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Conformal invariance and conserved quantity of Hamilton systems*

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This paper studies conformal invariance and conserved quantities of Hamilton system. The definition and the determining equation of conformal invariance for Hamilton system are provided. The relationship between the conformal invariance and the Lie symmetry are discussed, and the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of the system under the infinitesimal one-parameter transformation group is deduced. It gives the conserved quantities of the system and an example for illustration.

Keywords: Hamilton system, conformal invariance, determining equation, conserved quantity
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1. Introduction

The dynamics of Hamilton systems has been extensively applied to the field of modern mathematics, mechanics and physics.[1,2] The symmetry method is a modern method for finding conserved quantity of dynamical systems.[3] The symmetry and conserved quantity of dynamic systems have been paid attention these years. The symmetry and the conserved quantity for a dynamical system play an important role in the fields of modern science and technology. There are three main symmetry methods used to seek the conserved quantities of the dynamical system: Noether symmetry,[3,4] Lie symmetry[5−7] and Mei's form invariance.[8−12] We can obtain the Noether conserved quantity, the Hojman conserved quantity and a new type of conserved quantity, respectively.[13,14] In recent years, more and more attention has been paid to research into symmetry and the conserved quantity of Hamilton systems, and some important results have been gained.[15−24] Reference [25] studied conformal invariance, or conformal symmetry, of Birkhoff system under special infinitesimal transformations. This paper studies the conformal invariance and conserved quantity of Hamilton system under general infinitesimal transformations, and gives an example to illustrate the application of the result.

2. Conformal invariance of Hamilton systems

Here we study a mechanical system of N particles, and its configuration is determined by n generalized coordinates $q_s(s=1,\ldots,n)$. The Hamilton canonical equations of the system can be written as

$$\dot{q}_s = \frac{\partial H}{\partial p_s}\,,\ \dot{p}_s = -\frac{\partial H}{\partial q_s}\,,\ (s=1,\ldots,n), \tag{1}$$

where q_s and p_s are the canonical variables. Equation (1) can be expressed as follows

$$\dot{q}_s = g_s(t,\boldsymbol{q},\boldsymbol{p})\,,\ \dot{p}_s = h_s(t,\boldsymbol{q},\boldsymbol{p})\,,\quad (s=1,\ldots,n), \tag{2}$$

or

$$\boldsymbol{F}_s \equiv \begin{bmatrix} F_s^q \\ F_s^p \end{bmatrix} = \begin{bmatrix} \dot{q}_s - g_s(t,\boldsymbol{q},\boldsymbol{p}) \\ \dot{p}_s - h_s(t,\boldsymbol{q},\boldsymbol{p}) \end{bmatrix} = 0\,,\quad (s=1,\ldots,n). \tag{3}$$

The equations of motion of the nonsingular Lagrange system can be written in the canonical forms under the Legendre transformation. All of them are the Lagrange system, i.e. the holonomic conservative

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system, the holonomic system in which the generalized forces have generalized potential, and the systems of inverse problem of the Lagrange mechanics, so their equations of motion can always be written in the Hamilton canonical form of Eq.(3).

We introduce a single-parameter infinitesimal transformations group for time t, generalized coordinates q_s and generalized momentums p_s

$$t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s,$$
$$p_s^*(t^*) = p_s(t) + \Delta p_s. \tag{4}$$

Their expanding forms are

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}),$$
$$p_s^*(t^*) = p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \tag{5}$$

where ξ_0, ξ_s and η_s are called the infinitesimal generators of t, q_s and p_s, respectively, under the infinitesimal transformations (4), and ε is a small parameter.

We introduce an infinitesimal transformation vector of generator

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}, \tag{6}$$

and its first extension is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s} + (\dot{\eta}_s - \dot{p}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{p}_s}. \tag{7}$$

Definition If there is a nonsingular matrix $\boldsymbol{H}_s^k$ satisfying the following condition

$$X^{(1)}(\boldsymbol{F}_s) = \boldsymbol{H}_s^k(\boldsymbol{F}_k), \tag{8}$$

then this kind of invariance is called the conformal invariance of the Hamilton equations (3) under the single-parameter infinitesimal transformations (5).

Equation (8) is the determining equations of conformal invariance of the Hamilton system (3), and $\boldsymbol{H}_s^k$ is called conformal factor. If the generators ξ_0, ξ_s and η_s of the infinitesimal transformations (5) satisfy the determining equation (8), then the transformations are called conformal transformations.

