Vol 17 No 9, September 2008
1674-1056/2008/17(09)/3284-05

Chinese Physics B

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Phase transition in evolution of traffic flow with scale-free property*

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(Received 8 January 2007; revised manuscript received 28 November 2007)

This paper investigates the behaviour of traffic flow in traffic systems with a new model based on the NaSch model and cluster approximation of mean-field theory. The proposed model aims at constructing a mapping relationship between the microcosmic behaviour and the macroscopic property of traffic flow. Results demonstrate that scale-free phenomenon of the evolution network becomes obvious when the density value of traffic flow reaches at the critical point of phase transition from free flow to traffic congestion, and jamming is limited in this scale-free structure.

Keywords: phase transition, traffic flow, scale-free property, cellular automata
PACC: 0590, 0200

In recent years, the studies of networks, in the form of statistical theory, show that a great many real networks have complex network characteristics, such as small-world effect[1−3] and scale-free phenomenon,[3−9] etc. These networks include the World Wide Web,[10,11] Internet,[12] social networks,[13] metabolic networks,[14,15] scientific document citation networks,[16] and so on. Complex network theories aim to find the statistical properties that characterize the structure and behaviour of networked systems, and to predict what the behaviour of these systems will be.[3] In particular, research on the scale-free property has practical significance for the system's robustness, frangibility and attack tolerance.

The scale-free degree distribution means that the fraction of nodes having k degrees follows a power-law distribution: $p(k) \sim k^{-\gamma}$, where γ is called the degree exponent. The scale-free property indicates that a few nodes in a network possess large numbers of degrees, whereas most nodes have only a small quantity of degrees. The so-called scale-free networks are free of a characteristic scale and the distribution of degrees is exactly the same at all scales.[17] This phenomenon hints that the Matthew effect inhabits the evolution of many complex dynamic systems.

We know that the traffic network itself may be not a scale-free network. But from the space–time diagram of traffic flow, we can observe the scale-free phenomenon. In this paper, we use a new model based on the improved mean-field theory[18,19] and NaSch traffic model[19−21] to analyse the detail of the evolution of traffic flow and propose a new analysable view. The main purpose of our method is to construct relationship from the microcosmic to the macroscopic behaviours of traffic flow. The primary distributions include: (1) a new analysable method based on cluster approximation of mean-field theory is proposed; (2) we find that free flow becomes traffic congestion from a critical point at which the scale-free property first appears, and the scale-free property of the evolution network is of benefit to the flux of traffic. Our results could provide insights for understanding some complex phenomenon in the evolution of traffic flow.

The model employed here is based on NaSch traffic model which is one of the basic Cellular Automata models used to describe the traffic flow in a single lane. In the model, the road is depicted by a one-dimensional lattice of L cells numbered by $i = 1, 2, \ldots, L$, and each cell, called site, may either be occupied by one of M vehicles with an integer velocity $0, 1, \ldots, v_{\max}$, or empty. At each discrete time step t, all vehicles are updated in parallel with four consecutive steps: (A) accelerate with $v_i^{(t+1)} = \min\left(v_i^{(t)} + 1, v_{\max}\right)$; (B) decelerate with

*Project supported by the National Basic Research Program of China (973) (Grant No 2006CB705500), and the National Natural Science Foundation of China (Grant No 70671008).

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$v_i^{(t+1)} = \min\left(v_i^{(t)}, d-1\right)$, where d is the distance to the next vehicle ahead the vehicle i; (C) decrease the velocity by 1 with probability p if $v_i^{(t+1)} > 0$; (D) move the vehicle i forward $v_i^{(t+1)}$ sites. In this work, the periodic boundary condition is considered which means that all sites form a closed circle.

In our model, we define a consecutive fraction of $N < L$ sites as an unit called a configuration. At time t, the states of a configuration are represented by a set of binary digits, namely $S_t = (s_1, s_2, \ldots, s_N)$, where $s_i = 0$ $(i = 1, \ldots, N)$ denotes that the site is empty and $s_i = 1$ means that the site is occupied by a vehicle, and then all possible states construct a state space with size $\Omega = 2^N$. Suppose that each state of a configuration is regarded as a node. Then, when the state of a configuration transfers from S_t to S_{t+1}, a newborn node is produced if $S_t \neq S_{t+1}$, and a link from the newborn node to its precursor is created. As time lapses, all nodes and links construct a graph which represents an evolution network. For observing the influence of traffic flow on the behaviour of the evolution network, multiple links between a pair of nodes are allowed. In the following, we present a new method based on cluster approximation of mean-field theory to analyse the relationship between the characters of the evolution network and the traffic flow, and give the numerical results.

