Vol 17 No 9, September 2008
1674-1056/2008/17(09)/3180-05

Chinese Physics B

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Conformal invariance and Hojman conserved quantities of first order Lagrange systems*

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(Received 1 January 2008; revised manuscript received 27 January 2008)

In this paper the conformal invariance by infinitesimal transformations of first order Lagrange systems is discussed in detail. The necessary and sufficient conditions of conformal invariance and Lie symmetry simultaneously by the action of infinitesimal transformations are given. Then it gets the Hojman conserved quantities of conformal invariance by the infinitesimal transformations. Finally an example is given to illustrate the application of the results.

Keywords: first order Lagrange systems, infinitesimal transformation, conformal invariance, Hojman conserved quantities

PACC: 0320

1. Introduction

The research on symmetries and conserved quantities of mechanical systems possesses important theoretical and practical significance.[1−3] Conserved quantities, also known as first integrals, can simplify the solutions of dynamical systems. Sophus Lie introduced the notion of continuous groups, now known as Lie groups. Lie showed that the order of an ordinary differential equation could be reduced by one, if it is invariant under a one-parameter Lie groups of point transformations.[4−6] The symmetry method is a modern method for finding conserved quantities of dynamical systems.[7−21] In Ref.[1], all kinds of constrained dynamical systems were studied comprehensively and thoroughly on symmetries and quantities. In 1997, Galiullin *et al* discussed the symmetry of Birkhoff equations and its relation with the conformal invariance.[22] Recently conformal Hamiltonian systems were discussed by geometric methods, in which the author researched the geometric structure of conformal systems with symmetry and pointed out the difference of conformal systems from arbitrary symmetric systems.[23] The Lagrange systems in mechanics are usually of second order. But many systems of molecular biology, biochemistry, physiology, bionomy and social science are described by first order differential equations. And some of them can be expressed by first order Lagrange equations: such as the dynamical models of the variation of thickness in Lotka biochemistry vibrator and the alive quantities of biology in certain two-group ecosystems.[24] So there is theoretical and practical significance in researching the first order Lagrange systems. In this paper, the conformal invariance of first order Lagrange systems by infinitesimal point transformations is first defined, and we discussed the conformal invariance and their relations with Lie symmetry of first order Lagrange systems in detail. Then we seek the Hojman quantities of the first order Lagrange systems. Finally an example is given to illustrate the application of the results.

2. The differential equations of the systems

Generally speaking, the first order Lagrange systems can be expressed as follows:

$$F_s\left(t, q_k, \dot{q}_k\right) = 0, \qquad (s, k = 1, \ldots, n). \tag{1}$$

Equation (1) can be transformed into the Lagrange forms, that is

$$F_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} \qquad (s = 1, \ldots, n). \tag{2}$$

And more, Eq.(2) can be described by

$$F_s = A_{sk}\dot{q}_k + B_s\left(t, \boldsymbol{q}\right), \tag{3}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos 10372053, 10572021 and 10772025) and the National Natural Science Foundation of Henan province of China(Grant No 0311010900).

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where

$$A_{sk}=\frac{\partial^2 L}{\partial\dot{q}_s\partial q_k},\quad B_s=\frac{\partial^2 L}{\partial\dot{q}_s\partial t}-\frac{\partial L}{\partial q_s}. \tag{4}$$

Suppose

$$\det\left(\frac{\partial^2 L}{\partial\dot{q}_s\partial q_k}\right)\neq 0,$$

we can obtain from Eq.(1)

$$\dot{q}_k=\alpha_k\left(t,\boldsymbol{q}\right). \tag{5}$$

In addition, the conditions and methods of Eq.(1) are expressed in the form of Lagrange forms, see [24–26].

3. Infinitesimal transformations and the determining equations of conformal invariance

We take the infinitesimal transformations of t and q_s as

$$t^*=t+\Delta t,\quad q_s^*(t^*)=q_s(t)+\Delta q_s. \tag{6}$$

Or expanded it as

$$t^*=t+\varepsilon\xi_0\left(t,\boldsymbol{q}\right),\quad q_s^*\left(t^*\right)=q_s\left(t\right)+\varepsilon\xi_s\left(t,\boldsymbol{q}\right), \tag{7}$$

where ε is infinitesimal parameter, and $\xi_0\left(t,\boldsymbol{q}\right)$, $\xi_s\left(t,\boldsymbol{q}\right)$ are infinitesimal generators.