3. Conformal invariance and Lie symmetry

Theorem 1 For the Hamilton system (3), if the generators ξ_0, ξ_s and η_s of the infinitesimal transformations (5) satisfy the determining equation (8), and there is a nonsingular matrix $\boldsymbol{\beta}_s^k$ satisfying the following condition

$$X^{(1)}(\boldsymbol{F}_s) - X^{(1)}(\boldsymbol{F}_s)\mid_{F_s=0} = \boldsymbol{\beta}_s^k(\boldsymbol{F}_k),$$
$$(s, k = 1, \ldots, n), \tag{9}$$

then the necessary and sufficient condition that the conformal invariance would be Lie symmetry of system is

$$\boldsymbol{H}_s^k = \boldsymbol{\beta}_s^k. \tag{10}$$

Proof In fact, if there is a nonsingular matrix $\boldsymbol{\beta}_s^k$ satisfying equation (9), and the Lie symmetry of Hamilton equations (3) satisfies

$$X^{(1)}(\boldsymbol{F}_s)\mid_{F_s=0} = 0, \tag{11}$$

then equation (9) becomes

$$X^{(1)}(\boldsymbol{F}_s) = \boldsymbol{\beta}_s^k(\boldsymbol{F}_k). \tag{12}$$

According to the definition (8), the conformal factor of the system is $\boldsymbol{H}_s^k = \boldsymbol{\beta}_s^k$.

Vice versa, according to the definition (8) and the equation (9), we can easily verify that there is a relation as follows

$$(\boldsymbol{H}_s^k - \boldsymbol{\beta}_s^k)(\boldsymbol{F}_k) = X^{(1)}(\boldsymbol{F}_s)\mid_{F_s=0},$$
$$(s, k = 1, \ldots, n). \tag{13}$$

If $\boldsymbol{H}_s^k = \boldsymbol{\beta}_s^k$, then we have

$$X^{(1)}(\boldsymbol{F}_s)\mid_{F_s=0} = 0.$$

The system is Lie symmetrical.

4. Conformal factor of conformal invariance of Hamilton systems

In fact, taking

$$X^{(1)}(\boldsymbol{F}_s) - X^{(1)}(\boldsymbol{F}_s)\mid_{F_s=0}, \tag{14}$$

we can obtain the conformal factor of conformal invariance of Hamilton system.

Using

$$\dot{\xi}_s = \frac{\partial\xi_s}{\partial t} + \frac{\partial\xi_s}{\partial q_k}\dot{q}_k + \frac{\partial\xi_s}{\partial p_k}\dot{p}_k,$$
$$\dot{\eta}_s = \frac{\partial\eta_s}{\partial t} + \frac{\partial\eta_s}{\partial q_k}\dot{q}_k + \frac{\partial\eta_s}{\partial p_k}\dot{p}_k,$$
$$(s, k = 1, \ldots, n), \tag{15}$$

we have

$$\begin{aligned} & X^{(1)}(F_s^q) \\ =\ & \dot{\xi}_s - \dot{q}_s\dot{\xi}_0 - \frac{\partial g_s}{\partial t}\xi_0 - \frac{\partial g_s}{\partial q_k}\xi_k - \frac{\partial g_s}{\partial p_k}\eta_k \\ =\ & \frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_k}\dot{q}_k + \frac{\partial \xi_s}{\partial p_k}\dot{p}_k - \dot{q}_s\dot{\xi}_0 \\ & -\frac{\partial g_s}{\partial t}\xi_0 - \frac{\partial g_s}{\partial q_k}\xi_k - \frac{\partial g_s}{\partial p_k}\eta_k, \end{aligned} \tag{16}$$

$$\begin{aligned} & X^{(1)}(F_s^p) \\ =\ & \dot{\eta}_s - \dot{p}_s\dot{\xi}_0 - \frac{\partial h_s}{\partial t}\xi_0 - \frac{\partial h_s}{\partial q_k}\xi_k - \frac{\partial h_s}{\partial p_k}\eta_k \\ =\ & \frac{\partial \eta_s}{\partial t} + \frac{\partial \eta_s}{\partial q_k}\dot{q}_k + \frac{\partial \eta_s}{\partial p_k}\dot{p}_k - \dot{p}_s\dot{\xi}_0 - \frac{\partial h_s}{\partial t}\xi_0 \\ & -\frac{\partial h_s}{\partial q_k}\xi_k - \frac{\partial h_s}{\partial p_k}\eta_k, \end{aligned} \tag{17}$$

$$\begin{aligned} & X^{(1)}(F_s^q)|_{F_s^q=0} \\ =\ & \frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_k}g_k + \frac{\partial \xi_s}{\partial p_k}\dot{p}_k - g_s\dot{\xi}_0 - \frac{\partial g_s}{\partial t}\xi_0 \\ & -\frac{\partial g_s}{\partial q_k}\xi_k - \frac{\partial g_s}{\partial p_k}\eta_k, \end{aligned} \tag{18}$$

$$\begin{aligned} & X^{(1)}(F_s^p)|_{F_s^p=0} \\ =\ & \frac{\partial \eta_s}{\partial t} + \frac{\partial \eta_s}{\partial q_k}\dot{q}_k + \frac{\partial \eta_s}{\partial p_k}h_k - h_s\dot{\xi}_0 - \frac{\partial h_s}{\partial t}\xi_0 \\ & -\frac{\partial h_s}{\partial q_k}\xi_k - \frac{\partial h_s}{\partial p_k}\eta_k. \end{aligned} \tag{19}$$

