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Novel four-wing and eight-wing attractors using coupled chaotic Lorenz systems

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This paper presents the problem of generating four-wing (eight-wing) chaotic attractors. The adopted method consists in suitably coupling two (three) identical Lorenz systems. In analogy with the original Lorenz system, where the two wings of the butterfly attractor are located around the two equilibria with the unstable pair of complex-conjugate eigenvalues, this paper shows that the four wings (eight wings) of these novel attractors are located around the four (eight) equilibria with two (three) pairs of unstable complex-conjugate eigenvalues.

Keywords: chaotic attractors, multi-wing attractor, coupled Lorenz systems, dynamical behaviours
PACC: 0545

1. Introduction

Over the last four decades, chaotic phenomena in physics and engineering have been extensively studied.[1−5] An interesting topic is related to the development of techniques for obtaining chaotic attractors with complicated topological structures and complex shapes.[6] Therefore some efforts have been recently devoted to the generation of multi-scroll attractors,[7−14] multi-wing attractors[15−19] and spherical attractors.[20]

The generation of multi-scroll attractors and the corresponding hardware implementation are no longer a difficult task.[12,13] From the theoretical point of view, most of the developed methods share some basic features: the multiple equilibria are usually obtained by selecting nonlinearities with several breakpoints; the number of scrolls equals that of the equilibria; and the shape of the attractors is characterized by the fact that each scroll is a rotating cycle moving around the equilibrium point. On the other hand, the generation of multi-wing attractors seems to be a more challenging topic. The underlying idea is based on the pioneering Lorenz system described in Ref.[1], from which over the years several variations have been developed, including a recent modification which is able to produce a reverse butterfly-shape attractor.[21] Specifically, the methods proposed so far to generate multi-wing attractors are based on a generalized Lorenz-like system[16] or on a novel four-dimensional system.[17−19] These approaches share some basic features: the nonlinearities of these systems are smooth functions (i.e., quadratic or cubic cross-product terms); the number of wings is not equal to that of the equilibria; and the attractor possesses four wings in a butterfly shape, indicating a generalization of the butterfly attractor introduced by Lorenz in 1963.[1]

This paper presents the problem of generating four-wing (eight-wing) attractors, obtained by coupling two (three) identical Lorenz systems. In analogy with the original Lorenz system, where the two wings of the butterfly attractor are located around the two equilibria with the unstable pair of complex-conjugate eigenvalues, this paper shows that the four wings (eight wings) of the proposed attractors are located around the four (eight) equilibria with two (three) pairs of unstable complex-conjugate eigenvalues. Besides the similar property of the equilibria, the paper analyses other basic features of the four-wing (eight-wing) attractors, including invariance and symmetry properties. Finally, some pictures show the graceful shapes of the proposed attractors.

2. Four-wing attractor

The proposed system is obtained by coupling two identical Lorenz systems as follows:

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1),\\
\dot{y}_1 &= \rho x_1 - y_1 - x_1 z_1 + k_1(x_2 - y_2),\\
\dot{z}_1 &= x_1 y_1 - \beta z_1,\\
\dot{x}_2 &= \sigma(y_2 - x_2),\\
\dot{y}_2 &= \rho x_2 - y_2 - x_2 z_2 + k_2(x_1 - y_1),\\
\dot{z}_2 &= x_2 y_2 - \beta z_2,
\end{aligned} \tag{1}$$

