Vol 17 No 9, September 2008
1674-1056/2008/17(09)/3198-05

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

A study on entanglement dynamics for a four-qubit model*

Man Zhong-Xiao(满忠晓)† and Xia Yun-Jie(夏云杰)

College of Physics and Engineering, Qufu Normal University, Qufu 273165, China

(Received 14 January 2008; revised manuscript received 23 February 2008)

In this paper, we consider the entanglement dynamics of a four-qubit model [2006 *Phys. Rev.* A **74** 042328] where two entangled qubits a and b locally interact with separate qubits A and B via the spin-exchange-like Hamiltonian. We study the effect of purity of initial entangled state of qubits a, b on the entanglement evolution and its relation with energy transfer. Also, we find that the total bipartite entanglement of qubits a, b plus A, B is not a constant any longer when the initial entangled state of a, b is not pure, which is a complement to the result in the paper [2007 *J. Phys.* B **40** S45] for the pure case.

Keywords: entanglement dynamics, entanglement sudden death, purity
PACC: 0365, 0367

Entanglement is at the heart of quantum mechanics and attracts more and more attentions due to the powerful applications in quantum information process[1] and quantum computing.[2] It is therefore of increasing importance to understand entanglement from its role and manifestation in realistic systems[3−5] besides that from the mathematical point of view to formalize the theory of entanglement. Recently, Yu and Eberly[6] discovered that the entanglement of a pair of qubits exposed to local noisy environments can vanish completely in a finite time, in striking contrast with decoherence of an individual qubit that decays only asymptotically. Such a surprising phenomenon was termed entanglement sudden death (ESD). Because of its intrinsic and practical interests, ESD has attracted much attentions from both theoretical[7−14] and experimental[15] point of view. In Ref.[8], the authors dealt with a double Jaynes–Cummings model in which two initially entangled two-level atoms are independently coupled with separate cavity fields. Focusing on the atomic subsystem they found out that, depending on the type of atomic initial state, the entanglement may or may not exhibit ESD. The authors of Ref.[8] also formulated distinct forms of entanglement invariant for different initial states. The situation in which different qubits are coupled to the same environment was also investigated in Refs.[10, 11].

Although the extensive investigations in understanding the disentanglement, it is still unclear what reason causes this phenomenon and what is the physics behind.[11] In some cases, the entanglement evolution is companied with the energy transfer between the system considered and its environment. Hence, the connection between entanglement and energy has been also studied.[11,12] In Ref.[11], the authors considered the entanglement dynamics and energy transfer for both open and closed systems. In an open system where two qubits coupled with the same environment, i.e. for the Tavis–Cumming model, they showed that the dynamics of entanglement and its revival stem from the energy transfer between the system and environment.[11] For another closed quantum systems, in which the evolution of the system is insulated to its surroundings, they also proved the existence of ESD and the direct relation between entanglement evolution and energy transfer.[11] In Ref.[12], the authors studied a four qubit model where a qubit a (b) of the entangled qubit-pair ab interacts with qubit A (B) of the separate qubit-pair AB via spin-exchange-like Hamiltonian and there are no any interactions between the subsystems Aa and Bb. They showed that the entanglement and energy of subsystem AB are strictly related. However, their study[12] took pure Bell-like state as initial state of entangled qubit-pair ab and only considered the entanglement evolution (energy transfer) of the initially separate subsystem AB.

*Project supported by National Natural Science Foundation of China (Grant No 10774088).
†E-mail: manzhongxiao@163.com

In this paper, we shall consider the same model as that in Ref.[12] but pay additional attention on the effects of the purity r of the initial entangled state of ab. Also, we will show that the total entanglement amount of qubit-pairs AB and ab under evolution is related to r. That is, it is a preserved constant when the initial entangled state of ab is pure, i.e., $r = 1$, whereas it is variable when the initial state of ab is mixedly entangled.

To quantify the entanglement of a two-qubit system we adopt Wootters' concurrence.[16] The concurrence $C(\rho_{ab})$ for the density matrix ρ_{ab} of a two-qubit system ab is defined as

$$C(\rho_{ab}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (1)$$

where λ_i are the eigenvalues of the matrix $\zeta = \rho(\sigma_y^a \otimes \sigma_y^b)\rho^*(\sigma_y^a \otimes \sigma_y^b)$ arranged in decreasing order. Here $\sigma_y^{a(b)}$ is the y-Pauli matrix acting on qubit a (b) and ρ^* is the complex conjugation of ρ in the standard (computational) basis. For the separate state $C = 0$ whereas $C = 1$ for maximally entangled state.

