Vol 17 No 9, September 2008
1674-1056/2008/17(09)/3270-06

Chinese Physics B

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Hyperchaos of two coupled Bose–Einstein condensates with a three-body interaction

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(Received 2 January 2008; revised manuscript received 20 March 2008)

We investigate the dynamics of two tunnel-coupled Bose–Einstein condensates (BECs) in a double-well potential. The effects of the three-body recombination loss and the feeding of the condensates from the thermal cloud are studied in the case of attractive interatomic interaction. An imaginary three-body interaction term is considered and a two-mode approximation is used to derive three coupled equations which describe the total atomic numbers of the two condensates, the relative population and relative phase respectively. Theoretical analyses and numerical calculations demonstrate the existence of chaotic and hyperchaotic behaviour by using a periodically time-varying scattering length.

Keywords: Bose–Einstein condensate, hyperchaos, two-mode approximation，Lyapunov exponent

PACC: 0547, 3280P, 4265T

1. Introduction

It has been shown numerically[1] and experimentally[2] that a Bose–Einstein condensate (BEC) can occur in atomic traps even with attractive two-body interactions, as in ^{7}Li. A positive s-wave scattering length is associated with a repulsive two-body interaction and a negative scattering length is associated with an attractive two-body interaction for the atomic BEC.[3−5] In the experiment,[6] the scattering length can be adjusted from a repulsive range to an attractive one, where the atomic gas may finally become unstable and tend to collapse.[7,8] A condensed state of atoms with a negative s-wave scattering length would be metastable, as in the cases of the experiments for ^{7}Li[8] and ^{85}Rb.[9] In Ref.[10], a three-body recombination rate was studied. For the three-body combination loss, an imagination quintic term was added into the Gross–Pitaevskii (G–P) equation.[8,11,12] It was reported in Ref.[13] that the addition of a three-body term would increase the number of condensed atoms considerably, even when the strength of three-body interaction is very small compared with the strength of the two-body interaction.

When Bose–Einstein condensation occurs in an atomic gas with an attractive interaction,[2,3,8] a metastable BEC is observed, where an upper critical number N_c of the ground state atoms exists.[8] Once the atomic number in the condensate exceeds a critical value, the attractive interaction and the condensed atoms undergo a collapse, which leads their number to a decrease. As the number reduces to a value that is less than N_c, the collapse is stopped and the condensate starts to grow through feeding by the particle flux from the above-condensed atoms: the collapse and the growth taken place alternately. In the process of collapse and growth, three-body recombination plays an important role. The phenomenon of collapse and growth circulating has been verified by analysing the G–P equation.[8,14,15] The main purpose of the present paper is to demonstrate the chaotic and the hyperchaotic behaviour in BEC.

2. The two-mode approximation

The relation between the three-body recombination loss and condensate density $n(\boldsymbol{r},t)=|\psi|^2$ is described as[8] $n(\boldsymbol{r},t)=-\alpha_r n^3(\boldsymbol{r},t)$, where $\alpha_r=C\dfrac{\hbar}{m}a^4$ is the recombination rate with m being the atomic mass and C being a constant, which can be adjusted in a large range.[16,17] This could contribute an imaginary interaction term proportional to $|\psi|^4\psi$ to the G–P equation. Therefore, two tunnel-coupled BECs in a double-well potential with the recombination loss and the atomic feeding are governed by the modified

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G–P equation [8,17−19]

$$\mathrm{i}\hbar\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t}=\Big(-\frac{\hbar^2}{2m}\nabla^2+v_{\mathrm{ext}}(\boldsymbol{r},t)-g\,|\Psi(\boldsymbol{r},t)|^2\Big)\Psi(\boldsymbol{r},t)+\mathrm{i}(\gamma-\xi'\,|\Psi(\boldsymbol{r},t)|^4)\,\Psi(\boldsymbol{r},t). \quad (1)$$

