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Quantum entanglement in the mode-mode competition system*

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Considering a two-level atom interacting with the competing two-mode field, this paper investigates the entanglement between the two-level atom and the two-mode field by using the quantum reduced entropy, and that between the two-mode field by using the quantum relative entropy of entanglement. It shows that the two kinds of entanglement are dependent on the relative coupling strength of atom-field and the atomic distribution, and exhibit the periodical evolution. The maximal atom–field entanglement state can be prepared via the appropriate selection of system parameters and interaction time.

Keywords: mode-mode competition system, entanglement, reduced entropy, relative entropy
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1. Introduction

It is well known that entanglement plays an important role in quantum information theory. Quantum entanglement (QE) provides strong tests of quantum nonlocality,[1] and is also an useful resource for various kinds of quantum information processing, including teleportation,[2,3] cryptographic key distribution,[4] quantum error correction[5] and quantum computation.[6] For future applications like quantum networks or the quantum repeater[7] it is mandatory to achieve entanglement between separated quantum processors like atoms and photons which forms the interface between atomic quantum memories and photonic quantum communication channels.[8,9]. Therefore, it is of great importance to prepare quantum entangled states[10−16] in well-controlled experimental conditions and to quantify the entanglement to assess the efficacy of quantum information processing. If we want to quantify entangled states, we should know whether they are pure states or mixed states. That is, if the entangled states are pure states, then it is sufficient to use von Neumann entropy[17] for the reduced states.[18,19] For mixed states, the relative entropy of entanglement is a good measure, which tells us the amount of entanglement in the state with its distance from the disentangled set of states. In statistical terms, the more entangled a state is, the more it is distinguishable from a disentangled state.[20] The degree of entanglement for mixed states in various generalized Jaynes–Cummings models has been studied by the quantum relative entropy of entanglement.[21−32] On the other hand, it is worthy of noticing that in such models one assumes that a given mode acts only on one optical transition, i.e., the two modes do not act simultaneously on one transition. Semiclassically one knows that if an atom interacts with different fields, several nonlinear processes can take place at the same time, and a large body of literature has been devoted to the interaction of a bichromatic field with a two-level atom. The bichromatic fields lead to many new dynamical aspects as a result of mode-mode competition,[33−35] such as Stokes and anti-Stokes processes.[36] In Ref.[37], Iyotsna and Agarwal showed that the quantum effects were rather dominant in the mode-mode competition system. The authors have studied the atomic dynamical properties and the statistic properties of the photon in such system.[38−41] However, less attention has been paid to the study of the QE in the mode–mode competition system. In this paper, we shall investigate the entanglement between the two-level atom and the two-mode field by using the quantum reduced entropy, and that between the competing two-mode field by using the quantum relative entropy of entanglement, and discuss the influences of the relative competing strength

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of atom-field and the atomic distribution on the two kinds of entanglement.

2. Model and the reduced density matrix of the field and the atom

We consider the interaction between a two-level atom with states $|+\rangle$ and $|-\rangle$ separated by the frequency ω_0, and a two-mode field with frequencies ω_a and ω_b. We assume that one of the modes, a mode, interacts with the atom through a three-photon process and the other mode, b mode, interacts via a one-photon process. For simplicity, we further assume that two modes are in perfect resonance with the atomic transition, i.e. $\omega_0 = 3\omega_a = \omega_b$. The interaction Hamiltonian of the system under consideration in this paper in the rotating-wave approximation and in the interaction picture can be written as[37]

$$H_{\rm I} = \lambda(S_+ a^3 + S_- a^{\dagger 3}) + {\rm g}(S_+ b + S_- b^\dagger), \qquad (1)$$

where a, $a^\dagger$ and b, $b^\dagger$ are the annihilation and creation operators of the field modes a and b, $S_\pm$ are the atomic pseudospin operators, λ and g are the coupling constants for the three-photon process and for the single-photon process, respectively.