Introducing the vector forms as

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{8}$$

its first prolonged vectors are

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial\dot{q}_s}. \tag{9}$$

Definition 1 In the infinitesimal transformations of infinitesimal generators $\xi_0\left(t,\boldsymbol{q}\right)$ and $\xi_s\left(t,\boldsymbol{q}\right)$, if F_s satisfy the following expressions:

$$X^{(1)}(F_s)=\ell_s^l(F_l), \tag{10}$$

then the infinitesimal transformations have conformal invariance to the corresponding first order differential equations, where the expressions Eq.(10) are determining equations of Eq.(3), and ℓ_s^l is called the conformal factor.

4. The conformal factor of conformal invariance and Lie symmetry

In order to get the conformal factor, we need to suppose that the Hamilton equations have conformal invariance and Lie symmetry simultaneously under the infinitesimal transformations.

When Eq.(3) is Lie symmetric, we have

$$X^{(1)}F_s\left|_{F_s=0}\right.=0,\quad (s=1,\ldots,n). \tag{11}$$

To obtain the general expression of conformal invariance, we calculate the difference

$$X^{(1)}F_s-X^{(1)}F_s\left|_{F_s=0}\right.. \tag{12}$$

Owing to

$$\dot{\xi}_k=\frac{\partial\xi_k}{\partial t}+\frac{\partial\xi_k}{\partial q_l}\dot{q}_l,\quad (k=1,\ldots,n), \tag{13}$$

therefore,

$$\begin{aligned}X^{(1)}\left(F_s\right)=&A_{sk}\left(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0\right)+X^{(0)}\left(A_{sk}\right)\dot{q}_k\\&+X^{(0)}\left(B_s\right)\\=&A_{sk}\left(\frac{\partial\xi_k}{\partial q_l}\dot{q}_l+\frac{\partial\xi_k}{\partial t}-\dot{q}_k\dot{\xi}_0\right)\\&+X^{(0)}\left(A_{sk}\right)\dot{q}_k+X^{(0)}\left(B_s\right).\end{aligned} \tag{14}$$

Equivalently, we have

$$\begin{aligned}X^{(1)}F_s\left|_{F_s=0}\right.=&A_{sk}\left(\dot{\xi}_k-\alpha_k\dot{\xi}_0\right)+X^{(0)}\left(A_{sk}\right)\alpha_k\\&+X^{(0)}\left(B_s\right)\\=&A_{sk}\left(\frac{\partial\xi_k}{\partial q_l}\alpha_l+\frac{\partial\xi_k}{\partial t}-\alpha_k\dot{\xi}_0\right)\\&+X^{(0)}\left(A_{sk}\right)\alpha_k+X^{(0)}\left(B_s\right).\end{aligned} \tag{15}$$

Substituting Eqs.(14) and (15) into Eq.(12), we obtain

$$\begin{aligned}&X^{(1)}F_s-X^{(1)}F_s\left|_{F_s=0}\right.\\=&A_{sk}\left(\frac{\partial\xi_k}{\partial q_l}\left(\dot{q}_l-\alpha_l\right)-\left(\dot{q}_k-\alpha_k\right)\dot{\xi}_0\right)\\&+X^{(0)}\left(A_{sk}\right)\left(\dot{q}_k-\alpha_k\right).\end{aligned} \tag{16}$$

Due to

$$\begin{aligned}\dot{q}_l-\alpha_l=&\dot{q}_l+A^{lr}B_r\\=&A^{lr}\left(A_{rj}\dot{q}_j+B_r\right)=A^{lr}F_r,\end{aligned} \tag{17}$$

so

$$X^{(1)}F_s-X^{(1)}F_s\left|_{F_s=0}\right.$$

$$= \left[A_{sk} \left(\frac{\partial \xi_k}{\partial q_l} - \delta_l^k \dot{\xi}_0 \right) A^{lr} + X^{(0)} \left(A_{sk} \right) A^{kr} \right] F_r. \quad (18)$$