According to the model, the evolution process of a network can be determined by the transition probabilities $w_{i,j}(\rho)$, $i, j \in \Omega$ from configuration state (node) i to j, where $\rho = M/L$ is the vehicle density, a global constant in the closed system mentioned above. If the state of a configuration is regarded as a vector of N dimensions which represents the state of traffic in a local fraction of N sites, the transition probabilities $w_{i,j}(\rho)$ reveal the relationships between two states of traffic flow. Because each node represents a distinct state of a configuration, all nodes in Ω and the relationships of nodes construct a Markov chain. Suppose that there exists a probability distribution $\{\pi_k,\ k \in \Omega\}$ and the limit $\lim_{n\to\infty} p_{j,k}^{(n)} = \pi_k$ exists for all $j, k \in \Omega$, we say that the Markov chain with state space Ω has a final distribution. The final distribution implicates the emergence probability of each node in evolution of the network, and obviously it is also the token of the stationary state of a configuration which is the probability of finding in equilibrium a configuration in a given state denoted with velocity.[19]

It is difficult to calculate the final distribution of nodes, so we use the improved mean-field theories called cluster approximation[19,21] to represent the stationary state probability of a configuration firstly. The probability to find a configuration with N sites in state $\tau^{(N)} = (\tau_1, \ldots, \tau_N)$ is denoted by $P(\tau^{(N)})$. Here τ_j represents the occupation of site j, where $\tau_j = 0$ if site j is empty and $\tau_j = v$ if a vehicle with velocity v is in site j. Apparently, the stationary state probability is correlative with parameters: global vehicle density ρ, the probability of velocity decrease p and length N of a configuration. The cluster approximation means that $v_{\max}$ neighbouring sites to the given configuration's left and $v_{\max}$ sites to its right have to be taken into account. So, in the stationary state the master equation for configuration $\tau^{(N)} = (\tau_1, \ldots, \tau_N)$ of N sites has the following structure:[21]

$$P\left(\tau^{(N)}\right) = \sum\nolimits_{\tau^{(N+2v_{\max})}} w\left(\tau^{(N)}|\tau^{(N+2v_{\max})}\right) P\left(\tau^{(N+2v_{\max})}\right),$$

where $P(\tau^{(N)})$ is the probability for clusters at next time step; $\tau^{(N+2v_{\max})} = (\tau_{-v_{\max}+1}, \ldots, \tau_{N+v_{\max}})$ is the long clusters with $N+2v_{\max}$ sites; $w\left(\tau^{(N)}|\tau^{(N+2v_{\max})}\right)$ is the transition probabilities from a long state $\tau^{(N+2v_{\max})}$ at time step t to state $\tau^{(N)}$ at time step $t+1$, and $P\left(\tau^{(N+2v_{\max})}\right)$ is the probability of state $\tau^{(N+2v_{\max})}$ at time step t. Although it is very cumbersome to calculate the group of nonlinear equations, the results can be achieved with the help of some software such as MATHEMATICA.[22] Here we get $P\left(\tau^{(N)}\right)$ with the use of numerical simulations.

Once the probability $P\left(\tau^{(N)}\right)$ is gotten, the final distribution of node i can be calculated by the following equation:

$$P_i = \sum P\left(\tau_{(i)}^{(N)}\right),$$

where $\tau_{(i)}^{(N)}$ is the set of $\tau^{(N)}$ which can be mapped to the same binary sequence corresponding to node i with the following rules: if $\tau_j = 0$ the relevant binary bit is 0, else 1.

So, we can define $p(k)$ to be the probability that a node chosen uniformly at random has degree k with the following equation:

$$p(k) = \left(\sum\nolimits_{i=0}^{2^N} A_i\right)/2^N,$$

$$A_i = \begin{cases} 1, & nP_i = k, \\ 0, & nP_i \neq k, \end{cases}$$

where n is the number of time steps and P_i is the final distribution of node i. Because in every time step a

new link is created, multiple links between two nodes are considered in the model.

Figure 1 shows the numerical simulation results of above N-cluster approximation for the case $N = 10$, $L = 400$, $v_{\max} = 5$, $p = 0.5$, $\rho = 0.18$ and 10000 time steps. The results indicate that the distribution of links shows a power-law behaviour under certain conditions. These results are also in agreement with the measurements in our previous works.[17]

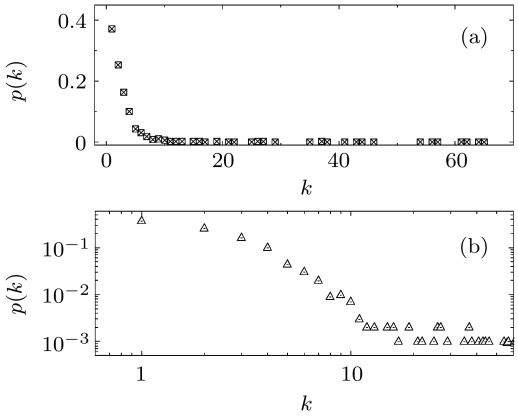


Fig.1. (a) Distribution probability $p(k)$ of links. (b) Log–log plot of $p(k)$ versus k.