We consider at time $t = 0$, the two-level atom is in a coherent superposition state of the excited state $|+\rangle$ and the ground state $|-\rangle$, the a mode field is in a Fock state $|3\rangle$ with just three photons which is the minimum number of photons required for the three-photon absorption, and the b mode field is in the vacuum state $|0\rangle$. Thus the initial state of the system is given by

$$\begin{aligned} &|\Psi_{fa}(0)\rangle \\ &= \cos\left(\frac{\theta}{2}\right)|3,0,+\rangle + {\rm e}^{-{\rm i}\varphi}\sin\left(\frac{\theta}{2}\right)|3,0,-\rangle, \end{aligned} \qquad (2)$$

where $0 \le \theta \le \pi$ denotes the atomic distribution, $0 \le \varphi \le 2\pi$ is the phase of the atom dipole.

The state vector at any time t can be expressed in terms of the complete set of the possible states in interaction picture as

$$|\Psi_{fa}(t)\rangle = |D\rangle|+\rangle + |E\rangle|-\rangle, \qquad (3)$$

with

$$\begin{aligned} |D\rangle &= B_1(t)|3,0\rangle + B_2(t)|0,1\rangle + B_3(t)|0,0\rangle, \\ |E\rangle &= A_0(t)|6,0\rangle + A_1(t)|3,1\rangle + A_2(t)|3,0\rangle \\ &\quad + A_3(t)|0,1\rangle + A_4(t)|0,2\rangle, \end{aligned} \qquad (4)$$

where the complex amplitudes are derived from the Schrödinger equation in the interaction picture

$${\rm i}\frac{\partial|\Psi_{fa}(t)\rangle}{\partial t} = H_{\rm I}|\Psi_{fa}(t)\rangle. \qquad (5)$$

At the time $t = 0$, $A_2(0) = {\rm e}^{-{\rm i}\varphi}\sin(\theta/2)$, $B_1(0) = \cos(\theta/2)$, and the rest of the coefficients are zero.

At any time $t > 0$ the density operator for the system is given by

$$\begin{aligned} \rho_{fa}(t) &= |\Psi_{fa}(t)\rangle\langle\Psi_{fa}(t)| \\ &= \begin{pmatrix} |D\rangle\langle D| & |E\rangle\langle D| \\ |D\rangle\langle E| & |E\rangle\langle E| \end{pmatrix}. \end{aligned} \qquad (6)$$

Tracing $\rho_{fa}(t)$ over the atom gives rise to the reduced field density operator for the system

$$\rho_f(t) = {\rm Tr}_a(\rho_{fa}(t)) = |D\rangle\langle D| + |E\rangle\langle E|. \qquad (7)$$

The eigenvalues and eigenstates of the reduced field density operator can be written as

$$\begin{aligned} \lambda_f^\pm(t) &= \langle D|D\rangle \pm {\rm e}^{\mp\alpha}|\langle D|E\rangle| \\ &= \langle E|E\rangle \pm {\rm e}^{\pm\alpha}|\langle D|E\rangle|, \end{aligned} \qquad (8)$$

$$\begin{aligned} |\psi_f^\pm(t)\rangle &= \frac{1}{\sqrt{2\lambda_f^\pm(t)\cosh(\alpha)}} \\ &\quad \times\{{\rm e}^{\frac{1}{2}({\rm i}\beta\pm\alpha)}|D\rangle \pm {\rm e}^{-\frac{1}{2}({\rm i}\beta\pm\alpha)}|E\rangle\}, \end{aligned} \qquad (9)$$

where

$$\begin{aligned} \alpha &= \sinh^{-1}\left(\frac{\langle D|D\rangle - \langle E|E\rangle}{2|\langle D|E\rangle|}\right), \\ \beta &= \tan^{-1}\left(\frac{{\rm Im}(\langle D|E\rangle)}{{\rm Re}(\langle D|E\rangle)}\right). \end{aligned} \qquad (10)$$

At the same time, tracing $\rho_{fa}(t)$ over the two-mode field gives rise to the reduced atom density operator for the system

$$\rho_a(t) = {\rm Tr}_f(\rho_{fa}(t)) = \begin{pmatrix} \langle D|D\rangle & \langle E|D\rangle \\ \langle D|E\rangle & \langle E|E\rangle \end{pmatrix}. \qquad (11)$$