where σ and ρ are the so-called Prandtl and Rayleigh numbers, respectively, whereas $k_1(x_2-y_2)$ and $k_2(x_1-y_1)$ are two linear coupling terms scaled by the parameters $k_1 > 0$ and $k_2 > 0$. For $\rho > 1$ the original third-order Lorenz system possesses three equilibria: the origin (0, 0, 0), characterized by three real eigenvalues, and the equilibria $(\sqrt{\beta(\rho-1)},\ \sqrt{\beta(\rho-1)},\ \rho-1)$ and $(-\sqrt{\beta(\rho-1)},\ -\sqrt{\beta(\rho-1)},\ \rho-1)$, characterized by two complex-conjugate eigenvalues. The original Lorenz system exhibits the well-known butterfly attractor for σ=10, ρ=28 and β=8/3.[1] Now an analysis of system (1) is conducted, showing that its equilibria are symmetric (with respect to certain coordinate planes) and can be grouped into three sets (according to their local stability properties). Namely, by solving the equilibrium equations derived from Eq.(1), it can be seen that the coupling coefficients k_1 and k_2 do not affect the values of the equilibria. Namely, the equilibria of system (1) are obtained by combining the equilibria of the first subsystem, that is $S_0^{\mathrm{I}} = (0,0,0)$, $S_1^{\mathrm{I}} = (\sqrt{\beta(\rho-1)},\ \sqrt{\beta(\rho-1)},\ \rho-1)$, $S_2^{\mathrm{I}} = (-\sqrt{\beta(\rho-1)},\ -\sqrt{\beta(\rho-1)},\ \rho-1)$ with the equilibria of the second subsystem, that is $S_0^{\mathrm{II}} = (0,0,0)$, $S_1^{\mathrm{II}} = (\sqrt{\beta(\rho-1)},\ \sqrt{\beta(\rho-1)},\ \rho-1)$ and $S_2^{\mathrm{II}} = (-\sqrt{\beta(\rho-1)},\ -\sqrt{\beta(\rho-1)},\ \rho-1)$. Thus system (1) possesses nine equilibrium points in the form

$$S_0 = (S_0^{\mathrm{I}}, S_0^{\mathrm{II}}), \tag{2}$$

$$\begin{aligned}
S_1 &= (S_0^{\mathrm{I}}, S_1^{\mathrm{II}}), \quad S_2 = (S_0^{\mathrm{I}}, S_2^{\mathrm{II}}),\\
S_3 &= (S_1^{\mathrm{I}}, S_0^{\mathrm{II}}), \quad S_4 = (S_2^{\mathrm{I}}, S_0^{\mathrm{II}}),
\end{aligned} \tag{3}$$

$$\begin{aligned}
S_5 &= (S_1^{\mathrm{I}}, S_1^{\mathrm{II}}), \quad S_6 = (S_1^{\mathrm{I}}, S_2^{\mathrm{II}}),\\
S_7 &= (S_2^{\mathrm{I}}, S_1^{\mathrm{II}}) \ \text{ and } \ S_8 = (S_2^{\mathrm{I}}, S_2^{\mathrm{II}}).
\end{aligned} \tag{4}$$

Referring to the invariance and symmetry properties, we note that system (1) is invariant under the transformation:

$$(x_1, y_1, z_1; x_2, y_2, z_2) \to (-x_1, -y_1, z_1; -x_2, -y_2, z_2). \tag{5}$$

By considering $\Re^3$ subspaces, relationship (5) implies the existence of several types of symmetries. For example, by considering the subsystems in the $(x_1,\ x_2,\ z_1)$ coordinates and $(x_1,\ x_2,\ z_2)$ coordinates, and by extracting from Eq.(5) the transformations $(x_1,\ x_2,\ z_1) \to (-x_1,\ -x_2,\ z_1)$ and $(x_1,\ x_2,\ z_2) \to (-x_1, -x_2, z_2)$, it is clear that these subsystems are symmetrical with respect to the z_1 axis and z_2 axis, respectively. Using similar considerations we can readily derive other symmetries in $\Re^3$ subspaces. By virtue of the invariance (5), the equilibria (2)–(4) have symmetries with respect to certain coordinate planes. Namely, S_5 and S_8 are symmetric with respect to the hyperplane defined by z_1 and z_2 in $\Re^6$. Similarly, the two equilibria S_6 and S_7 are symmetric with respect to the hyperplane defined by z_1 and z_2 in $\Re^6$. Finally, the pair S_1 and S_2 as well as the pair S_3 and S_4 are symmetric pairs with respect to the hyperplane defined by z_1 and z_2 in $\Re^6$.

