary conditions are

$$\begin{aligned}&\Delta \Phi_{\mathrm{II}}^{(i)} = 0, \\ &z_i + \eta^{(i)}(x, y, t) \le z \le z_{i-1} + \eta^{(i-1)}(x, y, t), \\ &i = 1, 2, ..., N, \end{aligned} \quad (18)$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(1)}}{\partial z} = 0, \quad z = z_0 = 0, \quad (19)$$

$$\begin{aligned}&\frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial z} + \frac{\partial^2 \Phi_{\mathrm{I}}^{(i)}}{\partial z^2} \eta_{\mathrm{I}}^{(i)} = \frac{\partial \eta_{\mathrm{II}}^{(i)}}{\partial t} + \frac{\partial \eta_{\mathrm{I}}^{(i)}}{\partial x} \cdot \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial x} + \frac{\partial \eta_{\mathrm{I}}^{(i)}}{\partial y} \cdot \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial y} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \eta_{\mathrm{II}}^{(i)}, \\ &z = z_i, \quad i = 1, 2, ..., N-1, \end{aligned} \quad (20)$$

$$\begin{aligned}&\frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial z} + \frac{\partial^2 \Phi_{\mathrm{I}}^{(i)}}{\partial z^2} \eta_{\mathrm{I}}^{(i-1)} = \frac{\partial \eta_{\mathrm{II}}^{(i-1)}}{\partial t} + \frac{\partial \eta_{\mathrm{I}}^{(i-1)}}{\partial x} \cdot \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial x} + \frac{\partial \eta_{\mathrm{I}}^{(i-1)}}{\partial y} \cdot \frac{\partial \Phi_{\mathrm{I}}^{(i)}}{\partial y} + \boldsymbol{U}_i \cdot \nabla_{\mathrm{h}} \eta_{\mathrm{II}}^{(i-1)}, \\ &z = z_{i-1}, \quad i = 2, 3, ..., N, \end{aligned} \quad (21)$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(N)}}{\partial z} = 0 \quad , \quad z = z_N, \quad (22)$$

$$\rho^{(i)}\left\{g\eta_{\mathrm{II}}^{(i)}+\frac{\partial\Phi_{\mathrm{II}}^{(i)}}{\partial t}+\frac{\partial^2\Phi_{\mathrm{I}}^{(i)}}{\partial t\partial z}\eta_{\mathrm{I}}^{(i)}+\frac{1}{2}\nabla\Phi_{\mathrm{I}}^{(i)}\cdot\nabla\Phi_{\mathrm{I}}^{(i)}+\boldsymbol{U}_i\cdot\nabla_{\mathrm{h}}\Phi_{\mathrm{II}}^{(i)}+\left[\boldsymbol{U}_i\cdot\nabla_{\mathrm{h}}\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial z}\right]\eta_{\mathrm{I}}^{(i)}\right\}$$
$$=\rho^{(i+1)}\left\{g\eta_{\mathrm{II}}^{(i)}+\frac{\partial\Phi_{\mathrm{II}}^{(i+1)}}{\partial t}+\frac{\partial^2\Phi_{\mathrm{I}}^{(i+1)}}{\partial t\partial z}\eta_{\mathrm{I}}^{(i)}+\frac{1}{2}\nabla\Phi_{\mathrm{I}}^{(i+1)}\cdot\nabla\Phi_{\mathrm{I}}^{(i+1)}\right.$$
$$\left.+\boldsymbol{U}_{i+1}\cdot\nabla_{\mathrm{h}}\Phi_{\mathrm{II}}^{(i+1)}+\left[\boldsymbol{U}_{i+1}\cdot\nabla_{\mathrm{h}}\frac{\partial\Phi_{\mathrm{I}}^{(i+1)}}{\partial z}\right]\eta_{\mathrm{I}}^{(i)}\right\},\quad z=z_i,\quad i=1,2,...,N-1. \tag{23}$$

