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Weak Noether symmetry for nonholonomic systems of non-Chetaev type*

Xie Jia-Fang(解加芳), Gang Tie-Qiang(江铁强), and Mei Feng-Xiang(梅凤翔)†

Department of Mechanics, Faculty of Science, Beijing Institute of Technology, Beijing 100081, *China*

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Based on the weak Noether symmetry proposed by Mei F X, this paper discusses the weak Noether symmetry for nonholonomic system of non-Chetaev type, and presents expressions of three kinds of conserved quantities by weak Noether symmetry. Finally, the application of this new results is showed by a practical example.

Keywords: weak Noether symmetry, conserved quantity, nonholonomic systems of non-Chetaev type

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1. Introduction

Nonholonomic system dynamics, appeared about over one hundred years ago, has been an important branch in analytical mechanics. It is devoted to an exact description of actual problem in the dynamics of complicated systems. In 1894, Hertz first divided the constraints and the mechanical systems into two kinds: holonomic and nonholonomic.[1] Since then, many mathematicians and physicists, such as Appell, Volterra, Maggi, Chaplygin, have developed nonholonomic mechanics, and the dynamics of nonholonomic system has become a new field studied by many scientific researchers.[2−17]

The interplay between symmetries, conservation laws and variational principles for many kinds of constrained systems have been under consideration for a long time, starting from the celebrated work by Noether.[18] The research on the symmetries and conserved quantities of mechanical systems can help us to get some associated information about the dynamical problem. Our aim in this paper is to study weak Noether symmetry and three classes of conserved quantities (i.e. the Noether conserved quantity, Hojman conserved quantity, and a new-type conserved quantity) for nonholonomic systems of non-Chetaev type. Weak Noether symmetry is a new kind of symmetry proposed by Mei and Shui in 2006.[19] This work is a continuation and generalization of recently published pertinent works.

It is well known that Chetaev introduced a famous condition, i.e. Chetaev's condition.[20] In 1957 Novoselov presented an example of non-Chetaev type, and then established differential equations of motion for nonholonomic systems of non-Chetaev type.[21] It is a kind of nonholonomic system for which the restriction of nonholonomic constraints to virtual displacements does not follow Chetaev's condition. Nonholonomic systems in fact exhibit many interesting and important properties and there is some hope that it will give new conserved quantities via weak Noether symmetry. At the end of this paper we provide an example with two degrees of freedom to show some results.

2. Weak Noether symmetry for nonholonomic system of non-Chetaev type

2.1. Preliminaries

Let us consider a mechanical system determined by n generalized coordinates q_s $(s = 1, \ldots, n)$, and it is subjected to g ideal nonholonomic constraints of non-Chetaev type expressed by

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \quad (\beta = 1, \ldots, g), \tag{1}$$

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†Corresponding author. E-mail: meifx@bit.edu.cn

together with the restriction imposed on virtual displacements given by

$$f_{\beta s}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\delta q_s = 0, \tag{2}$$

in general, $f_{\beta s}$ and $\dfrac{\partial f_\beta}{\partial \dot{q}_s}$ have no relations. If

$$f_{\beta s} = \frac{\partial f_\beta}{\partial \dot{q}_s},$$

then the constraints of non-Chetaev type become Chetaev ones.

Here we take Routh's equations for nonholonomic systems of non-Chetaev type in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta f_{\beta s}, \quad (s = 1, \ldots, n), \tag{3}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian of system; $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are non-potential generalized forces and λ_β are constraint multipliers. In this case, Eq.(1) and Eq.(3) constitute a set of closed equations.

Supposing that the system is regular, from Eq.(3) we obtain

$$\ddot{q}_s = \ddot{q}_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda_\beta). \tag{4}$$

Notice that in order to eliminate λ_β in Eq.(4), one should take the derivative of Eq.(1) with respect to the time t:

$$\dot{f}_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0. \tag{5}$$

Substituting Eq.(4) into Eq.(5) we get the constraint multipliers $\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ as a function of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$, such that Eq.(4) will be

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),$$

and Eq.(3) can be rewritten as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s, \tag{6}$$

where

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})f_{\beta s}$$

is the explicit expression of invariants $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$, Eq.(6) is also called holonomic system corresponding to nonholonomic systems (1) and (3).