We can prove that the eigenvalues of the reduced atom density operator have the same expression as the reduced field density operator, i.e., $\lambda_a^\pm = \lambda_f^\pm = \lambda^\pm$, and the corresponding eigenstates of the reduced atom density operator can be written as

$$\begin{aligned} |\psi_a^\pm(t)\rangle &= \frac{1}{\sqrt{2\lambda^\pm(t)\cosh(\alpha)}} \\ &\quad \times\{{\rm e}^{\frac{1}{2}({\rm i}\beta\pm\alpha)}|+\rangle \pm {\rm e}^{-\frac{1}{2}({\rm i}\beta\pm\alpha)}|-\rangle\}. \end{aligned} \qquad (12)$$

According to the Schmidt decomposition, the state of system can be expressed as

$$|\Psi_{fa}(t)\rangle = \sqrt{\lambda^+}|\psi_f^+\rangle|\psi_a^+\rangle + \sqrt{\lambda^-}|\psi_f^-\rangle|\psi_a^-\rangle. \quad (13)$$

In what follows, we will discuss the entanglement properties of atom-field and the two-mode field by using the above results.

3. Entanglement between the two-level atom and the competing two-mode field

Since we have assumed that the two-level atom and the competing two-mode field are initially in a disentangled pure state, the quantum entanglement of atom-field can be measured by the reduced entropy of the atom or the field,

$$\begin{aligned} S(\rho_j) &= -\mathrm{Tr}_j\{\rho_j \log_2 \rho_j\}, \quad (j = a \text{ or } f) \\ &= -\lambda^+ \log_2 \lambda^+ - \lambda^- \log_2 \lambda^-. \end{aligned} \quad (14)$$

It reflects the degree of the entanglement between the atom and the field (DEAF): if it takes its minimum value to be zero, the atom and the field are disentangled; if it takes its maximum value to be 1, the atom and the field are maximally entangled.

We numerically calculate the entanglement of atom-field, and illustrate it according to Eq.(14) in Figs.1(a)–(c) for different λ/g with $\varphi = 0$, $\theta = \pi$, and in Figs.2(a)–(c) for different θ with $\lambda/g = 1$, $\varphi = 0$. It is found that λ/g and θ have strong influence on the DEAF.

Firstly, we discuss the influence of the relative coupling strength between the atom and the field (λ/g) on the DEAF when the atom is initially in the ground state $(\theta = \pi)$, as shown in Figs.1(a)–(c). In the case of $\lambda/g = 0.6$, $S(\rho_a)$ evolves with a period of $\tau = \sqrt{\pi}$ (Fig.1(a)). At the time $gt = n\tau$ $(n = 0, 1, 2, \ldots)$, $S(\rho_a)$ takes its minimum value, which means that the atom is disentangled from the field. This point can be quantitatively interpreted by using Eqs.(8) and (13). After a straightforward calculation, we can obtain $\lambda^+ = 1$ and $\lambda^- = 0$ in this case. The system state can be expressed in the form

$$|\Psi_{fa}(gt = n\tau)\rangle = |\psi_f^+\rangle \otimes |\psi_a^+\rangle. \quad (15)$$

Obviously, this is a disentangled direct product state. Therefore, $S(\rho_a)$ equals to zero. At the rest of the time, the system state cannot be written in the direct product form, so the atom and the field are entangled. In particular, at $gt = n\tau + 0.58/1.19$ $(n = 0, 1, 2, \ldots)$, $S(\rho_a)$ takes its maximum value 1. This means that the atom and the field are maximally entangled, and the system state is an EPR (Einstein–Podolsky–Rosen) entanglement state which is widely applied in quantum information processing and is crucial for long distance quantum communication

$$\begin{aligned} &|\Psi_{fa}(gt = n\tau + 0.58/1.19)\rangle \\ &= \frac{1}{\sqrt{2}}(|\psi_f^+\rangle|\psi_a^+\rangle + |\psi_f^-\rangle|\psi_a^-\rangle). \end{aligned} \quad (16)$$