Now, we assume that the first-order random interfacial wave solutions of equations (12)–(17) are

$$\eta_{\mathrm{I}}^{(i)}=\sum_{j=1}^{\infty}b_j^{(i)}\cos\psi_j,\quad i=1,2,...,N-1, \tag{24}$$

$$\Phi_{\mathrm{I}}^{(i)}=\sum_{j=1}^{\infty}\mathrm{A}_j^{(i)}(z)\sin\psi_j,\quad i=1,2,...,N, \tag{25}$$

where $b_j^{(i)}$ denotes the amplitude of the jth wave component of the ith layer density interface; $\psi_j=\boldsymbol{k}_j\cdot\boldsymbol{x}-\omega_j t+\varepsilon_j$, in which $\boldsymbol{x}=(x,y)$, ω_j is the angular frequency, $\boldsymbol{k}_j=(k_{jx},k_{jy})$ is the horizontal wavenumber vector with the magnitude k_j, and ε_j denotes the random phase. If ε_j is taken as an independent random variable and assumed to be uniformly distributed over the interval $(0,\ 2\pi)$, then we can infer that $\eta_j^{(i)}$ obeys the Gaussian distribution. Substituting (24)–(25) into the first-order equations and the corresponding boundary conditions, we have

$$\Phi_{\mathrm{I}}^{(1)}=-\sum_{j=1}^{\infty}\frac{b_j^{(1)}q_{1j}}{k_j\sinh(k_jh_1)}\cosh(k_jz)\sin\Psi_j, \tag{26}$$

$$\Phi_{\mathrm{I}}^{(i)}=\sum_{j=1}^{\infty}\frac{q_{ij}}{k_j}\frac{b_j^{(i-1)}\cosh(k_j(z-z_i))-b_j^{(i)}\cosh(k_j(z-z_{i-1}))}{\sinh(k_jh_i)}\sin\Psi_j,\quad i=2,3,...,N-1, \tag{27}$$

$$\Phi_{\mathrm{I}}^{(N)}=\sum_{j=1}^{\infty}\frac{b_j^{(N-1)}q_{N\,j}}{k_j\sinh(k_jh_N)}\cosh(k_j(z-z_N))\sin\Psi_j, \tag{28}$$

where $q_{ij}=\omega_j-\boldsymbol{U}_i\cdot\boldsymbol{k}_j$.

Substituting (24) and (26)–(28) into (17) and manipulating these equations, we obtain

$$a_j^{(1)}b_j^{(1)}+c_j^{(1)}b_j^{(2)}=0,\ j=1,2,\ldots, \tag{29}$$

$$c_j^{(i-1)}b_j^{(i-1)}+a_j^{(i)}b_j^{(i)}+c_j^{(i)}b_j^{(i+1)}=0,$$
$$i=2,3,\ldots,N-2;\ j=1,2,\ldots, \tag{30}$$

$$c_j^{(N-2)}b_j^{(N-2)}+a_j^{(N-1)}b_j^{(N-1)}=0,\ j=1,2,\ldots, \tag{31}$$

where

$$a_j^{(i)}=g\left(\rho^{(i)}-\rho^{(i+1)}\right)+\frac{1}{k_j}(\rho^{(i)}q_{ij}^2\coth(k_jh_i)$$
$$+\rho^{(i+1)}q_{(i+1),j}^2\coth k_jh_{i+1}),$$
$$i=1,2,\ldots,N-1;\ j=1,2,\ldots, \tag{32}$$

$$c_j^{(i)}=-\frac{\rho^{(i+1)}q_{(i+1),j}^2}{k_j\sinh(k_jh_{i+1})},$$
$$i=1,2,\ldots,N-2;\ j=1,2,\ldots. \tag{33}$$