The model we are interested in can be described as follows. Suppose a system of four qubits a, b, A, and B interacting via a spin-exchange-like Hamiltonian:

$$H = H_{aA} + H_{bB}, \quad (2)$$

where

$$H_{aA} = \frac{\omega_a}{2}\sigma_z^a + \frac{\omega_A}{2}\sigma_z^A + g_{aA}(\sigma_-^a\sigma_+^A + \sigma_+^a\sigma_-^A), \quad (3)$$

and

$$H_{bB} = \frac{\omega_b}{2}\sigma_z^b + \frac{\omega_B}{2}\sigma_z^B + g_{bB}(\sigma_-^b\sigma_+^B + \sigma_+^b\sigma_-^B). \quad (4)$$

For each qubit, the relevant Pauli operators are defined as

$$\begin{aligned}\sigma_z =& |1\rangle\langle 1| - |0\rangle\langle 0|,\\ \sigma_+ =& |1\rangle\langle 0|,\\ \sigma_- =& |0\rangle\langle 1|,\end{aligned} \quad (5)$$

and the interaction operators like $\sigma_-^a\sigma_+^A$ can be viewed as annihilating an excitation of subsystem a and creating one in A. Here, we assume the complete resonance condition $\omega_A = \omega_a = \omega_B = \omega_b = \omega$ and the same coupling strength $g_{aA} = g_{bB} = g$ for subsystems aA and bB.

For the initial state of qubit-pair ab, instead of Bell-like and Werner states,[17] we shall consider the following extended Werner-like states

$$\rho_{ab}(0) = r\,|\Phi\rangle_{abab}\langle\Phi| + \frac{1-r}{4}I_{ab} \quad (6)$$

with r the purity of the initial state of qubits ab, I_{ab} the 4×4 identity matrix and

$$|\Phi\rangle_{ab} = (\sin(\theta)\,|01\rangle + \cos(\theta)\,|10\rangle)_{ab}, \quad (7)$$

the Bell-like state. Obviously, the state in Eq.(6) reduces to the standard Werner state when $\theta = \pi/4$ and to Bell-like pure state when $r = 1$. By dealing with the above extended Werner-like state, we are able to study the effect of mixedness of the initial entangled state. Both the Bell-like state and Werner state, and so the extended Werner-like state, belong to the so-called X-class state[13] whose density matrix is of the form

$$\rho_{ab} = \begin{pmatrix} x & 0 & 0 & v \\ 0 & y & u & 0 \\ 0 & u^* & z & 0 \\ v^* & 0 & 0 & w \end{pmatrix} \quad (8)$$

with x, y, z, w real positive and u, v complex quantities. The X-class states have the property that the corresponding two-qubit density matrix preserves the X-form during the system evolution. For the X-state Eq.(8), the concurrence can be derived as

$$C(\rho_{ab}) = 2\max\{0, |u| - \sqrt{xw}, |v| - \sqrt{yz}\}. \quad (9)$$

The total system state at $t = 0$ is in the form

$$\rho_{abAB}(0) = \rho_{ab}(0) \otimes |00\rangle_{ABAB}\langle 00|, \quad (10)$$

where $\rho_{ab}(0)$ is given by Eq.(6) and $|00\rangle_{AB} \equiv |0\rangle_A \otimes |0\rangle_B$. After time t, the reduced density matrix $\rho_{ab}(t)$ can be obtained by tracing over the degrees of freedom of qubits A and B, which remains the X-form with

$$\begin{aligned} x =& \frac{1-r}{4} + r\Big((\cos^2(\theta)\sin^2(g_{aA}t)\\ &+ \sin^2(\theta)\sin^2(g_{bB}t)\Big),\\ y =& \frac{1-r}{4} + r\sin^2(\theta)\cos^2(g_{bB}t),\\ z =& \frac{1-r}{4} + r\cos^2(\theta)\cos^2(g_{aA}t),\\ w =& \frac{1-r}{4},\\ u =& r\sin\theta\cos\theta\cos(g_{aA}t)\cos(g_{bB}t),\\ v =& 0. \end{aligned} \quad (11)$$

By virtue of Eq.(9), we can get the concurrence $C(\rho_{ab}(t))$ of $\rho_{ab}(t)$ as

$$C(\rho_{ab}(t)) = 2\max\{0, |u| - \sqrt{xw}\}. \quad (12)$$

The energy of the system ab, $U(\rho_{ab}(t))$, can be obtained by the mean values of the relevant restrictions of the free Hamiltonian:

$$\begin{aligned} U(\rho_{ab}(t)) =& \mathrm{Tr}\{\rho_{ab}H_{ab}\} \\ =& \mathrm{Tr}\{\rho_{ab}\frac{\omega}{2}(\sigma_z^a+\sigma_z^b)\} \\ =& w-x. \end{aligned} \tag{13}$$

