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Thermodynamic stability of a weakly interacting Fermi gas trapped in a harmonic potential

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Based on the theoretical results derived from pseudopotential method and local approximation, this paper studies the thermodynamic stability of a weakly interacting Fermi gas trapped in a harmonic potential by using analytical method of thermodynamics. The effects of the interparticle interactions as well as external potential on the thermodynamic stability of the system are discussed. It is shown that the system is stable as for the complete average, but as for local parts, the system is unstable anywhere. This instability shows that the stability conditions of mechanics cannot be satisfied anywhere, and the stability conditions of thermostatics cannot be satisfied somewhere. In addition, the interactions and external potential have direct effects on the local stability of the system.

Keywords: Fermi gas, interaction, stability
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1. Introduction

There have been large numbers of fruits about the investigation of thermodynamic properties of quantum gas recently.[1−12] For example, Refs.[10] and [11] investigated magnetic quantum phase transition of cold atoms and phase diagram of two-species Bose–Einstein condensates respectively in an optical lattice. Reference [12] investigated dynamics of a bright soliton in Bose–Einstein condensates with time-dependent atomic scattering length in an expulsive parabolic potential. On the basis of these studies, there also have been some investigation fruits about stability of Fermi and Bose system. For example, interparticle interactions can make the condensate of Bose gas disintegrate and collapse.[13−16] Reference [17] indicated that degenerate Fermi gas may collapse under certain conditions. References [18] and [19] investigated the effects of the interparticle interactions on the thermodynamic stabilization of Bose and Fermi system respectively, and gave the particle-number-density conditions of instability of the system. Reference [20] also gave the instability conditions of a weakly interacting Fermi gas trapped in a weak magnetic field.

However, the study of theory and experiment on stability of Fermi system is still in its initial stages, thus it is very important to investigate the thermodynamic stability of the interacting system under different conditions. So far, researches of the thermodynamic stability have been limited to the weakly interacting system without external potential, and there has been no study of thermodynamic stability of a weakly interacting Fermi gas trapped in an external potential, especially in a harmonic potential. From the point of both the whole average and the local, this paper studies the thermodynamic stability of a weakly interacting Fermi gas trapped in a harmonic potential. This study has positive meaning for further understanding the thermodynamic properties of a weakly interacting Fermi gas trapped in a harmonic potential.

2. Stability of the whole average

We consider a weakly interacting system of spin-1/2 fermions, with volume V, particle number N and s-wave scattering length a, and the condition of weak interaction is satisfied here.

$$|a|/\lambda \ll 1, \quad |a|\, n^{1/3} \ll 1. \tag{1}$$

The system is trapped in a harmonic potential

$$V(r) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \tag{2}$$

where $\lambda = \sqrt{2\pi\hbar^2/mkT}$ (k is the Boltzmann constant, T is the temperature) is the thermal wave-

length, $n = N/V$ is the average density of particle number, ω_x, ω_y and ω_z are the angular frequencies of harmonic potential along three coordinate axes, m is the mass of a single particle, and x, y, z are the position coordinates of particles.

In order to keep the thermodynamic stability of the system, the stability conditions of mechanics and thermostatics must be satisfied at the same time.

$$\left(\frac{\partial P}{\partial V}\right)_T < 0, \tag{3}$$

$$C_V > 0. \tag{4}$$

(1) The condition of high temperatures ($T \gg T_{\rm F0}$)