The distribution of links for different densities with parameters $v_{\max} = 5$ and $p = 0.5$ is given in Fig.2. The inset shows that the distribution of links does not show an obvious power-law behaviour with small vehicle density and the power-law distribution appears only when the vehicle density is larger than a critical density value marked with ρ_c here. But with the increasing vehicle density, the power-law phenomenon becomes blurry.

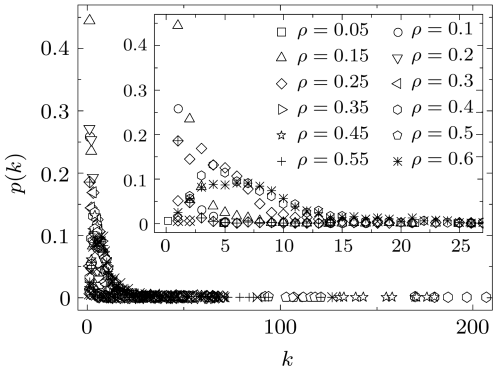


Fig.2. Distribution of links for different densities. The inset shows several plots of the representational densities in a small range of k.

The analysis of the data of results indicates that the distribution of links shows a power-law behaviour with $\rho \geq 0.08$ and whereas for $\rho > 0.5$ the power-law phenomenon disappears gradually. It is apparent from Fig.3 that the number of nodes having more links is few when vehicle density is small and a phase transition appears at the neighbourhood of $\rho = 0.08$.

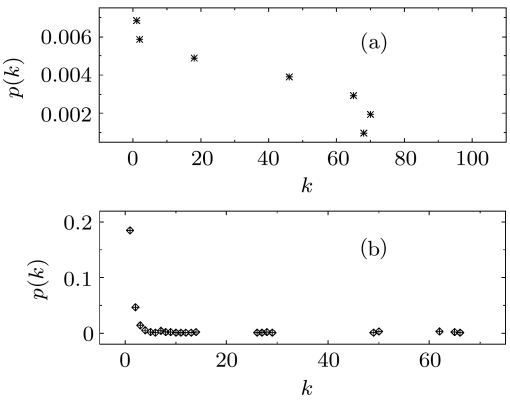


Fig.3. Distribution of links for (a) $\rho = 0.08$ and (b) $\rho = 0.09$.

Furthermore, Fig.4 shows the cases of $\rho = 0.45$ and $\rho = 0.5$. Note that when the vehicle density is larger than another critical value, the node which has the maximal distribution value will no longer be the one whose number of links is minimal. This tendency will be more evident with the increase of vehicle densities.

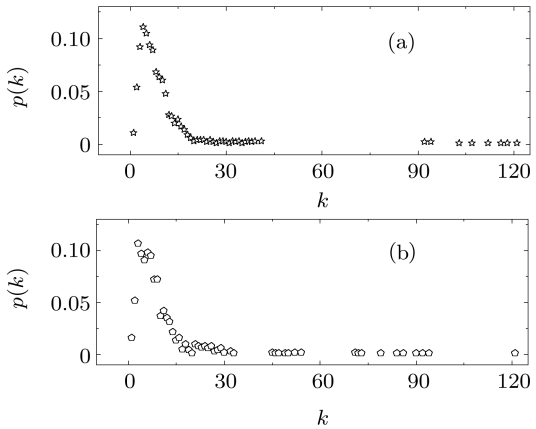


Fig.4. Distribution of links for (a) $\rho = 0.45$ and (b) $\rho = 0.5$.

The efficient performance of traffic network is mainly determined by how much traffic flows the network can support.[23] In order to detect the relationships between traffic flow and power-law phenomenon, we describe the average traffic flow as a function of vehicle density ρ by

$$f(\rho) \simeq \left(\sum_{t=1}^{T} \left(\left(\sum_{i=0}^{N-1} \tau_i \right) /N \right)_t \right) /T,$$

where τ_i is the velocity of a vehicle occupying site i in a configuration at time t, N is the length of the configuration and T is the number of all time steps. Figure 5 shows the plot of vehicle density against the traffic flow. It is obvious that the traffic flow increases with the increasing vehicle density when the density is small. However, when the traffic vehicle density exceeds a critical value, the traffic flow begins to drop down. This indicates that the traffic flow is free flow when the density is small and the traffic jams begin to emerge when the density is larger than the critical value. The above results indicate that there is a critical point of phase transition between free flow and traffic congestion. Comparing Fig.5 with Fig.2, we note that the critical density ρ_c at which the power-law phenomenon becomes obvious is just round about the point where the traffic flow is maximal.