It is known that the entanglement of Werner state is related to the purity r, that is if $r \leq r_C$ the state is separable. As for the extended Werner-like state considered here, different initial entanglement amount of qubits ab in terms of θ will correspond to different value of r_C. In Figs.1 and 2 we plot $C(\rho_{ab}(t))$ as functions of r and rescaled time gt for $\theta = \pi/4$ and $\theta = \pi/12$. We can see that for $\theta = \pi/4$, $r_C \simeq 0.3$ while for $\theta = \pi/12$, $r_C \simeq 0.5$. It is worth noting that for $r_C < r < 1$ (i.e., $r \neq 1$), $C(\rho_{ab}(t))$ can vanish in a finite time and remain zero before reviving, i.e., the ESD phenomenon occurs for the qubit pair ab. We can also see that ESD time is in inverse proportion to the purity r, that is the less the value r the longer the ESD time. Therefore, the behaviours of $C(\rho_{ab}(t))$ is related to the purity of initial entanglement of qubits ab. As for $r = 1$, i.e, initially the state of qubits ab is pure, the relations between entanglement evolution and energy transfer have shown in Ref.[12]. Here, to see the effects of r on the entanglement evolution and energy transfer, we plot $C(\rho_{ab}(t))$ (expressed by Eq.(12)) and $U(\rho_{ab}(t))$ (expressed by Eq.(13)) as a function of rescaled time gt for $\theta = \pi/4$ and different r in Fig.3. We can see that the variations of concurrence and energy are almost in-step not only for the case of $r = 1$ but also for $r = 0.8$.

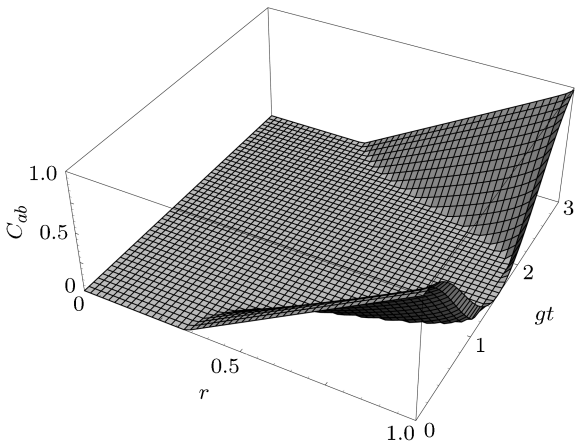


Fig.1. $C_{ab} \equiv C(\rho_{ab}(t))$ as functions of r and the dimensionless time gt for $\theta = \pi/4$.

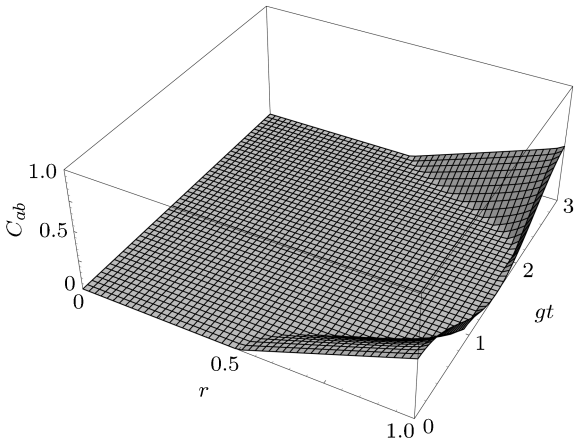


Fig.2. Same as in Fig.1 but for $\theta = \pi/12$.

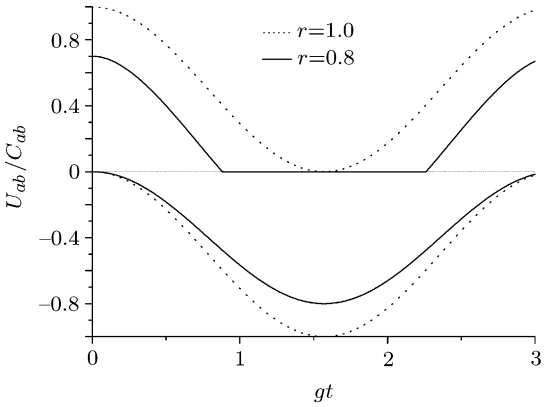


Fig.3. $C_{ab} \equiv C(\rho_{ab}(t))$ (upper part) and $U_{ab} \equiv U(\rho_{ab}(t))$ (lower part) as a function of the dimensionless time gt for $r = 0.8$ (solid line) and $r = 1$ (dot line) for $\theta = \pi/4$.