Now, we discuss the similar property of the equilibria (2)–(4). It is worth recalling that an equilibrium point is said to be a saddle point index r when its Jacobian possesses r eigenvalues with r positive real parts.[3] Moreover, if the Jacobian matrices have eigenvalues that can be grouped according to certain patterns, the corresponding equilibria are said to possess the property of similarity.[18] Based on these considerations, the equilibria (2)–(4) can be divided into three types, according to their local stability. Namely, by taking σ=10, ρ=28, β=8/3, and by computing the Jacobian for each of the nine equilibria (2)–(4), we can check that for $k_1 \in [0.005,\ 0.5]$ and $k_2 \in [0.005,\ 0.5]$ the set $\{S_0\}$ contains a *saddle point index 2*, the set $\{S_1,\ S_2,\ S_3,\ S_4\}$ contains *saddle points index 3* and the set $\{S_5,\ S_6,\ S_7,\ S_8\}$ contains *saddle points index 4*. Specifically, the equilibrium S_0 has *two* real eigenvalues with *two* positive real parts, the equilibria S_1, S_2, S_3 and S_4 have *three* eigenvalues with *three* positive real parts (including a pair of complex-conjugate eigenvalues) and S_5, S_6, S_7 and S_8 have *four* complex-conjugate eigenvalues with *four* positive real parts.

Now, four-wing attractors are generated from Eq.(1). By taking σ=10, ρ=28, β=8/3, $k_1 = 0.1$ and $k_2 = 0.01$, we find that system (1) generates the four-wing attractor reported in Fig.1(a).

The maximum Lyapunov exponent is positive (i.e., $\lambda_{\max}$=4.10), indicating that the proposed approach is able to produce chaotic behaviours. Finally, the four-wing attractor can be found in a wide range of system parameters. For example, by increasing the value of ρ and by taking σ=10, ρ=120, β=8/3 and $k_1 = k_2 = 0.1$, we can obtain the four-wing attractor reported in Fig.1(b).

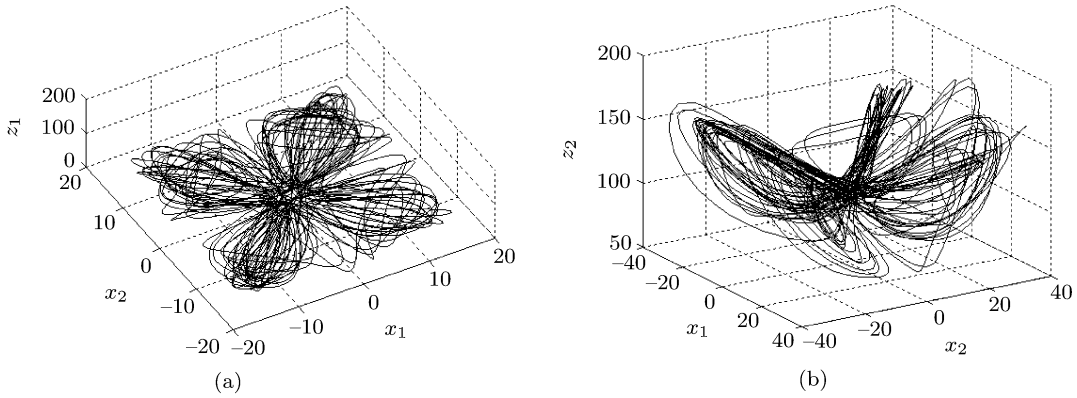


Fig.1. The four-wing attractor: (a) ρ=28; (b) ρ=120. Note the different scale of x_1 and x_2 in the two pictures.