When the temperature is much higher than the Fermi temperature $T_{\rm F0} = \hbar\omega(3N)^{1/3}/k$ ($\hbar = h/(2\pi)$ is Planck constant, $\omega = \sqrt[3]{\omega_x\omega_y\omega_z}$) of the system trapped in a harmonic potential, we have

$$f_\ell(z_0) \approx z_0, \quad F_{3/2,1/2,3/2}(z_0) \approx z_0^2/2^{3/2}, \tag{5}$$

$$F_{3/2,1/2,5/2}(z_0) \approx z_0^2/2^{5/2},$$

$$F_{3/2,1/2,1/2}(z_0) \approx z_0^2/2^{1/2}, \tag{6}$$

where

$$f_\ell(z_0) = \frac{1}{\Gamma(\ell)}\int_0^\infty \frac{t^{\ell-1}{\rm d}t}{z_0^{-1}{\rm e}^t + 1} \xrightarrow{z_0\le 1} \sum_{i=1}^\infty (-1)^{i-1}\frac{z_0^i}{i^\ell}, \tag{7}$$

$$F_{\alpha,\beta,\gamma} = \left[\int_0^\infty f_\alpha(z_0{\rm e}^{-t}) f_\beta(z_0{\rm e}^{-t}) t^{\gamma-1}{\rm d}t\right]/\Gamma(\gamma) \xrightarrow{z_0\le 1} \sum_{i,j=1}^\infty \frac{(-1)^{i+j}z_0^{i+j}}{i^\alpha j^\beta (i+j)^\gamma}, \tag{8}$$

$z_0 = N(\hbar\omega/(kT))^3$ is the fugacity of noninteracting system.

Based on Ref.[9], the total energy and heat capacity of the system can be calculated by using Eqs.(5) and (6). They are

$$U = NkT\left[3 + \frac{a}{\lambda}\frac{5z_0}{2^{5/2}}\right], \tag{9}$$

$$C_V = 3Nk\left[1 - \frac{a}{\lambda}\frac{5z_0}{2^{7/2}}\right]. \tag{10}$$

And the average pressure of the system is

$$P = \frac{2}{3}\frac{U}{V} = \frac{2}{3}\frac{N}{V}kT\left[3 + \frac{a}{\lambda}\frac{5z_0}{2^{5/2}}\right], \tag{11}$$

thus

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{2N}{V^2}kT\left[1 + \frac{a}{\lambda}\frac{5z_0}{3\times 2^{5/2}}\right]. \tag{12}$$

It is known from Eqs.(10) and (12) that under the condition of weak interactions, $C_V > 0, (\partial P/\partial V)_T < 0$, the system is thermodynamically stable as for the complete average.

(2) The condition of low temperatures ($T \ll T_{\rm F0}$)

At low temperatures, based on Ref.[9], we can obtain the total energy and heat capacity of the system,

$$U = \frac{3}{4}NE_{\rm F0}\left\{1 + \frac{2}{3}\pi^2\left(\frac{T}{T_{\rm F0}}\right)^2 + \frac{8192}{2835\pi^{3/2}}\frac{a}{\lambda_{\rm F0}}\left[1 + \frac{27\pi^2}{32}\left(\frac{T}{T_{\rm F0}}\right)^2\right], \tag{13}$$

$$C_V = Nk\pi^2\left[1 + \frac{128}{35\pi^{3/2}}\frac{a}{\lambda_{\rm F0}}\right]\frac{T}{T_{\rm F0}}, \tag{14}$$

where $E_{\rm F0} = kT_{\rm F0}, \lambda_{\rm F0} = (2\pi\hbar^2/(mkT_{\rm F0}))^{1/2}$. Thus,

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{\partial}{\partial V}\left(\frac{2}{3}\frac{U}{V}\right) = -\frac{2}{3}\frac{U}{V^2}. \tag{15}$$

It is known from Eqs.(14) and (15) that at low temperatures, the system is thermodynamically stable as for the complete average.

3. Local stability

(1) The condition of high temperatures ($T \gg T_{\rm F0}$)

From Ref.[9], we have the density of energy

$$u(r) = \frac{kT}{\lambda^3}\Big\{3f_{5/2}(z') + 2\beta V(r)f_{3/2}(z') - \frac{4a}{\lambda}[f_{3/2}^2(z') + \beta V(r)f_{3/2}(z')f_{1/2}(z')]\Big\}, \tag{16}$$

where $z' = z{\rm e}^{-\beta V(r)} \le z$, and $z \approx z_0\Big[1 + \frac{a}{\lambda}F_{3/2,1/2,3/2}(z_0)/f_2(z_0)\Big]$ is the fugacity of the system.