The physical reason for variation of the initial entanglement of qubits ab rests in the spreading out of entanglement over all of the system's degrees of freedom that may become entangled with each other in all possible ways through the underlying local interactions. Hence, in addition to consider the entanglement evolution of subsystem ab, it is also of interest to investigate the entanglement of its complement subsystem AB which may be the environment of the qubits ab in realistic system. We derive the concurrence $C(\rho_{AB}(t))$ for the reduced density matrix $\rho_{AB}(t)$ of qubits AB. The expression of $\rho_{AB}(t)$ at any time t can be obtained after tracing out the total system density matrix over the degrees of freedom of the qubits a, b. The matrix form of $\rho_{AB}(t)$ is the same as $\rho_{ab}(t)$ in Eq.(8) but with

the following matrix elements

$$\begin{aligned} x =& \frac{1-r}{4} + r\Big((\cos^2(\theta)\cos^2(g_{aA}t) \\ & + \sin^2(\theta)\cos^2(g_{bB}t)\Big), \\ y =& \frac{1-r}{4} + r\sin^2(\theta)\sin^2(g_{bB}t), \\ z =& \frac{1-r}{4} + r\cos^2(\theta)\sin^2(g_{aA}t), \\ w =& \frac{1-r}{4}, \\ u =& r\sin\theta\cos\theta\sin(g_{aA}t)\sin(g_{bB}t), \\ v =& 0. \end{aligned} \tag{14}$$

The concurrence $C(\rho_{AB}(t))$ of qubits AB is again given by Eq.(12). It has been proven in Ref.[8] that when $r = 1$ the total concurrence of qubits ab plus AB is a conserved constant $|\sin(2\theta)|$ which is the initial entanglement of qubits ab. Therefore, the loss of entanglement $C(\rho_{ab}(t))$ for the system ab is instantly compensated by concurrence gain in $C(\rho_{AB}(t))$ for the qubits AB, that is the entanglement gain in the pair AB is a process of entanglement transfer from ab to AB.[8] However, we find that for the arbitrary $r_C < r < 1$ the total concurrence TC$\equiv C(\rho_{AB}(t)) + C(\rho_{ab}(t))$ is not a constant any longer as shown in Fig.4 where we plot TC as functions of r and gt for the choice of $\theta = \pi/4$. A surprising phenomenon can be seen from the figure that the TC can exhibit ESD

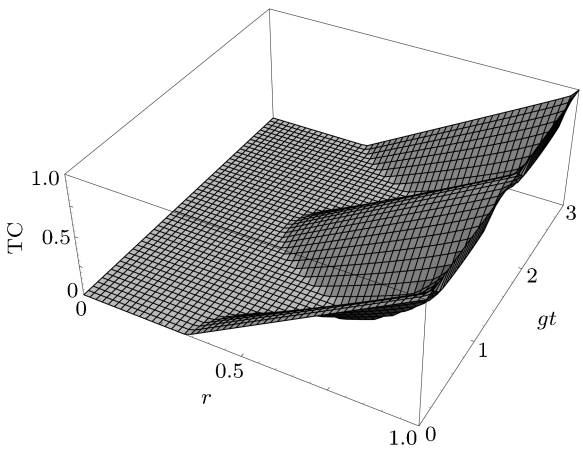


Fig.4. TC$\equiv C(\rho_{AB}(t)) + C(\rho_{ab}(t))$ as functions of the purity r and dimensionless time gt for $\theta = \pi/4$.

for the relatively small r. That is, both the entanglement of qubits AB and ab can simultaneously vanish and remain zero before recovering. We plot TC as a function of gt for $\theta = \pi/4$ and several $r_C < r < 1$ in Fig.5, where we can definitely see the ESD of TC for $r = 0.5$.

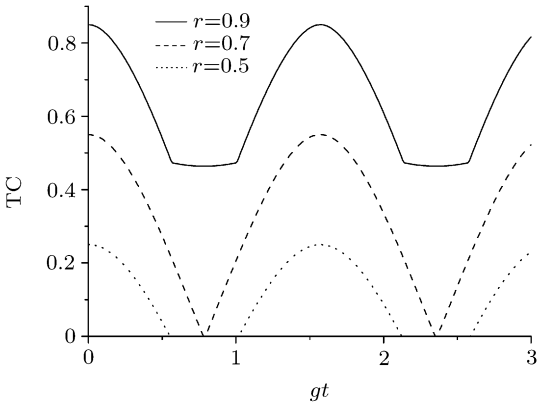


Fig.5. TC as a function of gt for $\theta = \pi/4$ and $r_C < r < 1$.