Now, we consider the projection of the attractor in Fig.1(a) onto the $(x_1,\ x_2)$ plane. Specifically, the projections of the equilibria (4) onto the $(x_1,\ x_2)$ plane are $\bar{S}_5 = (\sqrt{\beta(\rho-1)},\ \sqrt{\beta(\rho-1)})$, $\bar{S}_6 = (\sqrt{\beta(\rho-1)},\ -\sqrt{\beta(\rho-1)})$, $\bar{S}_7 = (-\sqrt{\beta(\rho-1)},\ \sqrt{\beta(\rho-1)})$, $\bar{S}_8 = (-\sqrt{\beta(\rho-1)},\ -\sqrt{\beta(\rho-1)})$, where $\bar{S}_i$ indicates the bi-dimensional projection of the corresponding six-dimensional S_i (i=5, 6, 7, 8). The expressions of $\bar{S}_i$ suggest that the pair $\{\bar{S}_5,\ \bar{S}_8\}$ and the pair $\{\bar{S}_6,\ \bar{S}_7\}$ are symmetric pairs with respect to the origin. Moreover, they are equilibria of the same type, that is, *saddle points index 4*. On the other hand, the projections of the equilibria (3) onto the $(x_1,\ x_2)$ plane are $\bar{S}_1 = (0,\ \sqrt{\beta(\rho-1)})$, $\bar{S}_2 = (0,\ -\sqrt{\beta(\rho-1)})$, $\bar{S}_3 = (\sqrt{\beta(\rho-1)},\ 0)$, $\bar{S}_4 = (-\sqrt{\beta(\rho-1)},\ 0)$. Therefore the pair $\{\bar{S}_1,\ \bar{S}_2\}$ and the pair $\{\bar{S}_3,\ \bar{S}_4\}$ are symmetric pairs with respect to the origin. Moreover, they are equilibria of the same type, i.e., *saddle points index 3*. Finally, the projection of the origin S_0 onto the $(x_1,\ x_2)$ plane is a *saddle point index 2*. For ρ=28 and β=8/3, the projections of the equilibria (4) onto $(x_1,\ x_2)$ are $\bar{S}_5 = (8.48,\ 8.48)$, $\bar{S}_6 = (8.48,\ -8.48)$, $\bar{S}_7 = (-8.48,\ 8.48)$ and $\bar{S}_8 = (-8.48,\ -8.48)$, whereas the projections of the equilibria (3) are $\bar{S}_1 = (0,\ 8.48)$, $\bar{S}_2 = (0,\ -8.48)$, $\bar{S}_3 = (8.48,\ 0)$ and $\bar{S}_4 = (-8.48,\ 0)$. By considering the $(x_1,\ x_2)$ plane depicted in Fig.1(a), we can unquestionably observe that the four wings are located around the points $\bar{S}_5$, $\bar{S}_6$, $\bar{S}_7$ and $\bar{S}_8$. The remaining points $\bar{S}_0$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$ are located outside the wings. The conclusion of the analysis is that, even though the unstable equilibria are nine, the four wings are located only around the *saddle points index 4*. Therefore an interesting analogy with the original Lorenz system can be established. Namely, in the original Lorenz system all the equilibria (i.e., *three*) are unstable, and the wings (*two*) are located only around the equilibria having the highest number of positive real parts (i.e., *two*, coming from the pair of complex-conjugate eigenvalues). Analogously, in the proposed system all the equilibria (i.e., *nine*) are unstable, and the wings (*four*) are located only around the equilibria having the highest number of positive real parts (i.e., *four*, coming from the two pairs of complex-conjugate eigenvalues).

3. Eight-wing attractor

Given three identical Lorenz systems, consider the system described by the following equations:

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1),\\
\dot{y}_1 &= \rho x_1 - y_1 - x_1 z_1 + k_1(x_3 - y_3),\\
\dot{z}_1 &= x_1 y_1 - \beta z_1,\\
\dot{x}_2 &= \sigma(y_2 - x_2),\\
\dot{y}_2 &= \rho x_2 - y_2 - x_2 z_2 + k_2(x_1 - y_1),\\
\dot{z}_2 &= x_2 y_2 - \beta z_2,\\
\dot{x}_3 &= \sigma(y_3 - x_3),\\
\dot{y}_3 &= \rho x_3 - y_3 - x_3 z_3 + k_3(x_2 - y_2),\\
\dot{z}_3 &= x_3 y_3 - \beta z_3,
\end{aligned} \tag{6}$$

where $k_1(x_3 - y_3)$, $k_2(x_1 - y_1)$, and $k_3(x_2 - y_2)$ are three linear coupling terms scaled by the parameters $k_1 > 0$, $k_2 > 0$, and $k_3 > 0$. An analysis similar to that conducted for system (1) demonstrates that system (6) possesses twenty-seven equilibrium points, obtained by combining the three equilibria of each subsystem with those of the remaining two subsystems. Additionally, system (6) possesses symmetry and invariance properties similar to those of system (1). Referring to the property of similarity, we focus

our attention on the eight equilibria described by

$$S_1 = (S_1^{\rm I}, S_1^{\rm II}, S_1^{\rm III}), \quad S_2 = (S_1^{\rm I}, S_1^{\rm II}, S_2^{\rm III}),$$
$$S_3 = (S_1^{\rm I}, S_2^{\rm II}, S_1^{\rm III}), \quad S_4 = (S_1^{\rm I}, S_2^{\rm II}, S_2^{\rm III}),$$
$$S_5 = (S_2^{\rm I}, S_1^{\rm II}, S_1^{\rm III}), \quad S_6 = (S_2^{\rm I}, S_1^{\rm II}, S_2^{\rm III}),$$
$$S_7 = (S_2^{\rm I}, S_2^{\rm II}, S_1^{\rm III}) \text{ and } S_8 = (S_2^{\rm I}, S_2^{\rm II}, S_2^{\rm III}), \qquad (7)$$

where $S_1^{\rm III} = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)$ and $S_2^{\rm III} = (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)$ are equilibria of the third subsystem. By computing the Jacobian for each of the equilibria (7) when σ=10, ρ=120, and β=8/3, we can verify that for k_1, k_2 and k_3 in the range [0.005, 0.5] the set $\{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$ contains *saddle points index 6*, i.e., the eight equilibria (7) are characterized by *six* complex-conjugate eigenvalues with *six* positive real parts. The resulting eight-wing attractor in the (x_1, x_2, x_3)-subspace is depicted in Fig.2(a) for $k_1 = k_2 = k_3$=0.1. In order to show that the attractor really possesses eight wings, different views are reported, which highlight the (x_1, x_2)-plane (Fig.2(b)), the (x_2, x_3)-plane (Fig.2(c)) and the (x_1, x_3)-plane (Fig.2(d)), respectively.

Now, the location of the wings with respect to the equilibria is analysed. Namely, even though the equilibria of system (5) are twenty-seven, the eight wings are located only around the *saddle points index 6*, i.e. around the eight equilibria given by $\bar{S}_i = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)})$, i= 1,...,8. Note that $\bar{S}_i$ is the three-dimensional projection of the corresponding nine-dimensional S_i in Eq.(7). When ρ=120 and β=8/3, the eight equilibria are $\bar{S}_i$ = (±17.82, ±17.82, ±17.82). They represent the eight vertices of a cube in the (x_1, x_2, x_3) subspace. With the help of the different views of the attractor reported in Figs.2(b)–2(d), we can verify that the eight wings are really located around the eight equilibria given by $\bar{S}_i$ = (±17.82, ±17.82, ±17.82).

Finally, note that the analogy with the original Lorenz system still holds for the eight-wing attractor. Namely, in system (6) all the equilibria (i.e., *twenty-seven*) are unstable, and the wings (*eight*) are located only around those equilibria having the highest number of positive real parts (i.e., *six*, coming from the three pairs of complex-conjugate eigenvalues).