We will study attractive atoms ^{85}Rb, where $g=\frac{4\pi\eta^2\,|a|}{m}$ and a is negative; $\xi'=(\alpha_r\eta)/2$ and γ represent the strength of the recombination loss in the condensates and the feeding by the particle flux from the nonequilibrium thermal cloud, respectively. They are two small positive constants in a practical situation. We consider the cigar-shaped potential $V_{\mathrm{ext}}(\boldsymbol{r},t)$ with a harmonic confinement in the transverse dimensions and a double-well potential in the longitudinal dimension, which includes a harmonic term.[19] When the radial frequency ω_r is high enough, we can approximate the transverse wavefunction as a harmonic-oscillator ground state,[20] thus Eq.(1) becomes a quasi-one-dimensional G–P equation[21]

$$\mathrm{i}\frac{\partial\overline{\Psi}(x,\tau)}{\partial\tau}=(-\nabla^2+v_{\mathrm{ext}}(x,\tau)-A\left|\overline{\Psi}(x,\tau)\right|^2)\overline{\Psi}(x,\tau)+\mathrm{i}(\gamma-\xi\left|\overline{\Psi}(x,\tau)\right|^4)\overline{\Psi}(x,\tau). \quad (2)$$

Here, the axial coordinate x is expressed in units of l_x and the dimensionless time $\tau=\omega_x t$, with $l_x=\sqrt{\frac{\eta}{m\omega_x}}$ being the harmonic-oscillator length, and ω_x the harmonic frequency in the double-well. The function $\overline{\Psi}=\frac{\Psi}{\overline{n}^{1/2}}$ is the normalized wavefunction with $\overline{n}=(4\,|a_s|\,\omega_r/(l_x\omega_x))^{-1}$. The parameters ω_x, ω_r and $|a_s|$ are on the order of $10^2\,\mathrm{s}^{-1}$, $10^3\,\mathrm{s}^{-1}$ and $10^{-9}\,\mathrm{m}$, respectively. The new three-body parameter reads $\xi=CA^4\,|a_s|^2/(48\pi^2 l_x^2)$ and the constant C=1000 will be used, as done in Borca's work.[10] The dimensionless s-wave scattering length is rewritten as $A=\frac{a}{a_s}$ with a_s being a fixed parameter.

To investigate the dynamics of BECs with the double-well potential, we seek for the solution of Eq.(2) in the form[18−26]

$$\overline{\Psi}(x,\tau)=\Psi_1(\tau)\Phi_1(x)+\Psi_2(\tau)\Phi_2(x). \quad (3)$$

When the two BECs are weakly coupled, the two-mode approximation can work well. The weak couple means that the spatial distribution functions $\Phi_1(x)$ and $\Phi_2(x)$ are localized in each well with a very weak overlap such that they satisfy the orthonormalization relation

$$\int\Phi_i^*(x)\Phi_j(x)\mathrm{d}x=\delta_{ij}\quad\text{for}\quad i,j=1,2. \quad (4)$$

This requires a higher barrier between the two wells and a smaller number of atoms in each well. Therefore, for a given double-well potential with the higher barrier the two-mode approximation can be safely used when the number of condensed atoms is less than the critical value N_{c}. In general, as a simple variational estimate, the spatial wavefunctions $\Phi_i(x)$ are assumed to be a Gaussian profile $\Phi(x)$ that obeys $\Phi(x)=\Phi(x-x_l)$, where $x=x_l$ is the centre of the ith wave packet[2] and determined by $\frac{\mathrm{d}V_{\mathrm{ext}}}{\mathrm{d}x}=0$. The width and the normalization constant of the Gaussian waves $\Phi(x-x_l)$ depend on the value of $\frac{\mathrm{d}^2v_{\mathrm{ext}}}{\mathrm{d}x^2}\,|_{x=x_l}$. Substituting expression (3) into Eq.(2) and using expression (4), after some integrations we obtain a pair of nonlinear equations,

$$\begin{aligned}\mathrm{i}\frac{\partial\Psi_1}{\partial\tau}=&(E_1-U_1\,|\Psi_1|^2)\,\Psi_1-K\Psi_2\\&+\mathrm{i}(\gamma-U_1'\,|\Psi_1|^4)\,\Psi_1,\\ \mathrm{i}\frac{\partial\Psi_2}{\partial\tau}=&(E_2-U_2\,|\Psi_2|^2)\,\Psi_2-K\Psi_1\\&+\mathrm{i}(\gamma-U_2'\,|\Psi_2|^4)\,\Psi_2,\end{aligned} \quad (5)$$

where E_i, U_i, K, and U_i' for i=1, 2 can be expressed by the overlapping integrations,

$$E_i=\int[|\nabla\Phi_i|^2+|\Phi_i|^2\,V_{\mathrm{ext}}]\mathrm{d}x, \quad (6)$$