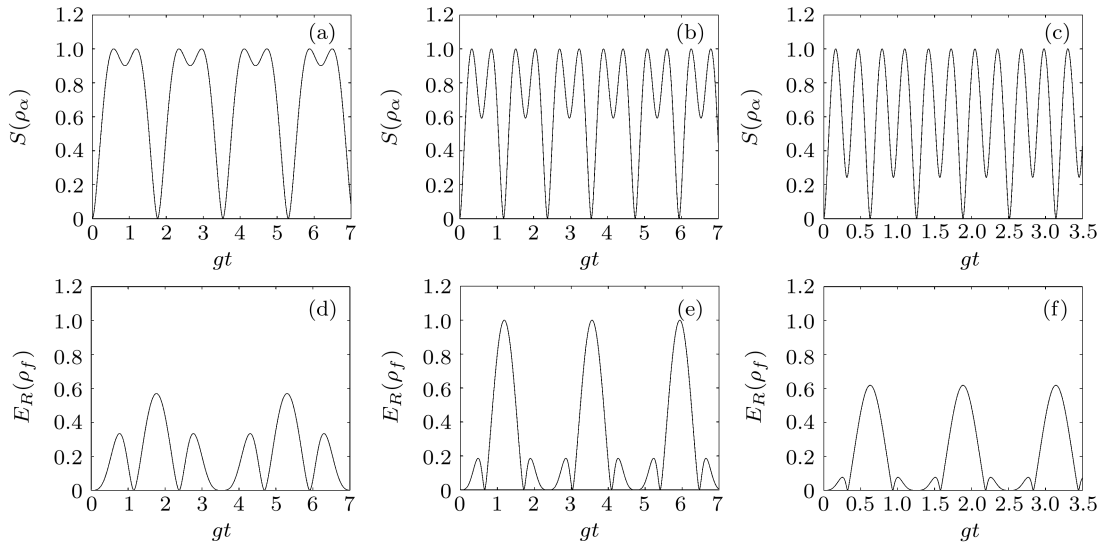


Fig.1. The time evolution of the degree of the entanglement for $\varphi = 0$ and $\theta = \pi$. (a), (d) $\lambda/g = 0.6$; (b), (e) $\lambda/g = 1$; (c), (f) $\lambda/g = 2$.