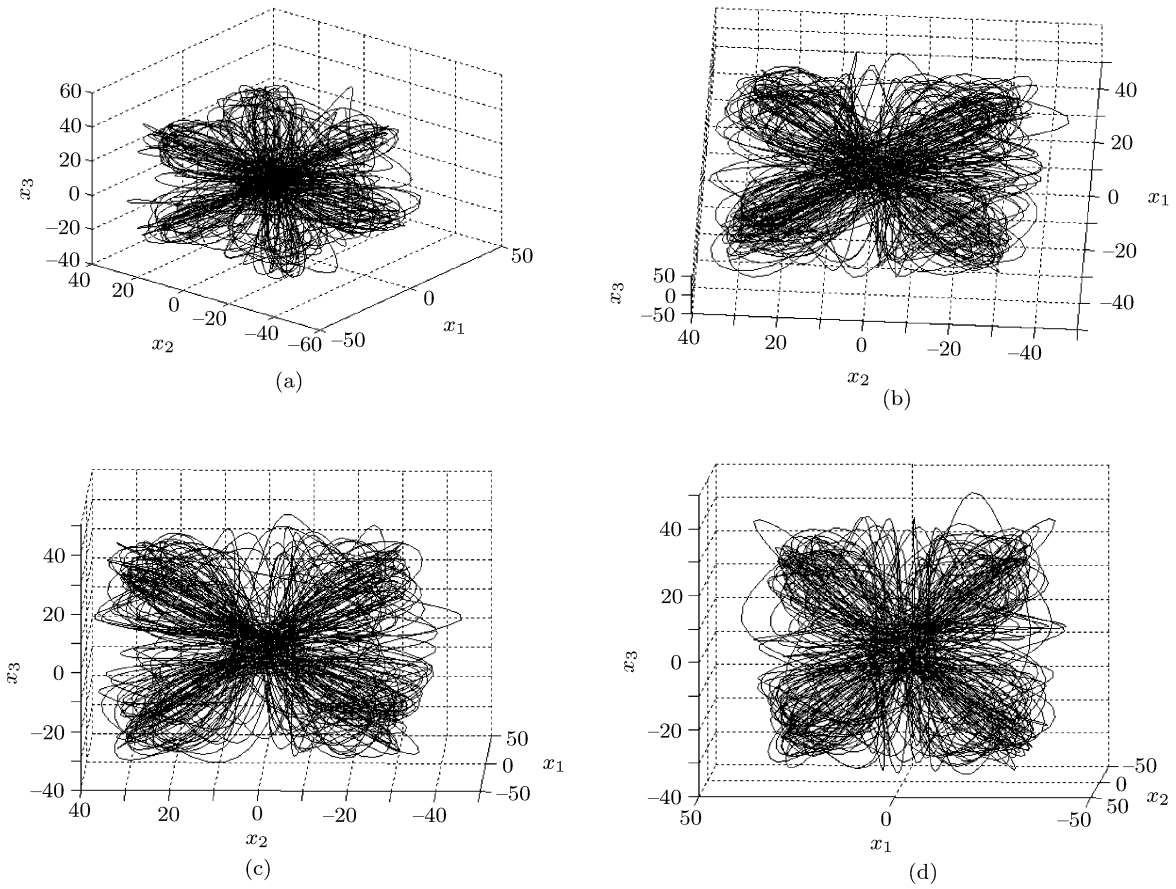


Fig.2. The eight-wing attractor: (a) view in the (x_1, x_2, x_3) subspace; (b) view highlighting the (x_1, x_2)-plane; (c) view highlighting the (x_2, x_3)-plane; (d) view highlighting the (x_1, x_3)-plane.

4. Conclusion

In this paper novel four-wing (eight-wing) attractors produced by two (three) coupled Lorenz systems have been described. In particular, the paper has shown that the equilibria have a number of symmetries with respect to specific coordinate planes and their local stability properties are analogous to that of a single Lorenz system. Note that the proposed approach generates only four-wing and eight-wing attractors, that is, it cannot be used to obtain an odd number of wings. Nevertheless, the proposed approach presents an interesting feature deriving from the use of coupled systems: it is straightforward and enables properties analogous to that of a single Lorenz system to be established.

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