$$U_i=A\int|\Phi_i|^4\mathrm{d}x, \quad (7)$$

$$K=-\int[|\nabla\Phi_1\cdot\nabla\Phi_2|+\Phi_1V_{\mathrm{ext}}\Phi_2]\mathrm{d}x, \quad (8)$$

and

$$U_i'=\xi\int|\Phi_i|^6\mathrm{d}x. \quad (9)$$

Therefore, there are some condensates when the trapped potential V_{ext} depends only on the spatial coordinates. Given the double-well potential, we can construct Gaussian waves $\Phi_i(x)$ and calculate the constants E_i, U_i, K and U_i'. Here, E_1 and E_2 represent the zero-point energies for each condensate; U_1 and U_2 are proportional to the mean-field interaction between atoms; K describes the tunnelling dynamics of the two condensates; and U_1' and U_2' denote the three-body interaction energies, which are proportional to ξ.

We set

$$\Psi_i(\tau)=\sqrt{N_i(\tau)}\exp[\mathrm{i}\theta_i(\tau)] \quad (10)$$

for i=1, 2 so that the time-dependent wavefunction $\overline{\psi}(x,\tau)$ contains $N_{1,2}$ and $\theta_{1,2}$, where $N_i(\tau)$ is the number of particles in the ith trap and expressed in units of $(4a_s\omega_r/(l_x\omega_x))^{-1}$ and $\theta_i(t)$ is the phase of state ψ_i. The normalization condition is $N_1+N_2=N$, where N is the total number of atoms of the two condensates with the same unit as that of N_i. We can make the function transformations[22] as follows:

$$\begin{aligned} u &= \Psi_2^*\Psi_1+\Psi_2\Psi_1^*,\\ v &= -\mathrm{i}(\Psi_2\Psi_1^*-\Psi_2^*\Psi_1),\\ z &= \Psi_2^*\Psi_2-\Psi_1^*\Psi_1, \end{aligned}$$

and

$$N=\Psi_2^*\Psi_2+\Psi_1^*\Psi_1, \tag{11}$$

where u and v characterize the coherence of the condensates, $z=\dfrac{N_2-N_1}{N}$ is the relative population between two condensates and N is time-dependent number of atoms; they are related by $u^2+v^2+z^2=N^2$. Inserting expressions (11) into Eq.(5) yields the group of equations

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\tau} =& v(\Delta E-\alpha N-\beta z)\\ &+u\left(2\gamma-\beta'\frac{N^2+z^2}{2}-\alpha' Nz\right),\\ \frac{\mathrm{d}v}{\mathrm{d}\tau} =& -2Kz-u(\Delta E-\alpha N-\beta z)\\ &+v\left(2\gamma-\beta'\frac{N^2+z^2}{2}-\alpha' Nz\right),\\ \frac{\mathrm{d}z}{\mathrm{d}\tau} =& 2Kv+2\gamma z-\alpha' N\frac{N^2+3z^2}{2}-\beta' z\frac{3N^2+z^2}{2},\\ \frac{\mathrm{d}N}{\mathrm{d}\tau} =& 2\gamma N-\alpha'\frac{3N^2+z^2}{2}-\beta' N\frac{N^2+3z^2}{2}. \end{aligned} \tag{12}$$

Here, we have defined the dimensionless parameters as $\Delta E=E_2-E_1$, $\alpha=\dfrac{U_2-U_1}{2}$, $\alpha'=\dfrac{U_2'-U_1'}{2}$, $\beta=\dfrac{U_2+U_1}{2}$ and $\beta'=\dfrac{U_2'+U_1'}{2}$.

For simplicity, we consider only the case of symmetric potential. In this case, we have $U_1=U_2=U=\beta$, $U_1'=U_2'=U'=\beta'$, $\Delta E=E_2-E_1=0$ and $\alpha=\alpha'=0$. By introducing the relative phase $\phi=\theta_1-\theta_2(\theta_i(\tau))$, Eq.(12) can be rewritten as[10]

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}\tau} &= 2\gamma N-U'N\frac{N^2+3z^2}{2},\\ \frac{\mathrm{d}z}{\mathrm{d}\tau} &= -2K\sqrt{N^2-z^2}\sin\phi+2\gamma z-U'z\frac{3N^2+z^2}{2},\\ \frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= -Uz+2K\frac{z}{\sqrt{N^2-z^2}}\cos\phi. \end{aligned} \tag{13}$$

When γ and U' are equal to zero, Eq.(13) are consistent with those obtained by Smerzi *et al.*[18] In their case, N is a constant. However, because of the three-body recombination loss and the atomic feeding from the above-condensate cloud, N is time-dependent in the present paper. Owing to the two imaginary interaction terms, chaos can exist in the system.