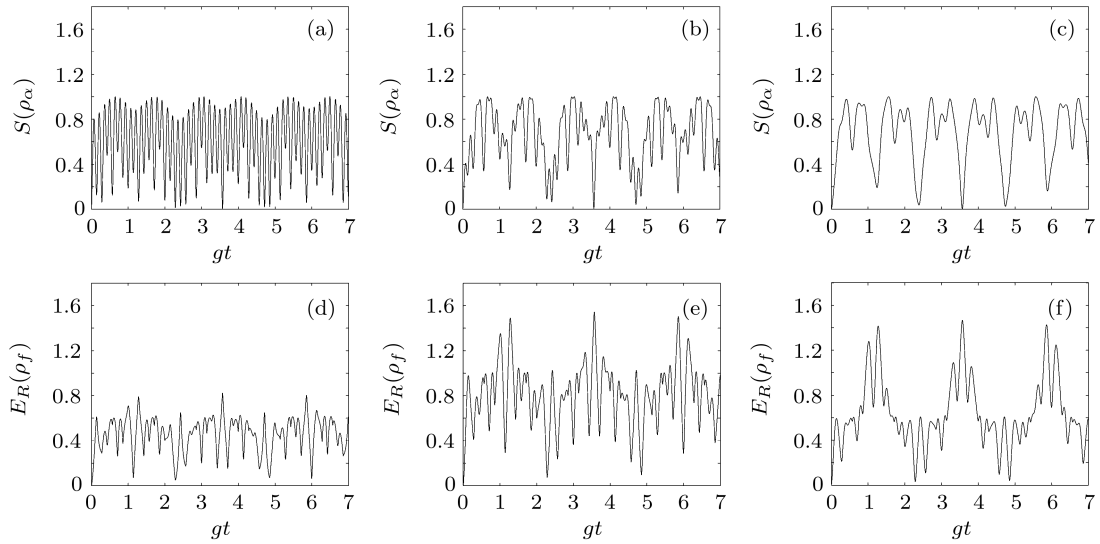


Fig.2. The time evolution of the degree of the entanglement for $\varphi = 0$ and $\lambda/g = 1$. (a), (d) $\theta = \pi/4$; (b), (e) $\theta = \pi/2$; (c), (f) $\theta = 3\pi/4$.

When λ/g increases to 1 and 2, we find that the larger the λ/g is, the shorter the period of $S(\rho_a)$ is, see Figs.1(b) and (c). Further study shows that, when λ/g keeps increasing to larger values, $S(\rho_a)$ oscillates rapidly. This stems from the fact that the increase in λ/g leads to a more strong interaction between the atom and the field.

Secondly, we examine the influence of atomic distribution θ on the DEAF, which is shown in Figs.2(a)–(c). The behaviour of $S(\rho_a)$ becomes more complicated, with the presence of local maxima and minima values. The presence of the local maxima and minima is due to the entanglement and disentanglement between the atom and the field. From these figures, one can see that the change of θ does not basically influence the periodicity of entanglement dynamics and the maximum of the DEAF; but suppresses the oscillation of $S(\rho_a)$, that is, the more the θ tends to π (the atom is initially in the ground state), the lower oscillation frequency of $S(\rho_a)$ is.

4. Entanglement between the competing two-mode field

Furthermore, we study the entanglement between the competing two-mode field. The above results show that the Hamiltonian given by Eq.(1) leads to the entanglement between the atom and the two-mode field, the state of the two-mode field may evolve into a mixed state. In this case, the degree of entanglement between the two-mode field cannot be measured by the reduced entropy. However, the relative entropy of entanglement is a good measure for the degree of entanglement of two subsystems in the mixed state, which is defined as[19]

$$E_R(\rho) = \min_{\sigma \in D} S(\rho||\sigma), \tag{17}$$

where D is the set of all disentangled states, and $S(\rho||\sigma) = \mathrm{Tr}[\rho(\log_2 \rho - \log_2 \sigma)]$ is the relative entropy of ρ with respect to σ. The relative entropy of entanglement plays an important role in understanding quantum entanglement,[19] which reduces to the von Neumann reduced entropy of either side for the bipartite pure states;[42] for general mixed states, the relative entropy of entanglement is usually difficult to calculate except for some specific states. Recently, the following theorem on the relative entropy of entanglement has been proved and it turns out to be quite suitable for the present analysis.

For a bipartite quantum state[21,43]

$$\rho = \sum_{n_1,n_2} a_{n_1,n_2} |\Phi_{n_1}\phi_{n_1}\rangle\langle\Phi_{n_2}\phi_{n_2}|, \tag{18}$$

the relative entropy of entanglement is given by

$$E_R(\rho) = -\sum_n a_{n,n} \log_2 a_{n,n} - S(\rho), \tag{19}$$

and the disentangled state ρ^* that minimizes the quan-

tum relative entropy is

$$\rho^* = \sum_n a_{n,n} |\Phi_n \phi_n\rangle\langle\Phi_n \phi_n|, \tag{20}$$

where $|\Phi_n\rangle$ and $|\phi_n\rangle$ are the orthogonal normalized states of each subsystem. $S(\rho)$ is the von Neumann entropy: $S(\rho) \equiv -\mathrm{Tr}(\rho \log_2 \rho)$.