From above equations (16)–(19), we obtain

$$\begin{aligned} & X^{(1)}(F_s^q) - X^{(1)}(F_s^q)\mid_{F_s^q=0} \\ =\ & \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k\dot{\xi}_0\right)(\dot{q}_k - g_k) \\ =\ & \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k\dot{\xi}_0\right)(F_k^q) \\ =\ & Q_s^k(F_k^q), \\ & X^{(1)}(F_s^p) - X^{(1)}(F_s^p)\mid_{F_s^p=0} \\ =\ & \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k\dot{\xi}_0\right)(\dot{p}_k - h_k) \\ =\ & \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k\dot{\xi}_0\right)(F_k^p) \\ =\ & P_s^k(F_k^p), \\ & (s,k=1,\ldots,n), \end{aligned} \tag{20}$$

or

$$\begin{aligned} & X^{(1)}(\boldsymbol{F}_s) - X^{(1)}(\boldsymbol{F}_s)\mid_{F_s=0} = \boldsymbol{\beta}_s^k(\boldsymbol{F}_k), \\ & (s,k=1,\ldots,n), \end{aligned} \tag{21}$$

where

$$\boldsymbol{F}_s = \begin{pmatrix} F_s^q \\ F_s^p \end{pmatrix}, \quad \boldsymbol{\beta}_s^k = \begin{pmatrix} Q_s^k & 0 \\ 0 & P_s^k \end{pmatrix},$$

$$Q_s^k = \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k\dot{\xi}_0\right), \quad P_s^k = \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k\dot{\xi}_0\right).$$

Furthermore, we can find the conformal factor that the conformal invariance would be Lie symmetry of Hamilton system

$$\boldsymbol{H}_s^k = \begin{pmatrix} Q_s^k & 0 \\ 0 & P_s^k \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} Q_s^k = \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k\dot{\xi}_0\right),\ P_s^k = \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k\dot{\xi}_0\right), \\ (s,k=1,\ldots,n). \end{aligned} \tag{23}$$

5. Conformal invariance and conserved quantity of Hamilton systems

The conformal invariance can also lead to a conserved quantity under certain conditions.

Theorem 2 For the Hamilton system (3), if the generators ξ_0, ξ_s and η_s of the infinitesimal transformations (5) satisfy the conformal factor (22) and (23), and there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following Lie symmetrical structure equation

$$-H\dot{\xi}_0 + \frac{\partial H}{\partial p_s}\eta_s + p_s\dot{\xi}_s - X^{(0)}(H) + \dot{G} = 0, \tag{24}$$

then the conformal invariance of Hamilton system (3) possesses the following conserved quantity

$$I = -H\xi_0 + p_s\xi_s + G = \text{const.} \tag{25}$$

Proof

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} =\ & -\dot{H}\xi_0 - H\dot{\xi}_0 + \dot{p}_s\xi_s + p_s\dot{\xi}_s + H\dot{\xi}_0 - \frac{\partial H}{\partial p_s}\eta_s \\ & -p_s\dot{\xi}_s + X^{(0)}(H) \\ =\ & \frac{\partial H}{\partial q_s}(\xi_s - \dot{q}_s\xi_0) + \dot{p}_s\left(\xi_s - \frac{\partial H}{\partial p_s}\xi_0\right) \\ =\ & 0. \end{aligned} \tag{26}$$

6. An illustrative example

Try to study the conformal invariance and conserved quantity of planar Kepler problem. The Hamiltonian of the planar motion of Kepler problem is

$$\begin{aligned} H =\ & \frac{1}{2m}(p_1^2 + p_2^2) - (q_1^2 + q_2^2)^{-1/2}, \\ & (m \ \text{ is a constant}), \end{aligned} \tag{27}$$

and the equations of motion are

$$\begin{aligned}
\dot{q}_1 &= \frac{p_1}{m}, \\
\dot{q}_2 &= \frac{p_2}{m}, \\
\dot{p}_1 &= -q_1(q_1^2+q_2^2)^{-3/2}, \\
\dot{p}_2 &= -q_2(q_1^2+q_2^2)^{-3/2}.
\end{aligned} \tag{28}$$