At high temperatures, z is very small, z' is smaller, and $f_\ell(z') \approx z'$, thus

$$u(r) \approx \frac{kT}{\lambda^3}\Big\{3 + 2\beta V(r) - \frac{4a}{\lambda}[1 + \beta V(r)]z{\rm e}^{-\beta V(r)}\Big\}z{\rm e}^{-\beta V(r)}. \tag{17}$$

We take an element of volume ΔV whose energy is $u(r)\Delta V$ in the neighbourhood of r and take it as research object. This element of volume contains many particles, just like Ref.[21], so that it can be considered as Fermi gas which has average potential, and the

statistics rules are still valid, so for the pressure p of this volume element, we have

$$\begin{aligned}&\left[\frac{\partial p}{\partial(\Delta V)}\right]_T\\ =&\frac{2}{3}\frac{\partial}{\partial(\Delta V)}[u(r)]_T\\ =&\frac{2}{3}\frac{kT}{\lambda^3}\left\{3+2\beta V(r)-\frac{8a}{\lambda}z[1+\beta V(r)]\mathrm{e}^{-\beta V(r)}\right\}\\ &\times\mathrm{e}^{-\beta V(r)}\frac{\mathrm{d}z}{\mathrm{d}(\Delta V)}.\end{aligned}\tag{18}$$

The fugacity of the system is $z=\exp[\beta\mu]$, and the chemical potential of the system μ is the same everywhere and it is independent of the volume ΔV, so the result of Eq.(18) is 0. Thus, for the whole system, the stability conditions of mechanics cannot be satisfied anywhere and the system is not mechanically stable anywhere.

Considering the local thermostatics stability, we also take an element of volume ΔV whose energy is $u(r)\Delta V$ in the neighbourhood of r. Taking the local volume element ΔV as research object and from the Eq.(17), we can obtain the local heat capacity

$$\begin{aligned}C_V=&\left(\frac{\partial u}{\partial T}\right)\Delta V=\frac{5}{2}\frac{k}{\lambda^3}z\mathrm{e}^{-\beta V(r)}\left\{3+\frac{12}{5}\frac{V(r)}{kT}\left[1+\frac{1}{3}\frac{V(r)}{kT}\right]+\frac{1}{5}\frac{T}{z}\frac{\mathrm{d}z}{\mathrm{d}T}\left[6+4\frac{V(r)}{kT}\right]\right.\\ &\left.-\frac{a}{5\lambda}z\mathrm{e}^{-\beta V(r)}\left[24+16\frac{V(r)}{kT}\left(2+\frac{V(r)}{kT}\right)+16\frac{T}{z}\frac{\mathrm{d}z}{\mathrm{d}T}\left(1+\frac{V(r)}{kT}\right)\right]\right\}\Delta V.\end{aligned}\tag{19}$$

At high temperatures, there are some approximate results:

$$z=z_0\left[1+\frac{a}{\lambda}\frac{z_0}{\sqrt{2}}\right],\tag{20}$$

$$z_0=\frac{N}{2}\left(\frac{\hbar\omega}{kT}\right)^3,\tag{21}$$

$$\frac{\mathrm{d}z}{\mathrm{d}T}=-\frac{3z_0}{T}\left[1+\frac{11}{12}\frac{a}{\lambda}\sqrt{2}z_0\right].\tag{22}$$