It should be noted that the density operator Eq.(7) can take the form of Eq.(18). Hence, the relative entropy of entanglement of the state given by Eq.(7) is written as

$$\begin{aligned} & E_R(\rho_f) \\ = & -\sum_n a_{n,n} \log_2 a_{n,n} - S(\rho_f) \\ = & -\sum_n a_{n,n} \log_2 a_{n,n} + \lambda^+ \log_2 \lambda^+ + \lambda^- \log_2 \lambda^-. \end{aligned} \tag{21}$$

The time evolution of the relative entropy of entanglement $E_R(\rho_f)$ given by Eq.(21) reflects the time evolution of the degree of entanglement between the two-mode field (DETMF), which is plotted in Figs.1(d)–(f) and Figs.2(d)–(f) for the parameter values being the same as in Figs.1(a)–(c) and Figs.2(a)–(c), respectively.

Firstly, we investigate the influence of λ/g on the DETMF. It can be clearly seen from Figs.1(d)–(f) that, (i) $E_R(\rho_f)$ exhibits the evolution quite periodically, the larger the λ/g the shorter the period of $E_R(\rho_f)$; (ii) when λ/g increases to 1, the maximum of $E_R(\rho_f)$ increases while the second minimum of $E_R(\rho_f)$ decreases; when λ/g keeps increasing to 2 (even larger than 2), both the maximum and the second minimum of $E_R(\rho_f)$ decrease. That is, only when the coupling constants of the two competing transitions are equivalent, the two-mode field can reach the maximal entangled state which has been applied widely in quantum information processing.

Comparing the Figs.1(a)–(c) with 1(d)–(f), we find that, (i)the evolution period of $E_R(\rho_f)$ is always twice the $S(\rho_a)$; (ii) at time $gt = (2k+1)\tau$ $(k = 0, 1, 2, \ldots)$ when the atom is disentangled from the two-mode field, the DETMF achieves its maximum, which means that the two-mode field is in a maximal entangled state. This comes from the fact that the DEAF and the DETMF can impair each other due to the interaction between the atom and the field; (iii) while at time $gt = 2k\tau$ $(n = 0, 1, 2, \ldots)$, both the DEAF and the DETMF equal to zero with the system reverting to the initial state.

Then we study the influence of atomic distribution θ on the DETMF which is displayed in Figs.2(d)–(f). It can be seen that the change of θ does not basically influence the periodicity of entanglement dynamics, but changes the maximum of the DETMF. When the atom is initially in the coherent superposition state of the excited state $|+\rangle$ and the ground state $|-\rangle$ with equal probability ($\theta = \pi/2$), the maximum of the DETMF can reach 1.54. That is, the more the atomic distribution θ tends to $\pi/2$, the stronger the entanglement between the two-mode field is. From another point of view, it reflects the atomic coherence control on the entanglement between the two-mode field, which has been shown to play an important role in inducing, enhancing and controlling entanglement.[44,45]

5. Conclusion

In this paper, we have studied the entanglement in the system of a two-level atom interacting with a competing two-mode field. The entanglement between the two-level atom and the two-mode field is investigated by using the quantum reduced entropy, and that between the two-mode field by using the quantum relative entropy of entanglement. Our results show that the relative competing strength λ/g and the atomic distribution θ have strong influence on the DEAF and the DETMF. Firstly, when the atom is initially in the ground state $|-\rangle$ ($\theta = \pi$), both the entanglement of atom-field and that of the two-mode field evolve periodically, and the period of the DEAF is twice the DETMF; the larger the λ/g the shorter the period of the DEAF and the DETMF. Secondly, the DEAF and the DETMF impair each other due to the two-level atom interacting with the two-mode field. Finally, the maximal entangled state of atom-field can be produced by choosing appropriately the system parameters and interaction time; the maximal entangled state of the two-mode field can be prepared when the two coupling constants of the competing transitions are equivalent($\lambda/g = 1$) and the atom is initially in the coherent superposition state of the excited state $|+\rangle$ and the ground state $|-\rangle$ with equal probability ($\theta = \pi/2$). Our results are important for experimental realization of the preparation of entangled states.