2.2. Weak Noether symmetry for the system

In this paper the conditions for dynamical symmetries will be based on infinitesimal transformations of time and coordinates

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \end{aligned} \tag{7}$$

where ε is an infinitesimal parameters, ξ_0, ξ_s are infinitesimal generators.

The basic idea of weak Noether symmetry had its origin in Noether symmetry because the second-derivative of the generalized coordinate $\ddot{q}_s$ in the Noether identity cannot be replaced by $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$. This brings the added difficulty to us of finding one more conserved quantity. In order to simplify the procedure to make the problem easier to solve, in 2006 Mei and Shui proposed the so-called weak Noether symmetry method and showed that applications of weak Noether symmetry compared with Noether symmetry are more extensive.

As is well known, Noether identity for corresponding holonomic system (6) is

$$\begin{aligned} &L\dot{\xi}_0 + \frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) \\ &+(Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \dot{G}_{\mathrm{N}} = 0. \end{aligned} \tag{8}$$

In this way, according to definition of weak Noether symmetry we therefore get the weak Noether identity of system (6) which is written in the form:

$$\begin{aligned} &L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right) \\ &+(Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}} = 0, \end{aligned} \tag{9}$$

where $\bar{\mathrm{d}}/\mathrm{d}t$ denotes

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}.$$

So we have the following definition.

Definition If the infinitesimal generators ξ_0, ξ_s, together with the gauge function G_{N}, hold Eq.(9), then the symmetry is called the weak Noether symmetry of corresponding holonomic system (6).

If ξ_0, ξ_s also, in addition to hold Eq.(9), satisfy the restriction equation

$$f_{\beta s}(\xi_s - \dot{q}_s\xi_0) = 0, \tag{10}$$

then the symmetry is called weak Noether symmetry of nonholonomic systems (1) and (3) of non-Chetaev type.

2.3. Example

In this section we try to give an example to illustrate the difference between Noether symmetry and weak Noether symmetry.

Here we choose a system of non-Chetaev type with two degrees of freedom as

$$\begin{aligned} &L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - q_1, \\ &f = \dot{q}_2 - t\dot{q}_1 = 0, \\ &\delta q_1 - \delta q_2 = 0. \end{aligned}$$

By making use of Eq.(3) it is easy to get

$$\ddot{q}_1 + 1 = \lambda, \quad \ddot{q}_2 = -\lambda.$$

Taking the derivative with respect to f we have

$$\lambda = \frac{t - \dot{q}_1}{1 + t}.$$

Therefore, we obtain

$$\ddot{q}_1 = \frac{t - \dot{q}_1}{1 + t} - 1, \quad \ddot{q}_2 = -\frac{t - \dot{q}_1}{1 + t}.$$

For this system, the weak Noether identity (9) reads as follows:

$$\begin{aligned} &L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \dot{q}_1\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 - \dot{q}_1\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right) \\ &+\dot{q}_2\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 - \dot{q}_2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right) \\ &-\xi_1 + \left(\frac{t - \dot{q}_1}{1 + t}\right)(\xi_1 - \dot{q}_1\xi_0) \\ &-\left(\frac{t - \dot{q}_1}{1 + t}\right)(\xi_2 - \dot{q}_2\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}} = 0. \end{aligned}$$

If we take the generators in the form:

$$\begin{aligned} &\xi_0 = 0, \quad \xi_1 = \xi_2 = 1, \\ &\xi_0 = 0, \quad \xi_1 = \xi_2 = \dot{q}_1 + \dot{q}_2 + t, \end{aligned}$$

it is easy to see that the two sets of generators are not only weakly Noether symmetric but also Noether symmetric. Nevertheless, for the following generators:

$$\xi_0 = 0, \quad \xi_1 = \xi_2 = (\dot{q}_1 + \dot{q}_2 + t)^2, \qquad \text{(a)}$$

according to Eq.(8), we get the Noether identity

$$\begin{aligned} &(\dot{q}_1 + \dot{q}_2)2(\dot{q}_1 + \dot{q}_2 + t)(\ddot{q}_1 + \ddot{q}_2 + 1) \\ &-(\dot{q}_1 + \dot{q}_2 + t)^2 + \dot{G}_{\mathrm{N}} = 0, \end{aligned}$$