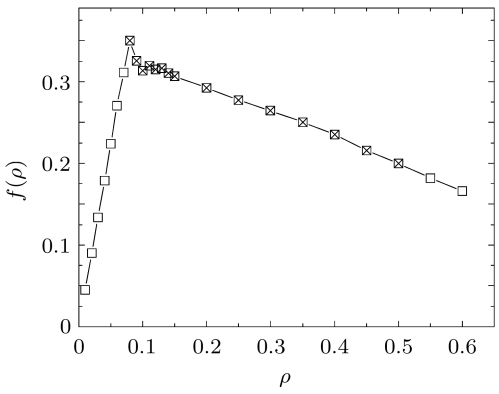


Fig.5. The plot of vehicle densities against the traffic flow.

Changing the value of the parameter $v_{\max}$, we can also find that the emergence of power-law phenomenon is round the point where the traffic flow reaches its peak value. For example,.we can observe that the traffic flow obtains its maximal value at the position of $\rho = 0.1$, as shown in Fig.6 which plots the instance of $v_{\max} = 4$. Synchronously, the power-law behaviour of links does also appear at this point. A number of numerical results and simulations demonstrate that the distribution $p(k)$ of links in evolution of traffic flow follows a power-law when the density of vehicle exceeds the critical value ρ_c at which the traffic flow reaches its peak value.

As mentioned above, when the density is larger than the critical value ρ_c, the traffic jams begin to emerge. However, from Fig.5 and the top plot of Fig.6, we notice that the downtrend of traffic flow at the right side of ρ_c is slower than the uptrend at the left. This illuminates that the scale-free distribution of genealogical relationships among the nodes is of benefit to the flux of traffic.

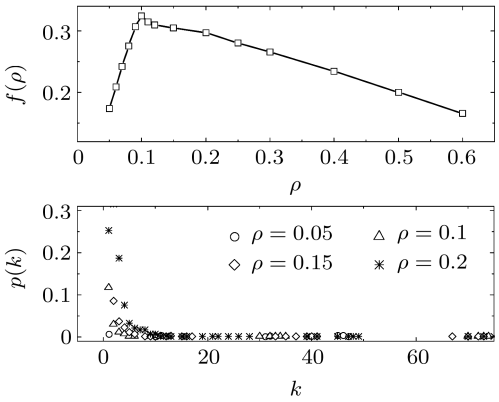


Fig.6. The plot of vehicle densities against the traffic flow (top), and distribution $p(k)$ of links (bottom) with $v_{\max}$.

Figure 7 gives the further explanation for this viewpoint. From Fig.7(a) we can note that the nodes which have large number of links have small average density when the global density $\rho = M/L$ is small and the evolution network has scale-free structure. Here the average density ρ^* of a node is the density of the state corresponding to that node in a configuration. The small density is in favour of enhancement to vehicle's velocity and then is propitious to enhance the traffic flow. Whereas, when the level of jamming is higher (Fig.7(b)), there are no nodes having large number of links in network, but more nodes have large average density.

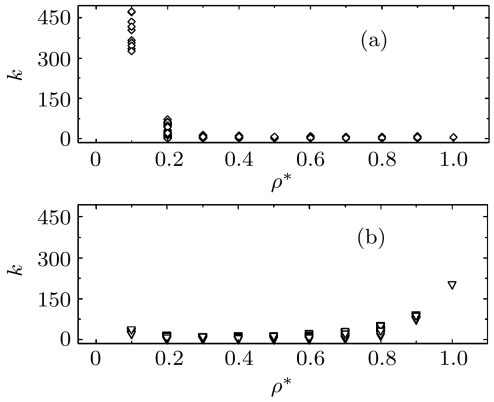


Fig.7. The distribution circumstances of nodes with different number of links and average density. The abscissa is the average density ρ^* of a state corresponding with a node. (a) The case of global density $\rho = 0.2$. (b) The case of global density $\rho = 0.6$.

On the other hand, Fig.7(a) also indicates that the emergence probabilities of the nodes having small

average density are larger. This implies that traffic flow always progresses to the direction of jamming avoidance and jamming is limited in this scale-free structure. The results above also coincide with those in the Refs.[23, 24].

The scale-free property presented above is the macroscopic exhibition of traffic. It constructs a mapping relationship between the microcosmic and the macroscopic behaviour of traffic flow and may be a way to study traffic flow from the entire view point.

In summary, we have studied the evolution of traffic flow by a new method based on the NaSch model and presented a new method to analyse its characters. Our aim is to investigate the macroscopic property of traffic flow and construct a map from its detail to its entire behaviours. We find that the density value at which the power-law phenomenon becomes obvious is just round the point where traffic flow is maximal. Furthermore, the results also show that the scale-free property of the evolution network is of benefit to the flux of traffic, and the jamming is limited in this scale-free structure. However, how to determine the influence on the probability of velocity decrease is still an open issues. It will be the subject of our further study.

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