3. Numerical simulation

We calculate the Lyapunov exponent spectrum of BEC system by using the algorithm presented by Wolf *et al.*[27] Figure 1(a) shows the Lyapunov exponent λ as a function of the s-wave scattering length A. The parameters are U=0.7A, k=3.9, γ=0.00016, and U'=0.000005A^4. Figure 1(b) shows the maximal Lyapunov exponent $\lambda_{\max}$ as a function of the tunnelling dynamics of the two condensates k. The parameters are U=0.7A, γ=0.00016, U'=0.000005A^4, and A=2.1. The middle point-drawing line represents the value of zero. We find that the BEC system can enter into chaotic and hyperchaotic states.

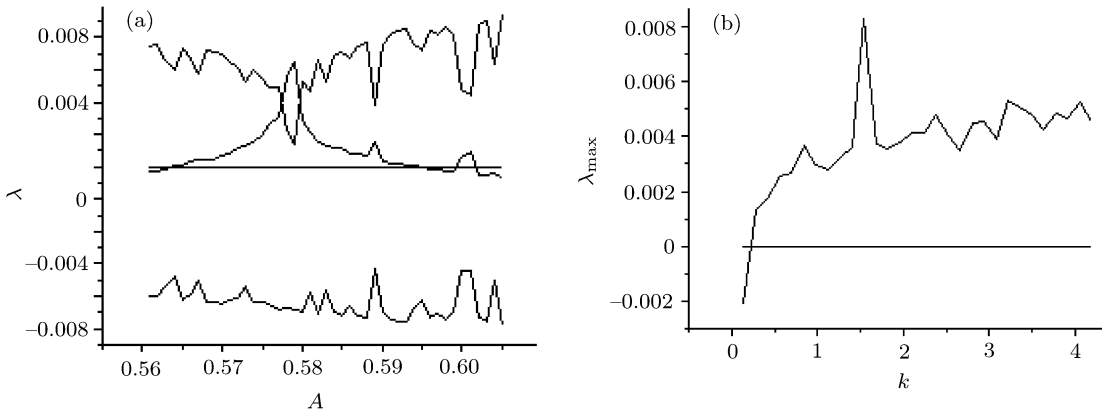
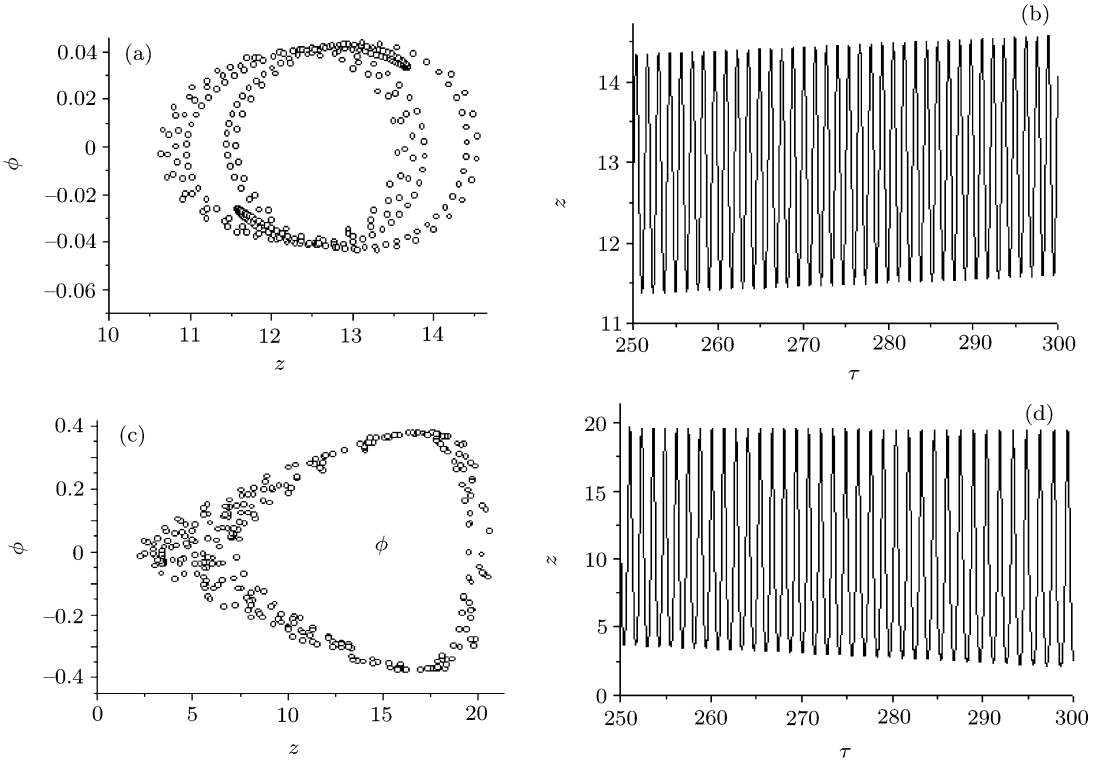