and its Killing equations are

$$\begin{aligned} \frac{\partial G_{\mathrm{N}}}{\partial \dot{q}_1} &= -2(\dot{q}_1 + \dot{q}_2)(\dot{q}_1 + \dot{q}_2 + t), \\ \frac{\partial G_{\mathrm{N}}}{\partial \dot{q}_2} &= -2(\dot{q}_1 + \dot{q}_2)(\dot{q}_1 + \dot{q}_2 + t), \end{aligned}$$

$$\begin{aligned} &\frac{\partial G_{\mathrm{N}}}{\partial t} + \frac{\partial G_{\mathrm{N}}}{\partial q_1}\dot{q}_1 + \frac{\partial G_{\mathrm{N}}}{\partial q_2}\dot{q}_2 \\ &= (\dot{q}_1 + \dot{q}_2 + t)^2 - 2(\dot{q}_1 + \dot{q}_2)(\dot{q}_1 + \dot{q}_2 + t). \end{aligned}$$

This is an inconsistent system of equations, so from here we can see that the method of solving Killing equations to meet the requirement of Noether identity sometimes will not do. In other words, it has failed to look for the Noether conserved quantity. However, for the generators given in Eq.(a) we can find the weakly Noether conserved quantity. This is described in section 4.

In what follows we will present three kinds of conserved quantities, i.e. the Noether conserved quantity, Hojman conserved quantity and a new-type conserved quantity, for the aforementioned system. The results are embodied by a few propositions.

3. Three kinds of conserved quantities

3.1. Noether conserved quantity

Proposition 1 If the infinitesimal generators ξ_0, ξ_s are both satisfied with Eq.(9) and Eq.(10), then the weak Noether symmetry of the systems (1) and (3) can deduce the following Noether conserved quantity:

$$I_{\mathrm{N}} = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{N}} = \text{const.}, \qquad (11)$$

however, we call Eq.(11) the Noether conserved quantity of the corresponding holonomic system (6) if ξ_0, ξ_s only admits Eq.(9) and does not admit Eq.(10).

3.2. Hojman conserved quantity

In a similar way, we can arrive at Hojman conserved quantity.

For the transformations (7), taking $\xi_0 = 0$, implying that time t is unchangeable, we get the determining equation of Lie symmetry

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \qquad (12)$$

In this way, the weak Noether identity (9) will become

$$\begin{aligned} &\frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + (Q_s + \Lambda_s)\xi_s + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}} \\ &= 0. \end{aligned} \qquad (13)$$

As a consequence, we obtain another proposition.

Proposition 2 If the infinitesimal generator ξ_s satisfies Eqs.(12) and (13), and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{14}$$

then Hojman conserved quantity of corresponding holonomic system (6) is

$$\begin{aligned} I_{\mathrm{H}} &= \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu \xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ &= \text{const.}. \end{aligned} \tag{15}$$

3.3. New-type conserved quantity

It is familiar for us that the criterion equation of form invariance for corresponding holonomic system (6) is

$$E_s\{X^{(1)}(L)\} = X^{(1)}(Q_s + \Lambda_s), \tag{16}$$

where

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s},$$

and thereby we obtain Proposition 3.

Proposition 3 If the generators ξ_0, ξ_s of weak Noether symmetry hold Eq.(16) and a gauge function $G_{\mathrm{F}} = G_{\mathrm{F}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ exists, which holds structure equation

$$\begin{aligned} &\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \tilde{X}^{(1)}\{\tilde{X}^{(1)}(L)\} \\ &+\tilde{X}^{(1)}(Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{F}} \\ &= 0, \end{aligned} \tag{17}$$

then new-type conserved quantity deduced by weak Noether symmetry for corresponding holonomic system (6) is

$$\begin{aligned} I_{\mathrm{F}} &= \tilde{X}^{(1)}(L)\xi_0 + \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{F}} \\ &= \text{const.}, \end{aligned} \tag{18}$$

where

$$\tilde{X}^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}.$$

Now we have obtained three kinds of conserved quantities by weak Noether symmetry.

To illustrate the above propositions, with that, in next section, we will show how to utilize weak Noether symmetry to find out conservation laws.