4. Second-order random interfacial wave solutions

Similar to the procedure used for the first-order approximation, the solution of second-order equation can be obtained as follows:

$$\Phi_{\mathrm{II}}^{(1)} = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}[A_{1jl}^{-}\cosh(|\boldsymbol{k}_j-\boldsymbol{k}_l|z)\sin(\psi_j-\psi_l)+A_{1jl}^{+}\cosh(|\boldsymbol{k}_j+\boldsymbol{k}_l|z)\sin(\psi_j+\psi_l)], \tag{34}$$

$$\begin{aligned}\Phi_{\mathrm{II}}^{(i)} =& \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\{[A_{1ijl}^{-}\cosh(|\boldsymbol{k}_j-\boldsymbol{k}_l|(z-z_i))-A_{2ijl}^{-}\cosh(|\boldsymbol{k}_j-\boldsymbol{k}_l|(z-z_{i-1}))]\sin(\psi_j-\psi_l)\\ &+[A_{1ijl}^{+}\cosh(|\boldsymbol{k}_j+\boldsymbol{k}_l|(z-z_i))-A_{2ijl}^{+}\cosh(|\boldsymbol{k}_j+\boldsymbol{k}_l|(z-z_{i-1}))]\sin(\psi_j+\psi_l)\},\\ & i=2,3,...,N-1,\end{aligned} \tag{35}$$

$$\begin{aligned}\Phi_{\mathrm{II}}^{(N)} =& \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}[A_{Njl}^{-}\cosh(|\boldsymbol{k}_j-\boldsymbol{k}_l|(z-z_N))\sin(\psi_j-\psi_l)\\ &+A_{Njl}^{+}\cosh(|\boldsymbol{k}_j+\boldsymbol{k}_l|(z-z_N))\sin(\psi_j+\psi_l)],\end{aligned} \tag{36}$$

$$\eta_{\mathrm{II}}^{(i)} = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}[d_{ijl}^{-}\cos(\psi_j-\psi_l)+d_{ijl}^{+}\cos(\psi_j-\psi_l)], \quad i=1,2,...,N-1, \tag{37}$$

$$d_{ijl}^{-} = 0, \quad j,l=1,2,... \text{ and } j=l, \quad i=1,2,...,N-1, \tag{38}$$

$$\begin{cases} m_{1jl}^{-}d_{1jl}^{-}+n_{1jl}^{-}d_{2jl}^{-}=p_{1jl}^{-}\ ,\\ n_{(i-1)jl}^{-}d_{(i-1)jl}^{-}+m_{ijl}^{-}d_{ijl}^{-}+n_{ijl}^{-}d_{(i+1)jl}^{-}=p_{ijl}^{-}\ , \quad i=2,3,...,N-2\ ,\\ n_{(N-2)jl}^{-}d_{(N-2)jl}^{-}+m_{(N-1)jl}^{-}d_{(N-1)jl}^{-}=p_{(N-1)jl}^{-}\ , \end{cases}$$

$$j\ ,l\ =\ 1,2,... \quad \text{and} \quad j\neq l, \tag{39}$$

$$\begin{cases} m_{1jl}^{+}d_{1jl}^{+}+n_{1jl}^{+}d_{2jl}^{+}=p_{1jl}^{+}\ ,\\ n_{(i-1)jl}^{+}d_{(i-1)jl}^{+}+m_{ijl}^{+}d_{ijl}^{+}+n_{ijl}^{+}d_{(i+1)jl}^{+}=p_{ijl}^{+}\ , \quad i=2,3,...,N-2\ ,\\ n_{(N-2)jl}^{+}d_{(N-2)jl}^{+}+m_{(N-1)jl}^{+}d_{(N-1)jl}^{+}=p_{(N-1)jl}^{+}\ , \end{cases}$$

$$j\ ,l\ =\ 1,2,..., \tag{40}$$

where the coefficients $A_{1jl}^{\pm}$, $A_{1ijl}^{\pm}$, $A_{2ijl}^{\pm}$, $A_{Njl}^{\pm}$, $m_{ijl}^{\pm}$, $n_{ijl}^{\pm}$ and $p_{ijl}^{\pm}$ are given at the end of the paper in the Appendix.