References

[1] Braunstein L and Caves C M 1998 *Phys. Rev. Lett.* **61** 662

[2] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895

[3] Bouwmeester D *et al* 1997 *Nature* **390** 575

[4] Deutsch D, Ekert A, Jozsa R, Macchiavello C, Popescu S and Sanpera A 1996 *Phys. Rev. Lett.* **77** 2818

[5] Shor P W 1995 *Phys. Rev.* A **52** 2493

[6] Deutsch D 1985 *Proc. R. Soc.* London *Ser.* A **400** 97

[7] Briegel H J, Durr W, Cirac J I and Zoller P 1998 *Phys. Rev. Lett.* **81** 5932

[8] Matsukevich D N and Kuzmich A 2004 *Science* **306** 663

[9] Julsgaard B, Sherson J, Cirac J I, Fiurasek J A and Polzik E S 2004 *Nature* **432** 482

[10] Zhou Q P, Fang M F, Liu X J, Chen X M, Wu Q and Xu H Z 2004 *Chin. Phys.* **13** 1881

[11] Zhou Q P, Fang M F, Liu X J, Chen X M and Wu Q 2004 *Chin. Phys. Lett.* **21** 1692

[12] Clark S G and Parkins A S 2003 *Phys. Rev. Lett.* **90** 4

[13] Wu Y and Deng L 2004 *Opt. Lett.* **29** 1144

[14] Wu Y, Payne M G, Hagley E W and Deng L 2004 *Phys. Rev.* A **69** 063803

[15] Wu Y, Payne M G, Hagley E W and Deng L 2004 *Phys. Rev.* A **70** 063812

[16] Wu Y 2001 *Phys. Rev.* A **63** 052303

[17] von Neumann J 1932 *Mathematische Grundlagen der Quanttenmechanik*

[18] Bennett C H, Divincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev.* A **54** 3824

[19] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275

[20] Vedral V 2002 *Rev. Mod. Phys.* **74** 197

[21] Wu S and Zhang Y D 2000 quant-ph/0004018

[22] Wang C Z and Fang M F 2002 *Acta Phys. Sin.* **51** 1989

[23] Wang C Z and Fang M F 2003 *Chin. Phys.* **12** 287

[24] Liu X J, Fang M F and Zhou Q P 2004 *Chin. Phys. Lett.* **21** 1071

[25] Liu X J and Fang M F 2003 *Chin. Phys.* **12** 971

[26] Zeng K and Fang M F 2005 *Chin. Phys.* **14** 2009

[27] Jin L J and Fang M F 2006 *Chin. Phys.* **15** 2012

[28] Cai X, Huang G M and Li G X 2005 *Chin. Phys.* **14** 223

[29] Liu T K, Wang J S, Feng J and Zhan M S 2005 *Chin. Phys.* **14** 536

[30] Gao Y F, Feng J and Wang J S 2005 *Chin. Phys.* **14** 980

[31] Liu T K 2006 *Chin. Phys.* **15** 542

[32] Shigeru F and Mahmoud A A 2001 *J. Phys.* A: *Math. Gen.* **34** 6851

[33] Agarwal G S and Nayak N 1986 *Phys. Rev.* A **33** 391

[34] Zhu Y 1990 *Phys. Rev.* A **41** 6574

[35] Agarwal G S and Nayak N 1982 *Phys. Rev.* A **25** 2056

[36] Bloembergen N 1965 *Nonliear Optics* (New York: Benjamin)

[37] Jyotsna I V and Agarwal G S 1994 *Phys. Rev.* A **50** 1770

[38] Wu Q and Fang M F 2004 *Chin. Phys.* **13** 1432

[39] Wu Q and Fang M F 2005 *Commun. Theor. Phys.* **43** 515

[40] Wu Q and Fang M F 2006 *Acta Quant. Opt.* **12** 8 (in Chinese)

[41] Wu Q and Fang M F 2006 *J. Natural Science of Hunan Normal University* **29** 49 (in Chinese)

[42] Vedral V and Plenio M B 1998 *Phys. Rev.* A **57** 1619

[43] Rains E M 1999 *Phys. Rev.* A **60** 179

[44] Li F L, Xiong H and Zubairy M S 2005 *Phys. Rev.* A **72** 010303(R)

[45] Wang X Y, Du S D and Chen X S 2006 *J. Phys.* B **39** 3805