In conclusion, we have considered the entanglement evolution of a pair of entangled qubits a, b interacting respectively with another two independent qubits A, B via spin-exchange-like Hamiltonian. Instead of pure Bell-like state, we take a class of extended Werner-like state as the initial state of the entangled qubits a, b. We study effects of the purity r of initial entanglement of qubits a, b on the its entanglement evolution and energy transfer. Also, by integrating the entanglement of the qubit-pair A, B, we find that the total entanglement of them is related to time t for $r_C < r < 1$. This result is a complement to the finding of Ref.[8] where the authors proved that for $r = 1$ the total entanglement of them is a conserved constant which is equal to the initial entanglement of qubits a, b. We also show that for relatively small r, this total entanglement can vanish in finite time and remain zero before reviving, i.e., exhibit the ESD phenomenon. The understanding of entanglement behaviours in realistic system is a precondition for its application in practice, thus studying the entanglement dynamics in various situation, such as that has been done here, proves to be very necessary.

References

[1] Cai Q Y and Li B W 2004 *Phys. Rev.* A **69** 054301
Cai Q Y 2003 *Phys. Rev. Lett.* **91** 109801
Long G L and Liu X S 2002 *Phys. Rev.* A **65** 032302
Deng F G and Long G L 2003 *Phys. Rev.* A **68** 042315
Deng F G, Long G L and Liu X S 2003 *Phys. Rev.* A **68** 042317

[2] Long G L and Xiao L 2004 *Phys. Rev.* A **69** 052303
Zhou Z W, Han Y J and Guo G C 2006 *Phys. Rev.* A **74** 052334
Zhou Z W, Yu B, Zhou X, Feldman M J and Guo G C 2004 *Phys. Rev. Lett.* **93** 010501
[3] Fang M F, Jiang C L and Wu Z Z 2006 *Acta Phys. Sin.* **55** 4647 (in Chinese)
[4] Hao Y J and Liang J Q 2006 *Chin. Phys.* **15** 1161
[5] Li G X, Tan H T and Xia H X 2005 *Chin. Phys.* **14** 1382
[6] Yu T and Eberly J H 2004 *Phys. Rev. Lett.* **93** 140404
[7] Eberly J H and Yu T 2007 *Science* **316** 555
Yu T and Eberly J H 2006 *Phys. Rev. Lett.* **97** 140403
Yu T and Eberly J H 2006 *Opt. Commun.* **264** 393
[8] Yönaç M, Yu T and Eberly J H 2006 *J. Phys.* B **39** S621
Yönaç M, Yu T and Eberly J H 2007 *J. Phys.* B **40** S45
[9] Sainz I and Björk G 2007 *Phys. Rev.* A **76** 042313
[10] Ficek and Tanaś R 2006 *Phys. Rev.* A **74** 024304
Liu R F and Chen C C 2006 *Phys. Rev.* A **74** 024102
[11] Cui H T, Li K and Yi X X 2007 *Phys. Lett.* A **365** 44
[12] Cavalcanti D, Oliveira J G, Peixoto de Faria J G, Terra Cunha M O and Santos M F 2006 *Phys. Rev.* A **74** 042328
[13] Yu T and Eberly J H 2007 *Quantum Inf. Comput.* **7** 459
[14] Sun Z, Wang X G and Sun C P 2007 *Phys. Rev.* A **75** 062312
Yi X X and Wang W 2007 *Phys. Rev.* A **76** 042109
Wang W, Fu L B and Yi X X 2007 *Phys. Rev.* A **75** 045601
Ma Z H and Wang X G 2007 *Phys. Rev.* A **75** 014304
Carvalho A R R, Mintert F, Palzer S and Buchleitner A 2007 *Eur. Phys. J.* D **41** 425
Ban M, Kitajima S and Shibata F 2005 *J. Phys.* A **38** 7161
Bellomo B, Franco R Lo and Compagno G 2007 *Phys. Rev. Lett.* **99** 160502
[15] Almeida M P, de Melo F, Hor-Meyll M, Salles A, Walborn S P, Souto Ribeiro P H and Davidovich L 2007 *Science* **316** 579
[16] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
[17] Werner R F 1989 *Phys. Rev.* A **40** 4277