Let

$$\beta_s^r = A_{sk} \left(\frac{\partial \xi_k}{\partial q_l} - \delta_l^k \dot{\xi}_0 \right) A^{lr} + X^{(0)} \left(A_{sk} \right) A^{kr},$$

so that Eq.(18) is changed into

$$X^{(1)} F_s - X^{(1)} F_s \left|_{F_s=0} \right. = \beta_s^r F_r. \quad (19)$$

When Eq.(3) have conformal invariance and Lie symmetry simultaneously by the action of the infinitesimal point transformations of one-parameter group Eq.(7), we can obtain

$$\left(\ell_s^r - \beta_s^r \right) F_r = X^{(1)} F_s \left|_{F_s=0} \right. = 0 \,, \quad (20)$$

from Eqs.(10) and (11). So we obtain the following equality from Eq.(20).

$$\ell_s^l = \beta_s^l. \quad (21)$$

Thus, from the above discussion there is the following proposition.

Proposition 1 To Eq.(3), the necessary and sufficient conditions of its conformal invariance which is Lie symmetric simultaneously by the action of the infinitesimal one-parameter groups are that there exists the conformal factor

$$\ell_s^r = A_{sk} \left(\frac{\partial \xi_k}{\partial q_l} - \delta_l^k \dot{\xi}_0 \right) A^{lr} + X^{(0)} \left(A_{sk} \right) A^{kr}. \quad (22)$$

5. Conformal invariance and Hojman conserved quantities

By the action of special infinitesimal transformations, when there exists the conformal factor which makes the first order equations have conformal invariance and Lie symmetry simultaneously, then the Hojman conserved quantities can be educed. Assume the special infinitesimal transformations with time invariance.

$$t^* = t, \qquad q_s^* \left(t^* \right) = q_s \left(t \right) + \varepsilon \xi_s \left(t, \boldsymbol{q} \right). \quad (23)$$

So by the special infinitesimal transformations with time invariance, the conformal factor is predigested in the following form

$$\ell_s^r = A_{sk} \frac{\partial \xi_k}{\partial q_l} A^{lr} + X^{(0)} \left(A_{sk} \right) A^{kr}. \quad (24)$$

Proposition 2 To the infinitesimal transformations, when there exists conformal factor Eq.(24), and the infinitesimal generators $\xi_s \left(t, \boldsymbol{q} \right)$ satisfy the following equations:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s - \frac{\partial \alpha_s}{\partial q_k} \xi_k = 0, \quad (25)$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \alpha_s \frac{\partial}{\partial q_s},$$

if we can find out some function $\mu = \mu \left(t, \boldsymbol{q} \right)$ which satisfies

$$\frac{\partial \alpha_s}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \quad (26)$$

so the Hojman conserved quantities can be induced from first order Lagrange systems

$$I_{\mathrm{H}} = \frac{1}{\mu} \frac{\partial}{\partial q_s} \left(\mu \xi_s \right) = \mathrm{const}. \quad (27)$$

The proof of Proposition 2 is evident; see also Ref.[1].

6. Illustrative example

There exists the Lagrangian of singular systems

$$L = \dot{q}_1 q_2 - \dot{q}_2 q_1 + a \left(q_1^2 + q_2^2 \right), \quad (28)$$

where a is a constant. So the differential equations are

$$\begin{aligned} F_s =& \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} \\ =& \begin{pmatrix} F_1 = 2\dot{q}_2 - 2aq_1 \\ F_2 = -2\dot{q}_1 - 2aq_2 \end{pmatrix} = 0. \end{aligned} \quad (29)$$

When the infinitesimal generators are

$$\begin{aligned} \xi_1 &= \left(q_1^2 + q_2^2 \right) \cos(at), \\ \xi_2 &= \left(q_1^2 + q_2^2 \right) \sin(at), \end{aligned} \quad (30)$$

so

$$\begin{aligned} X^{(1)} =& \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \left(\dot{\xi}_s - \dot{q}_s \xi_0 \right) \frac{\partial}{\partial \dot{q}_s} \\ =& \xi_s \frac{\partial}{\partial q_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s}. \end{aligned} \quad (31)$$

