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Exact bound state solutions of the Klein–Gordon particle in Hulthén potential

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In this paper, the Klein–Gordon equation with the spherical symmetric Hulthén potential is turned into a hypergeometric equation and is solved in the framework of function analysis exactly. The corresponding bound state solutions are expressed in terms of the hypergeometric function, and the energy spectrum of the bound states is obtained as a solution to a given equation by boundary constraints.

Keywords: Hulthén potential, Klein–Gordon equation, bound state, hypergeometric function
PACC: 0365G, 1110Q, 1240Q

1. Introduction

It is well known that when a particle is in a strong potential field, the relativistic effect must be considered.[1] Taking the relativistic effect into account, one could apply the Klein–Gordon equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle. Therefore, it is interesting to find the exact solutions of the Klein–Gordon equation and Dirac equation for a particle in some typical potential. The Hulthén potential is one of the important short range potentials[2,3] and has been applied to a number of areas such as nuclear and particle physics, atomic physics, condensed matter and chemical physics.[4−7] This naive potential explains the electronic properties of F′-colour centres in alkali halides quite well.[8] Within a non-relativistic framework the model of three-dimensional delta-function well could be considered as a Hulthén potential with the radius of the force going down to zero.[9] Moreover, the relativistic effect for a particle under the action of this potential could become important, especially for a strong coupling system. Dominguez-Adame has solved the s-wave Klein–Gordon equation of the Hulthén potential and examined the effects of strong coupling on bound states.[10] Talukdar *et al* have given the scattering state solution of the s-wave Klein–Gordon equation with Hulthén potential for regular and irregular boundary conditions.[11] Hu and Su have obtained the exact solution of s-wave Dirac equation for Hulthén potential.[12] All these works are completed in the case that each scalar potential of them is equal to or stronger than its vector potential. Recently, Guo *et al* have investigated the Dirac equation with special Hulthén problem for the more general case.[13] Qiang *et al* have derived the ℓ-state solution to the Klein–Gordon equation for the general Hulthén potential within an approximation of the centrifugal potential for any ℓ-states.[14] Haouat and Chetouani have solved the (3+1)-dimensional Dirac equation in the presence of the radial pseudoscalar Hulthén potential by using the usual approximation of the centrifugal potential.[15] As we know, the Klein–Gordon equation with Hulthén problem has not yet been solved for the more general case.

It is the purpose of the present article to solve the Klein–Gordon equation with Hulthén potential for the more general case. Since the Hulthén potential is independent of the time, we could separate a Schrödinger-like equation satisfied by the radial component r from the four-dimensional covariant Klein–Gordon equation. Then by making an adequate change of the radial variable we obtain a second-order differential equation that resembles to the hypergeometric equation. Finally, the bound state solutions are expressed in terms of the hypergeometric function, and the energy spectrum of the bound states is obtained as a solution to a given equation by boundary constraints.

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2. Solutions of the Klein–Gordon equation with Hulthén potential

The Klein–Gordon equation, minimally coupled to a vector potential A_μ can be written as[16]

$$\left[(\partial_\mu - \mathrm{i}eA_\mu)^2 - 1\right]\psi = 0. \tag{1}$$

For simplicity, here and thereafter we choose to work in natural units $\hbar = c = m = 1$. When the vector potential A^μ is spherical symmetric and does not depend on time, i.e. $A^\mu = V(r)$, we have $\psi = \psi(r)\exp(-\mathrm{i}Et)$. Therefore, in the spherical coordinates, the problem of solving the Klein–Gordon equation (1) is reduced to that of finding the solution to the second-order differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\phi(r) + \left[(E - V(r))^2 - 1\right]\phi(r) = 0, \tag{2}$$

where

$$\phi(r) = \frac{\psi(r)}{r}. \tag{3}$$

The spherically symmetric Hulthén potential is

$$V(r) = -V_0\frac{\mathrm{e}^{-r/a}}{1-\mathrm{e}^{-r/a}}, \tag{4}$$

where a represents the spatial range and $1/a$ can be considered as the screening parameter. For small r compared to a, the Hulthén potential approaches to the Coulomb potential whereas for large r it approaches to zero exponentially.[15]

Substituting Eq.(4) into Eq.(2) yields

$$\frac{\mathrm{d}^2\phi(r)}{\mathrm{d}r^2} + \left[\left(E + V_0\frac{\mathrm{e}^{-r/a}}{1-\mathrm{e}^{-r/a}}\right)^2 - 1\right]\phi(r) = 0. \tag{5}$$

Introducing a non-dimensional transformation

$$y = \mathrm{e}^{-r/a}, \quad (r\to 0,\ \ y\to 1;\ \ r\to\infty,\ \ y\to 0). \tag{6}$$

Equation (5) becomes

$$y^2\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} + y\frac{\mathrm{d}\phi(y)}{\mathrm{d}y} + a^2\left[\left(E + \frac{V_0 y}{1-y}\right)^2 - 1\right]\phi(y) = 0. \tag{7}$$