4. Example

Let us consider the example in the section 2.3. For Eq.(a), one can find the gauge function by Eq.(9):

$$G_{\mathrm{N}} = t(\dot{q}_1 + \dot{q}_2 + t)^2,$$

in this case one get Noether conserved quantity by Eq.(11) in the form

$$\begin{aligned} I_{\mathrm{N}} &= \dot{q}_1(\dot{q}_1 + \dot{q}_2 + t)^2 + \dot{q}_2(\dot{q}_1 + \dot{q}_2 + t)^2 \\ &\quad +t(\dot{q}_1 + \dot{q}_2 + t)^2 \\ &= (\dot{q}_1 + \dot{q}_2 + t)^3 \\ &= \text{const.}. \end{aligned}$$

Besides solution (a), we can still give a set of solutions to Eq.(9),

$$\begin{aligned} &\xi_0 = 0, \\ &\xi_1 = \xi_2 = q_1 + q_2 + \frac{1}{2}t^2 - (\dot{q}_1 + \dot{q}_2 + t)t, \\ &G_{\mathrm{N}} = t\left\{q_1 + q_2 + \frac{1}{2}t^2 - (\dot{q}_1 + \dot{q}_2 + t)t\right\}. \end{aligned} \tag{b}$$

In the same way, we obtain

$$\begin{aligned} I_{\mathrm{N}} &= (\dot{q}_1 + \dot{q}_2 + t)\left\{q_1 + q_2 + \frac{1}{2}t^2 - (\dot{q}_1 + \dot{q}_2 + t)t\right\} \\ &= \text{const.}. \end{aligned}$$

In order to show that solutions (a) and (b) are also Lie symmetric, by determining equations (12) of Lie symmetry, we achieve

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 &= -\frac{1}{1+t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 &= -\frac{1}{1+t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2. \end{aligned} \tag{19}$$

Obviously, generators (a) and (b) are both satisfied with Eq.(19), it is to say that Eqs.(a) and (b) are Lie symmetric. Equation (14) gives

$$-\frac{1}{1+t} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln \mu = 0,$$

from which it follows

$$\mu = 1 + t,$$

therefore, for generators (a) by Eq.(15) we have the Hojman conserved quantity

$$I_{\mathrm{H}} = \frac{\partial \xi_1}{\partial q_1} + \frac{\partial \xi_2}{\partial q_2} = 0.$$

Similarly, for generators (b) we have

$$I_{\mathrm{H}} = 2.$$

An analogous analysis can be performed for finding new-type conserved quantity, in this case as long as generator (a) is concerned, we easily obtain

$$\begin{aligned}&\tilde{X}^{(1)}(L) = -(\dot{q}_1 + \dot{q}_2 + t)^2,\\ &E_s\{\tilde{X}^{(1)}(L)\} = 0,\\ &\tilde{X}^{(1)}\{\tilde{X}^{(1)}(L)\} = 0,\end{aligned} \tag{20}$$

so Eq.(a) is form invariant. Substituting Eq.(20) into Eq.(17) we obtain

$$G_{\mathrm{F}} = 0,$$

it then follows that the new-type conserved quantity from Eq.(18) is given by

$$\begin{aligned}I_{\mathrm{F}} \;=\; & -2(\dot{q}_1 + \dot{q}_2 + t)(\dot{q}_1 + \dot{q}_2 + t)^2\\ & -2(\dot{q}_1 + \dot{q}_2 + t)(\dot{q}_1 + \dot{q}_2 + t)^2\\ =\; & -4(\dot{q}_1 + \dot{q}_2 + t)^3 = \text{const}..\end{aligned}$$

But, in the case of generators (b) we cannot find new-type conserved quantity. As a matter of fact,

$$\begin{aligned}&\tilde{X}^{(1)}(L) = (\dot{q}_1 + \dot{q}_2 + t)t - \left(q_1 + q_2 + \frac{1}{2}t^2\right),\\ &E_s\{\tilde{X}^{(1)}(L)\} = 4,\end{aligned}$$

$$\begin{aligned}&\tilde{X}^{(1)}\{\tilde{X}^{(1)}(L)\}\\ &= 2\left(\dot{q}_1 + \dot{q}_2 + t\right)t - 2\left(q_1 + q_2 + \frac{1}{2}t^2\right),\end{aligned}$$

and

$$\tilde{X}^{(1)}(Q_s + \Lambda_s) = 0;$$

this shows that Eq.(16) cannot be established, this means the generators (b) is not form invariant. Consequently, we cannot present the new-type conserved quantity via Eq.(b).

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