5. Discussion

Two special cases of the solutions presented here can be noted as follows:

(1) When assumption N=2 is satisfied, the first-order and second-order random interfacial wave solutions are exactly the same as the expressions derived by Song.[4]

(2) Taking $U_i = 0$ $(i = 1, 2, \ldots, N)$, the present results are exactly reduced to those derived by Chen[5] for second-order random interfacial wave solutions in N-layer density-stratified fluid, as expected.

6. Conclusions

The second-order solutions for random interfacial wave in N-layer density-stratified fluid flowing as steady uniform currents contained between two rigid walls at the top and the bottom are derived by an expansion approach, used by Longuet–Higgins[1] and Sharma and Dean[2] in the study of random surface waves. From the results, it is noted that the first-order random wave solutions are the superposition of many linear waves with different amplitudes, wave numbers and frequencies, and these solutions represent not only the linear interactions between the waves and currents, but also the linear modulations on the amplitudes among the first-order wave components with same wave numbers and frequencies between the adjacent density interfaces. It is also noticed that the second-order solutions describe not only the second-order wave-wave interactions among both the wave components on the same density interface and the wave components between the adjacent density interfaces, but also the second-order wave-current interactions between the waves and the steady uniform currents. In addition, the results of the present work include those derived by Song,[4] and Chen and Song[5] as special cases.

Appendix

$$A^{-}_{1jl}=\begin{cases}-\dfrac{C^{-}_{1jl}}{k^{-}_{jl}\sinh(k^{-}_{jl}h_1)}, & \text{for } j\neq l,\\ 0 & \text{for } j=l,\end{cases}$$

$$A^{+}_{1jl}=\frac{C^{+}_{1jl}}{k^{+}_{jl}\sinh(k^{+}_{jl}h_1)}\ ,$$

$$A^{-}_{kijl}=\begin{cases}\dfrac{C^{-}_{kijl}}{k^{-}_{jl}\sinh(k^{-}_{jl}h_i)}, & \text{for } j\neq l,\\ 0, & \text{for } j=l,\end{cases}\qquad i=2,3,...,N-1;\ k=1,2,$$

$$A^{+}_{kijl}=-\frac{C^{+}_{kijl}}{k^{+}_{jl}\sinh(k^{+}_{jl}h_i)}\ ,\qquad i=2,3,...,N-1;\ k=1,2,$$

$$A^{-}_{Njl}=\begin{cases}\dfrac{C^{-}_{Njl}}{k^{-}_{jl}\sinh(k^{-}_{jl}h_N)}, & \text{for } j\neq l,\\ 0, & \text{for } j=l,\end{cases}$$

$$A^{+}_{Njl}=\frac{C^{+}_{Njl}}{k^{+}_{jl}\sinh(k^{+}_{jl}h_N)}\ ,$$

$$m^{\pm}_{ijl}=g\left(\rho^{(i)}-\rho^{(i+1)}\right)+\left[\frac{\rho^{(i)}\left(q_{ij}\pm q_{il}\right)^2}{\tanh\left(k^{\pm}_{jl}h_i\right)}+\frac{\rho^{(i+1)}\left(q_{(i+1)j}\pm q_{(i+1)l}\right)^2}{\tanh\left(k^{\pm}_{jl}h_{i+1}\right)}\right]\frac{1}{k^{\pm}_{jl}},\quad i=1,2,...,N-1,$$

$$n^{\pm}_{ijl}=-\frac{\rho^{(i+1)}\left(q_{(i+1)j}\pm q_{(i+1)l}\right)^2}{k^{\pm}_{jl}\sinh\left(k^{\pm}_{jl}h_{i+1}\right)},\quad i=1,2,...,N-2\ ,$$