Fig.1. The Lyapunov exponent λ as a function of the s-wave scattering length A with U=0.7A, k=3.9, γ=0.00016, and U'=0.000005A^4, (a) and the maximal Lyapunov exponent $\lambda_{\max}$ as a function of the tunnelling dynamics of the two condensates k with U=0.7A, γ=0.00016, U'=0.000005A^4, and A=2.1 (b).

Using the fourth Runge–Kutta algorithm we solve Eq.(13) numerically, and illustrate the attractors in an equivalent phase of (z, ϕ) in Fig.2. In order to avoid transient chaos, the values of z and ϕ in the initial values of 10000 steps are eliminated, the values of z and ϕ in the final 20000 steps remain in Fig.2(a)–2(h). And the values of z and ϕ in the initial values of 40000 steps are eliminated; the values of z and ϕ in the final 10000 steps remain in Fig.2(i)–2(j). Figure 2 shows the final attractors, the time series and the power spectra. The parameters in Fig.2 are listed as follows: U=0.7A, γ=0.00016, and U'=0.000005A^4. Figure 2(a) and 2(b) for the initial condition: (N, z, ϕ)=(24, 10, 0), A=0.53, and k=3.9. The three Lyapunov exponents are λ_1=0.0168, λ_2=–0.00604 and λ_3=–0.01116, respectively. The BEC system lies in a chaotic state as the maximal Lyapunov exponent is positive. Figures 2(c) and 2(d) are for the initial condition: (N, z, ϕ)=(24,10,0), A=0.7, and k=3.9. The three Lyapunov exponents are λ_1=0.02015, λ_2=0.00415 and λ_3=–0.02699 respectively. The BEC lies in a hyperchaotic state as two Lyapunov exponents are positive. Figures 2(e) and 2(f) are for the initial condition: (N, z, ϕ)=(24,10,0), A=0.9, and k=3.9. The three Lyapunov exponents are λ_1=0.03072, λ_2=–0.00608 and λ_3=–0.03287 respectively. The BEC system is in a chaotic state. Figure 2(g) and 2(h) are for the initial condition: (N, z, ϕ)=(24, 10, 0), A=3.0, and k=3.9. The three Lyapunov exponents are λ_1=0.0194, λ_2=–0.03182 and λ_3=–0.03695 respectively. The shown BEC is in a chaotic state. Figures 2(i) and 2(j) are for the initial condition: (N, z, ϕ)=(24, 10, 0), A=2.1, and k=0.14. The BEC is in a period-1 state. Figure 2(b) is the time series of Fig.2(a). Figure 2(d) is the time series of Fig.2(c). Figure 2(f) is the power spectrum of Fig.2(e). Figure 2(h) is the power spectrum of Fig.2(g). Figure 2(j) is the time series of Fig.2(i). Parameter $z = \dfrac{N_2 - N_1}{N}$ is the relative population between two condensates, where N is the time-dependent number of atoms and $\phi = \theta_1 - \theta_2$ is the relative phase.