Furthermore, we have

$$
\begin{aligned}
X^{(1)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} =& \left(\xi_s \frac{\partial}{\partial q_s} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s}\right) \begin{pmatrix} F_1 = 2\dot{q}_2 - 2aq_1 \\ F_1 = -2\dot{q}_1 - 2aq_2 \end{pmatrix} \\
=& \begin{pmatrix} 2\dot{\xi}_2 - 2a\xi_1 \\ -2\dot{\xi}_2 - 2a\xi_2 \end{pmatrix} \\
=& \begin{pmatrix} 2\left[(2q_1\dot{q}_1 + 2q_2\dot{q}_2)\sin(at) + a\left(q_1^2 + q_2^2\right)\cos(at)\right] - 2a\left(q_1^2 + q_2^2\right)\cos(at) \\ -2\left[(2q_1\dot{q}_1 + 2q_2\dot{q}_2)\cos(at) - a\left(q_1^2 + q_2^2\right)\sin(at)\right] - 2a\left(q_1^2 + q_2^2\right)\sin(at) \end{pmatrix} \\
=& \begin{pmatrix} 2q_2\sin(at) & -2q_1\sin(at) \\ -2q_2\cos(at) & 2q_1\cos(at) \end{pmatrix} \begin{pmatrix} 2\dot{q}_2 - 2aq_1 \\ -2\dot{q}_1 - 2aq_2 \end{pmatrix}.
\end{aligned} \tag{32}
$$

Therefore, the conformal factor is

$$
\ell_s^r = \begin{pmatrix} 2q_2\sin(at) & -2q_1\sin(at) \\ -2q_2\cos(at) & 2q_1\cos(at) \end{pmatrix}. \tag{33}
$$

Utilizing Eq.(22) of Proposition 1, we can also obtain

$$
\begin{aligned}
\ell_s^r =& A_{sk}\left(\frac{\partial \xi_k}{\partial q_l} - \delta_l^k \dot{\xi}_0\right) A^{lr} + X^{(0)}\left(A_{sk}\right) A^{kr} \\
=& \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2q_1\cos(at) & 2q_2\cos(at) \\ 2q_1\sin(at) & 2q_2\sin(at) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
=& \begin{pmatrix} 2q_2\sin(at) & -2q_1\sin(at) \\ -2q_2\cos(at) & 2q_1\cos(at) \end{pmatrix}.
\end{aligned} \tag{34}
$$

By all appearance, the result is same to Eq.(33). Therefore the determining equations of conformal invariance of Eq.(29) by the action of special infinitesimal transformations are

$$
X^{(1)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 2q_2\sin(at) & -2q_1\sin(at) \\ -2q_2\cos(at) & 2q_1\cos(at) \end{pmatrix} \begin{pmatrix} 2\dot{q}_2 - 2aq_1 \\ -2\dot{q}_1 - 2aq_2 \end{pmatrix}. \tag{35}
$$

Here Eq.(29) by the action of Eq.(30) has conformal invariance and Lie symmetry simultaneously. Substituting Eq.(30) into Eqs.(25) and (26), we obtain

$$
\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0. \tag{36}
$$

When

$$
\mu = 1,
$$

consequently we obtain the Hojman conserved quantities

$$
I_{\mathrm{N}} = \frac{\partial \xi_1}{\partial q_1} + \frac{\partial \xi_2}{\partial q_2} = 2\left(q_2\cos(at) + q_1\sin(at)\right) = \text{const.} \tag{37}
$$

7. Conclusion

When first-order Lagrange systems have conformal invariance by the action of infinitesimal transformations, we can obtain the determining equations and conformal factor. The existence of conformal factor is the sufficient and necessary condition of conformal invariance and Lie symmetry simultaneously. Furthermore, when the conformal invariance satisfies some condition, we can also induce the corresponding conserved quantities.

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