In the brace of the Eq.(19), we take $z\approx z_0$, $\mathrm{d}z/\mathrm{d}T\approx -3z_0/T$, so the Eq.(19) can be simplified as follows:

$$\begin{aligned}C_V=&\frac{1}{2}\frac{k}{\lambda^3}z\mathrm{e}^{-\beta V(r)}\left\{-3+4\left[\frac{V(r)}{kT}\right]^2\right.\\ &\left.+8\frac{a}{\lambda}z\mathrm{e}^{-\beta V(r)}\left[3+2\frac{V(r)}{kT}\left(1-\frac{V(r)}{kT}\right)\right]\right\}\Delta V.\end{aligned}\tag{23}$$

It is known from the Eq.(23) that in the space region of $\dfrac{V(r)}{kT}<1$ and $\left[\dfrac{V(r)}{kT}\right]^2>\dfrac{3}{4}$, the expression $C_V>0$ is satisfied, which means that in the space region the system is thermostable everywhere. However, in the space region of $\dfrac{V(r)}{kT}<1$ and $\left[\dfrac{V(r)}{kT}\right]^2<\dfrac{3}{4}$, the expression $C_V<0$ is satisfied, which means that in the space region the system is not thermostable anywhere and the higher temperature grows, the larger the unstable space region will become. In addition, the attractive interactions aggravate this instability, but the repulsive interactions weaken the instability. The condition of potential centre can be attributed to this kind of instance that the thermostatics stability is not satisfied. Thus, the potential energy has a direct effect on the local thermostatics stability and $V(r)=\dfrac{\sqrt{3}}{2}kT$ is the transition point of potential energy of the local thermostatics stability. At the point of $V(r)=\dfrac{\sqrt{3}}{2}kT$ and its neighbourhood, the repulsive interactions make the thermostatics stability satisfied, but the thermostatics stability is not satisfied because of the attractive interactions

(2) The condition of low temperatures ($T\ll T_{\mathrm{F}0}$)

From the Eq.(16), for the pressure p of the local ΔV, we have

$$\begin{aligned}&\left[\frac{\partial p}{\partial(\Delta V)}\right]_T\\ =&\frac{2}{3}\frac{\partial}{\partial(\Delta V)}[u(r)]_T\\ =&\frac{kT}{\lambda^3}\left\{3f_{3/2}(z')+2\beta V(r)f_{1/2}(z')\right.\\ &-\frac{4a}{\lambda}[2f_{3/2}(z')f_{1/2}(z')\\ &\left.+\beta V(r)(f_{1/2}^2(z')+f_{3/2}(z')f_{-1/2}(z'))]\right\}\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}(\Delta V)}.\end{aligned}\tag{24}$$

For the same reason, the stability conditions of mechanics cannot be satisfied for the whole system anywhere and the system is not mechanically stable anywhere because of $\mathrm{d}z/\mathrm{d}(\Delta V)=0$.

At low temperatures, in the neighbourhood of $r \to 0$, $z' = z\mathrm{e}^{-\beta V(r)}$ is larger, thus

$$f_\ell(z') \approx \frac{(\ln z')^\ell}{\Gamma(\ell+1)}, \tag{25}$$

$$f_3(z_0) = \frac{N}{2}\left(\frac{\hbar\omega}{kT}\right)^3, \tag{26}$$

$$z = z_0\left[1 + \frac{1024}{945}\frac{a}{\lambda}(\ln z_0)^{3/2}\right]. \tag{27}$$

Considering the local thermostatics stability again, we still take an element of volume ΔV whose energy is $u(r)\Delta V$ in the neighbourhood of r. Taking the local volume element ΔVas research object, we can obtain the local heat capacity on the basis of the Eq.(16),

$$\begin{aligned} C_V = \frac{k}{\lambda^3}\Bigg\{ & \frac{15}{2}f_{5/2}(z') + \frac{6V(r)}{kT}f_{3/2}(z') + \frac{2V(r)}{k}f_{1/2}(z')\left[\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}T} + \frac{V(r)}{kT^2}\right] + f_{3/2}(z')\frac{3T}{z}\frac{\mathrm{d}z}{\mathrm{d}T} \\ & -\frac{a}{\lambda}\Bigg[12f_{3/2}^2(z') + \frac{16V(r)}{kT}f_{3/2}(z')f_{1/2}(z') + \frac{8T}{z}\frac{\mathrm{d}z}{\mathrm{d}T}f_{3/2}(z')f_{1/2}(z') + \frac{4V(r)}{k}\left(\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}T} + \frac{V(r)}{kT^2}\right) \\ & \times(f_{1/2}^2(z') + f_{3/2}(z')f_{-1/2}(z'))\Bigg]\Bigg\}\Delta V. \end{aligned} \tag{28}$$