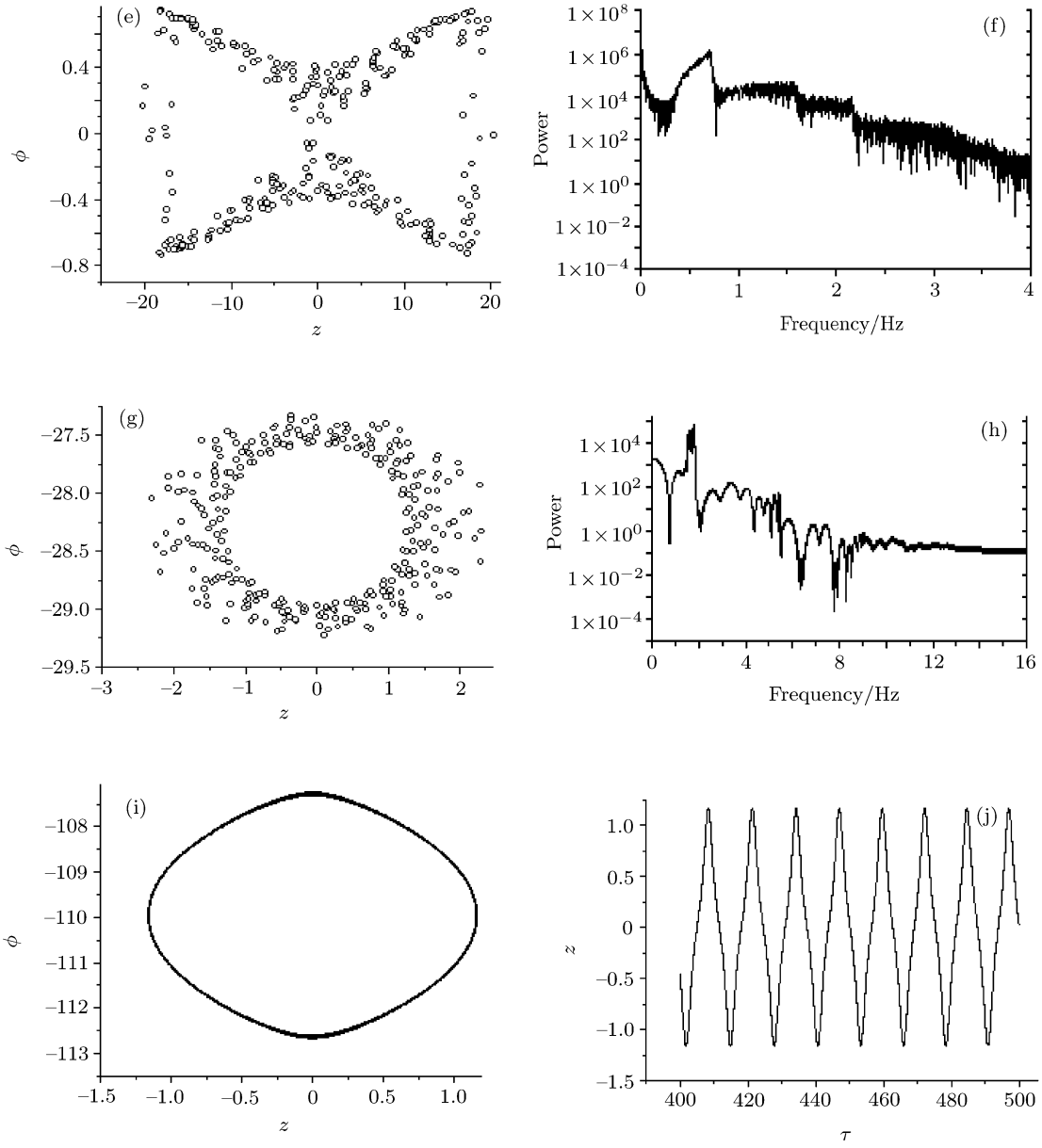


Fig.2. The attractors, the time series and the power spectra with U=0.7A, γ=0.00016, U'=0.000005A^4 and different values of A and k: A=0.53 and k=3.9 (a) (b), A=0.7 and k=3.9 (c) (d), A=0.9 and k=3.9 (e) (f), A=3.0 and k=3.9 (g) (h), A=2.1 and k=0.14 (i) (j).

4. Conclusion

We have investigated the chaotic and the hyperchaotic behaviour of two tunnel-coupled BECs with a three-body interaction. The double-well potentials are considered and the two-mode approximation is adopted. In this way the G–P equation is simplified into three coupled differential equations with two- and three-body interaction terms. The stationary and non-stationary features of the system have been studied analytically and numerically from these equations.

In the recent advances in applications of BEC, quantum computation with BEC atoms in Mott insulating states is an interesting subject.[28] However, chaos is associated with quantum entanglement[29] and quantum error correcting,[30] which are all key subjects in quantum computation. Therefore it is very important to investigate the chaos and the hyperchaos in BEC.

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