$$\begin{aligned}p^{\pm}_{1jl}=&-\frac{1}{4}\rho^{(1)}b^{(1)}_{j}b^{(1)}_{l}\left\{\varepsilon_{1jl}+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{q_{1j}q_{1l}}{S_{1j}S_{1l}}\mp q_{1j}q_{1l}-2q^2_{1l}+2q_{1j}\boldsymbol{U}_1\cdot\boldsymbol{k}_j\right\}\\&+\frac{1}{4}\rho^{(2)}\left\{b^{(1)}_{j}b^{(1)}_{l}\varepsilon_{2jl}+b^{(2)}_{j}b^{(2)}_{l}\tilde{\varepsilon}_{2jl}-2b^{(1)}_{j}b^{(2)}_{l}\sigma_{2jl}+\Big[\lambda_{2jl}\mp b^{(1)}_{j}b^{(1)}_{l}q_{2j}q_{2l}-2b^{(1)}_{j}b^{(1)}_{l}q^2_{2l}\right.\\&\left.+\ 2b^{(1)}_{j}b^{(1)}_{l}q_{2j}\boldsymbol{U}_2\cdot\boldsymbol{k}_j\Big]\right\},\\p^{\pm}_{ijl}=&-\frac{1}{4}\rho^{(i)}\left\{b^{(i-1)}_{j}b^{(i-1)}_{l}\tilde{\varepsilon}_{ijl}+b^{(i)}_{j}b^{(i)}_{l}\varepsilon_{ijl}-2b^{(i-1)}_{j}b^{(i)}_{l}\sigma_{ijl}+\Big[\lambda_{ijl}\mp b^{(i)}_{j}b^{(i)}_{l}q_{ij}q_{il}-2b^{(i)}_{j}b^{(i)}_{l}q^2_{il}\right.\\&\left.+\ 2b^{(i)}_{j}b^{(i)}_{l}q_{ij}\boldsymbol{U}_i\cdot\boldsymbol{k}_j\Big]\right\}+\frac{1}{4}\rho^{(i+1)}\left\{b^{(i)}_{j}b^{(i)}_{l}\varepsilon_{(i+1)jl}+b^{(i+1)}_{j}b^{(i+1)}_{l}\tilde{\varepsilon}_{(i+1)jl}-2b^{(i)}_{j}b^{(i+1)}_{l}\sigma_{(i+1)jl}\right.\\&\left.+\Big[\lambda_{(i+1)jl}\mp b^{(i)}_{j}b^{(i)}_{l}q_{(i+1)j}q_{(i+1)l}-2b^{(i)}_{j}b^{(i)}_{l}q^2_{(i+1)l}+2b^{(i)}_{j}b^{(i)}_{l}q_{(i+1)j}\boldsymbol{U}_{i+1}\cdot\boldsymbol{k}_j\Big]\right\},\\&i=2,3,\ldots,N-2,\\p^{\pm}_{(N-1)jl}=&-\frac{1}{4}\rho^{(N-1)}\left\{b^{(N-2)}_{j}b^{(N-2)}_{l}\tilde{\varepsilon}_{(N-1)jl}+b^{(N-1)}_{j}b^{(N-1)}_{l}\varepsilon_{(N-1)jl}-2b^{(N-2)}_{j}b^{(N-1)}_{l}\sigma_{(N-1)jl}\right.\\&+\Big[\lambda_{(N-1)jl}\mp b^{(N-1)}_{j}b^{(N-1)}_{l}q_{(N-1)j}q_{(N-1)l}-2b^{(N-1)}_{j}b^{(N-1)}_{l}q^2_{(N-1)l}\\&\left.+2b^{(N-1)}_{j}b^{(N-1)}_{l}q_{(N-1)j}\boldsymbol{U}_{N-1}\cdot\boldsymbol{k}_j\Big]\right\}+\frac{1}{4}\rho^{(N)}b^{(N-1)}_{j}b^{(N-1)}_{l}\\&\times\left\{\varepsilon_{Njl}+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{q_{Nj}q_{Nl}}{S_{Nj}S_{Nl}}\mp q_{Nj}q_{Nl}-2q^2_{Nl}+2q_{Nj}\boldsymbol{U}_N\cdot\boldsymbol{k}_j\right\},\end{aligned}$$