Taking

$$\begin{aligned}
&\xi_0 = 0, \quad \xi_1 = -q_2, \quad \xi_2 = q_1, \\
&\eta_1 = -p_2, \quad \eta_2 = -p_1,
\end{aligned} \tag{29}$$

we have

$$\begin{aligned}
X^{(1)} = {} & \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s} \\
& + (\dot{\eta}_s - \dot{p}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{p}_s} \\
= {} & \left(\xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s} + \dot{\xi}_s\frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s\frac{\partial}{\partial \dot{p}_s}\right),
\end{aligned} \tag{30}$$

$$\begin{aligned}
& X^{(1)}\begin{pmatrix} F_1^q \\ F_2^q \end{pmatrix} \\
&= \left(\xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s} + \dot{\xi}_s\frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s\frac{\partial}{\partial \dot{p}_s}\right)\begin{pmatrix} \dot{q}_1 - \dfrac{p_1}{m} \\ \dot{q}_2 - \dfrac{p_2}{m} \end{pmatrix} \\
&= \begin{pmatrix} -(\eta_1/m) + \dot{\xi}_1 \\ -(\eta_2/m) + \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} (p_2/m) - \dot{q}_2 \\ -(p_1/m) + \dot{q}_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \dot{q}_1 - \dfrac{p_1}{m} \\ \dot{q}_2 - \dfrac{p_2}{m} \end{pmatrix},
\end{aligned} \tag{31}$$

$$\begin{aligned}
X^{(1)}\begin{pmatrix} F_1^p \\ F_2^p \end{pmatrix} &= \left(\xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s} + \dot{\xi}_s\frac{\partial}{\partial \dot{q}_s} + \dot{\eta}_s\frac{\partial}{\partial \dot{p}_s}\right)\begin{pmatrix} \dot{p}_1 + q_1(q_1^2+q_2^2)^{-3/2} \\ \dot{p}_2 + q_2(q_1^2+q_2^2)^{-3/2} \end{pmatrix} \\
&= \begin{pmatrix} \dot{\eta}_1 + \xi_1\left[(q_1^2+q_2^2)^{-3/2} - 3q_1^2(q_1^2+q_2^2)^{-5/2}\right] - 3\xi_2 q_1 q_2(q_1^2+q_2^2)^{-5/2} \\ \dot{\eta}_2 - 3\xi_1 q_1 q_2(q_1^2+q_2^2)^{-5/2} + \xi_2\left[(q_1^2+q_2^2)^{-3/2} - 3q_2^2(q_1^2+q_2^2)^{-5/2}\right] \end{pmatrix} \\
&= \begin{pmatrix} -\dot{p}_2 - q_2(q_1^2+q_2^2)^{-3/2} \\ \dot{p}_1 + q_1(q_1^2+q_2^2)^{-3/2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \dot{p}_1 + q_1(q_1^2+q_2^2)^{-3/2} \\ \dot{p}_2 + q_2(q_1^2+q_2^2)^{-3/2} \end{pmatrix}.
\end{aligned} \tag{32}$$

Conformal factor is

$$\boldsymbol{Q}_s^k = \boldsymbol{P}_s^k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

From Eq.(23), we can also obtain conformal factor

$$Q_s^k = \left(\frac{\partial \xi_s}{\partial q_k} - \delta_s^k\dot{\xi}_0\right) = \frac{\partial \xi_s}{\partial q_k}, \tag{34}$$

$$P_s^k = \left(\frac{\partial \eta_s}{\partial p_k} - \delta_s^k\dot{\xi}_0\right) = \frac{\partial \eta_s}{\partial p_k}. \tag{35}$$

The determining equations of conformal invariance is

$$X^{(1)}\begin{pmatrix} F_s^q \\ F_s^p \end{pmatrix} = \begin{pmatrix} Q_s^k & 0 \\ 0 & P_s^k \end{pmatrix}\begin{pmatrix} F_k^q \\ F_k^p \end{pmatrix}. \tag{36}$$

Obviously, the conformal invariance would be also Lie symmetry of the system. Substituting equation (29) into Lie symmetrical structure equation (24), we obtain

$$\dot{G} = 0. \tag{37}$$

Taking $G = 0$, the conserved quantity (25) gives

$$I = q_1p_2 - q_2p_1 = \text{const.} \tag{38}$$

7. Conclusion

For a Hamilton system, by using the definition of conformal invariance and the Lie symmetrical determining equations, we can obtain the conformal factor of determining equations of conformal invariance. The conformal factor is also the necessary and sufficient condition that the conformal invariance would be Lie symmetry of the system. The conformal invariance can also lead to a conserved quantity under certain condition.

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