Using the Eqs.(25), (26) and (27), and taking some approximation as follows,

$$\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}T} = -\frac{1}{T}\left[1 + (3N)^{1/3}\frac{\hbar\omega}{kT}\right] \approx -(3N)^{1/3}\frac{\hbar\omega}{kT^2}, \tag{29}$$

we will have the local heat capacity

$$\begin{aligned} C_V = \frac{4k}{\lambda^3}\frac{(\ln z')^{1/2}}{\sqrt{\pi}}\Bigg\{ & (\ln z')^2 + \frac{\ln z'}{2}\left[\frac{4V(r)}{kT} - \frac{\hbar\omega\sqrt[3]{3N}}{kT}\right] + \frac{V(r)}{(kT)^2}\left[V(r) - \hbar\omega\sqrt[3]{3N}\right] \\ & -\frac{16}{3\sqrt{\pi}}\frac{a}{\lambda}(\ln z')^{1/2}\left[(\ln z')^2 + \frac{\ln z'}{kT}(2V(r) - \hbar\omega\sqrt[3]{3N}) + \frac{V(r)}{(kT)^2}(V(r) - \hbar\omega\sqrt[3]{3N})\right]\Bigg\}\Delta V. \end{aligned} \tag{30}$$

In the neighbourhood of $r \to 0$, we take $\ln z' = \ln z - \dfrac{V(r)}{kT} \approx \ln z$, and consider $\dfrac{V(r)}{kT} \ll \dfrac{\hbar\omega}{kT}\sqrt[3]{3N}$, so the Eq.(30) can be written as

$$\begin{aligned} C_V = \frac{4k}{\lambda^3}\frac{(\ln z)^{1/2}}{\sqrt{\pi}}\Bigg\{ & (\ln z)^2 - \left[\frac{\ln z}{2} + \frac{V(r)}{kT}\right]\frac{\hbar\omega\sqrt[3]{3N}}{kT} \\ & -\frac{16}{3\sqrt{\pi}}\frac{a}{\lambda}(\ln z)^{1/2}\left[(\ln z)^2 - \left(\ln z + \frac{V(r)}{kT}\right)\frac{\hbar\omega\sqrt[3]{3N}}{kT}\right]\Bigg\}\Delta V. \end{aligned} \tag{31}$$

Thus, in the neighbourhood of potential centre ($r \to 0$) where

$$(\ln z)^2 < \left[\frac{1}{2}\ln z + \frac{V(r)}{kT}\right]\frac{\hbar\omega}{kT}\sqrt[3]{3N} \tag{32}$$

is satisfied, the condition of thermostatics stability cannot be satisfied for the attractive interactions.

Especially, at the point of r =0, we know from the Eq.(32) that when

$$\ln z < \frac{\hbar\omega}{2kT}\sqrt[3]{3N} \tag{33}$$

is satisfied, the thermostatics stability will definitely not appear at the centre of potential for attractive interactions. Substituting Eqs.(25), (26) and (27) into (33), we will obtain the condition of the thermostatics instability, which can be expressed as

$$1 + \frac{1024}{945}a(3N)^{1/6}\sqrt{\frac{m\omega}{h}} < \frac{1}{2}. \tag{34}$$

When attractive interaction a, total number of particles N, mass of single particle m, and restricting frequency ω can be satisfied in the Eq.(34), the thermostatics stability will not appear at the centre of potential.

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