The bound-state solution of Eq.(7) satisfies the boundary conditions

$$\phi(0) = 0, \quad (y\to 0, \ \ r\to\infty), \tag{8}$$

and

$$\phi(1) = 0, \quad (y\to 1, \ \ r\to 0). \tag{9}$$

Using the trial function

$$\phi(y) = y^\mu(1-y)^\lambda H(y), \tag{10}$$

we find that Eq.(7) could be turned into a hypergeometric differential equation

$$\begin{aligned} &y(1-y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}H(y) + [(2\mu+1) \\ &-(2\mu+2\lambda+1)y]\frac{\mathrm{d}}{\mathrm{d}y}H(y) \\ &-(\mu+\lambda+\nu)(\mu+\lambda-\nu)H(y) = 0. \end{aligned} \tag{11}$$

Where

$$\mu = \mathrm{i}ak, \quad k^2 = \left(E^2 - 1\right), \tag{12}$$

$$\nu = a\sqrt{1-(E-V_0)^2}, \tag{13}$$

$$\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1-4a^2V_0^2}. \tag{14}$$

For the bound state solution, the parameter λ must be real, this demands $a < \dfrac{1}{2V_0}$. The general solution of Eq.(11) can be expressed as

$$\begin{aligned} &H(y) \\ &= AF(\mu+\lambda+\nu,\ \mu+\lambda-\nu, 1+2\mu, y) \\ &\quad +By^{-2\mu}F(-\mu+\lambda+\nu, -\mu+\lambda-\nu, 1-2\mu, y). \end{aligned} \tag{15}$$

For this to be a physically acceptable solution to the problem, it must satisfy the appropriate boundary condition. The first boundary condition of Eq.(7) demands that the parameter B in Eq.(15) must be zero. Then, Eq.(15) becomes

$$H(y) = AF(\mu+\lambda+\nu, \mu+\lambda-\nu, 1+2\mu, y). \tag{16}$$

But for the second boundary condition, the behaviour of Eq.(16) near $y = 1 (r\to 0)$ needs to be analysed carefully. For this we transform Eq.(16) into the following form:[17]

$$H(y) = CF(\mu+\lambda+\nu, \mu+\lambda-\nu, 2\lambda, 1-y) + D(1-y)^{1-2\lambda}F(1+\mu-\lambda-\nu, 1+\mu-\lambda+\nu, 2-2\lambda, 1-y). \tag{17}$$

Where

$$C = \frac{\Gamma(1+2\mu)\,\Gamma(1-2\lambda)}{\Gamma(1+\mu-\lambda-\nu)\,\Gamma(1+\mu-\lambda+\nu)}, \quad (18)$$

$$D = \frac{\Gamma(1+2\mu)\,\Gamma(2\lambda-1)}{\Gamma(\mu+\lambda+\nu)\,\Gamma(\mu+\lambda-\nu)}. \quad (19)$$

Since $y = 1 (r \to 0)$, $F(\mu+\lambda+\nu, \mu+\lambda-\nu, 2\lambda, 1-y) = 1$, we find that when 2λ is a positive integer the term of $\Gamma(1-2\lambda)$ in Eq.(18) inclines to ∞, one of the necessary and sufficient condition of bound-state for Eq.(17) is obtained as

$$1+\mu-\lambda-\nu = -n, \quad (n = 0, 1, 2, \ldots). \quad (20)$$

But using the analysis process in Ref.[3], we get the conclusion that when $n = 0$ ($y \to 1, r \to 0$), the second term in Eq.(17) is divergent. Therefore, the condition $n = 0$ should be excluded from Eq.(20). Then the bound-state condition for Eq.(11) is

$$1+\mu-\lambda-\nu = -n, \quad (n = 1, 2, \ldots). \quad (21)$$

Substituting Eqs.(12), (13) and (14) into Eq.(21), we obtain the bound-state energy equation as

$$\frac{1}{2}\left(\sqrt{1-4V_0^2a^2}-1\right) + a\left[\sqrt{1-(E-V_0)^2}-\sqrt{1-E^2}\right] = n, \quad (n = 1, 2, \ldots). \quad (22)$$

Also, the bound state solution of Eq.(5) could be expressed as

$$\phi(r) = \mathrm{e}^{-\mu r/a}\left(1-\mathrm{e}^{-r/a}\right)^{\lambda} \times AF(n+2\mu+1, n+2\mu+1-2\nu, 1+2\mu, \mathrm{e}^{-r/a}). \quad (23)$$

As showing above, the bound state solution of the Klein–Gordon equation with the Hulthén potential exists for the more general case, the energy spectrum and the bound state wavefunctions are given by Eqs.(22) and (23), respectively. For the bound state solution, the parameter λ must be real and this demands parameter $a < \frac{1}{2V_0}$. Recently, some studies of the relativistic scattering state and the transmission resonance on one-dimensional symmetric and well-behaved potentials have been carried out, such as the Woods-Saxon one,[18] and the cusp one.[19,20] It should be mentioned that Hulthén potential is a spherically symmetric and well-behaved potential. Therefore, the relativistic scattering state and the transmission resonance of Hulthén potential could be treated with the similar method and procedure used in this study.

3. Conclusions

In conclusion, the Klein–Gordon equation with the spherically symmetric Hulthén potential is turned into a hypergeometric equation and is solved in the framework of function analysis. It is shown that for the more general case, the bound state solution of the Klein–Gordon equation with Hulthén potential exists and can be expressed in terms of the hypergeometric functions. The energy spectrum of boundary states is obtained as a solution to a given equation by boundary constraints.

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