where

$$
\begin{aligned}
C^{\pm}_{1jl} =& \left(q_{1j} \pm q_{1l}\right) d^{\pm}_{1jl} + \frac{1}{4} b^{(1)}_{j} b^{(1)}_{l} \left[q_{1j}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/S_{1j} + q_{1l}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/S_{1l}\right] ,\\
C^{\pm}_{1ijl} =& \left(q_{ij} \pm q_{il}\right) d^{\pm}_{(i-1)jl} - \frac{1}{4} b^{(i-1)}_{j} b^{(i-1)}_{l} \left[q_{ij}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/S_{ij} + q_{il}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/S_{il}\right]\\
&+ \frac{1}{2} b^{(i-1)}_{j} b^{(i)}_{l} q_{il}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/R_{il},\\
C^{\pm}_{2ijl} =& \left(q_{ij} \pm q_{il}\right) d^{\pm}_{ijl} + \frac{1}{4} b^{(i)}_{j} b^{(i)}_{l} \left[q_{ij}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/S_{ij} + q_{il}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/S_{il}\right]\\
&- \frac{1}{2} b^{(i-1)}_{j} b^{(i)}_{l} q_{ij}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/R_{ij},\\
C^{\pm}_{Njl} =& q_{Nj} d^{\pm}_{(N-1)jl} - \frac{1}{4} b^{(N-1)}_{j} b^{(N-1)}_{l} \left[q_{Nj}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/S_{Nj} + q_{Nl}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/S_{Nl}\right] ,\\
\varepsilon_{ijl} =& \frac{\left(q_{ij} \pm q_{il}\right)\left[q_{ij}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/S_{ij} + q_{il}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/S_{il}\right]}{k^{\pm}_{jl} \tanh(k^{\pm}_{jl} h_i)}, \quad i = 1, 2, \ldots, N,\\
\tilde{\varepsilon}_{ijl} =& \frac{\left(q_{ij} \pm q_{il}\right)\left[q_{ij}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/S_{ij} + q_{il}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/S_{il}\right]}{k^{\pm}_{jl} \sinh(k^{\pm}_{jl} h_i)}, \quad i = 2, 3, \ldots, N-1,\\
\sigma_{ijl} =& \left(q_{ij} \pm q_{il}\right)\left[\frac{q_{il}\left(\boldsymbol{k}_j \cdot \boldsymbol{k}_l \pm k_l^2\right)/R_{il}}{k^{\pm}_{jl} \tanh(k^{\pm}_{jl} h_i)} + \frac{q_{ij}\left(k_j^2 \pm \boldsymbol{k}_j \cdot \boldsymbol{k}_l\right)/R_{ij}}{k^{\pm}_{jl} \sinh(k^{\pm}_{jl} h_i)}\right], \quad i = 2, 3, \ldots, N-1,\\
\lambda_{ijl} =& \boldsymbol{k}_j \cdot \boldsymbol{k}_l q_{ij} q_{il}\left(b^{(i-1)}_{j}/S_{ij} - b^{(i)}_{j}/R_{ij}\right)\left(b^{(i-1)}_{l}/S_{il} - b^{(i)}_{l}/R_{il}\right), \quad i = 2, 3, \ldots, N-1,\\
k^{\pm}_{jl} =& \left|\boldsymbol{k}_j \pm \boldsymbol{k}_l\right|, \quad k_j = \left|\boldsymbol{k}_j\right|, \quad k_l = \left|\boldsymbol{k}_l\right|,\\
q_{ij} =& \omega_j - \boldsymbol{U}_i \cdot \boldsymbol{k}_j, \quad S_{ij} = k_j \tanh(k_j h_i) , \quad R_{ij} = k_j \sinh(k_j h_i) .
\end{aligned}
$$

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