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Could one single dichotomous noise cause resonant activation for exit time over potential barrier?*

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This paper studies the mean first passage time (or exit time, or escape time) over the non-fluctuating potential barrier for a system driven only by a dichotomous noise. It finds that the dichotomous noise can make the particles escape over the potential barrier, in some circumstances; but in other circumstances, it can not. In the case that the particles escape over the potential barrier, a resonant activation phenomenon for the mean first passage time over the potential barrier is obtained.

Keywords: Resonant activation, mean first passage time, dichotomous noise
PACC: 0540

1. Introduction

The problem of the escape of a Brownian particle out of a potential well (or over the potential barrier) with thermal fluctuation, first characterized by Kramers,[1] plays a central role in the description of many processes in physics, chemistry, and biology.[2,3] Kramers' law predicts an escape rate of the form $\tilde{\gamma} = \exp(-U/k\tilde{T})$, where U is the depth of the well (or the height of the barrier), $\tilde{T}$ the temperature, and k the Boltzmann constant (now the mean first passage time (MFPT) T (or exit time or escape time), corresponding to the Kramers' escape rate has the form $T = 1/\tilde{\gamma} = \exp(U/k\tilde{T})$). In the case of a non-adiabatically driven Brownian particle, the aforementioned scenario may change significantly, and much work has been devoted to the study of this non-equilibrium phenomenon.[4−11] It has been shown that in the presence of non-adiabatic driving the MFPT of the particle from the potential well (or over the potential barrier) can be significantly reduced, a phenomenon named resonant activation (RA).

The RA phenomenon was first observed in a current-biased Josephson tunnel junction,[4] and further studied by a number of other authors.[5−8,10,12−39] Some earlier studies of activation of MFPT from potential well (or over potential barrier) were restricted to limiting cases, i.e., slow[40] or fast[40,41] fluctuations, or small fluctuations.[42] Owing to using approximate treatments in Refs.[40–42], the RA phenomenon can not be observed. But in Refs.[4–8,10,12–39], in the absence of approximate treatments as in Refs.[40–42], the escape time (i.e. MFPT) from a potential well (or over a potential barrier) was found to reveal the RA phenomenon. Now, the RA phenomenon has been reported to appear in chemistry, biology, physics and other science theoretically[5−8,10,12−31,35−39] and experimentally.[4,32−34] Very recently, Lepri *et al*[35] theoretically investigated the possibility of observing RA in the hopping dynamics of two-mode semiconductor lasers, and presented a series of simulations of a rate-equation model under random and periodic modulation of the bias current; Fiasconaro[36] studied the co-occurrence of RA and noise-enhanced stability in a model of cancer growth in the presence of immune response; Rotoli[37] considered the RA for a Josephson junction embedded in a shell circuit consisting of one capacitive and one inductive element; Bag *et al*[38] considered the escape through an unstable limit cycle with thermal fluctuation and found the RA phenomenon; Gommers[39] investigated the RA in a nonadiabatically driven optical lattice.

But, for the RA phenomenon in Refs.[5–8,10,12–39], what the authors considered are all the situation when the fluctuating potential barrier is driven by an additive Gaussian white noise. In Ref.[43], we have investigated the case when the driving additive noise

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is a dichotomous noise. But in Ref.[43], the potential barrier is fluctuating. One unavoidably wants to ask a question, i.e., how is the situation if the potential barrier is not fluctuating? In this paper, we will investigate the escape for the case when the potential barrier is not fluctuating (and only driven by an additive dichotomous noise), and try to get the RA phenomenon.

2. Model and its MFPT's equations

In this section, we will study the escape for the MFPT over the non-fluctuating potential barrier for a system which is only driven by a dichotomous noise. The model we consider here obeys the Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\partial_x U(x) + \xi(t), \tag{1}$$

where $\partial_x U(x)$ denotes the derivative of the non-fluctuating potential $U(x)$ (see Fig.1), $\xi(t)$ is a dichotomous noise which takes values a and b (the transition rate from a to b and vice versa are λ). The probability density and statistical properties of $\xi(t)$ are derived in the Appendix A.

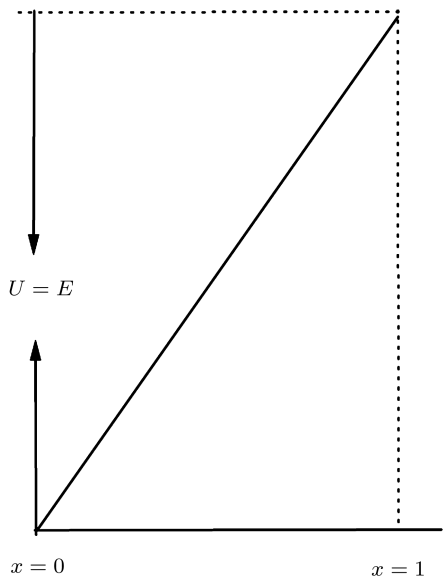


Fig.1. The non-fluctuating potential barrier for Eq.(1).

The master equation of Eq.(1) is[44]

$$\partial_t \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \hat{L}_1 & \lambda \\ \lambda & \hat{L}_2 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \tag{2}$$

where $\hat{L}_1 = -\lambda + \partial_x(E_1 - a)$, and $\hat{L}_2 = -\lambda + \partial_x(E_1 - b)$. $P_1 = P(x,t,a,E)$ and $P_2 = P(x,t,b,E)$. $P_1 = P(x,t,a,E)$ represents that the particle is at x, the potential is in $\partial_x U = E$ configuration and the noise in $\xi(t) = a$ configuration. There is the same understanding for P_2. We start with the particle at the bottom ($x = 0$), so the initial condition is $\sum_i P_i(x,0) = \delta(x)$ $(i = 1, 2)$.

The equations of the MFPT for Eq.(2) are[44]

$$\begin{aligned} (-E + a)\partial_x T_1 - \lambda T_1 + \lambda T_2 + 1/2 = 0, \\ (-E + b)\partial_x T_2 - \lambda T_2 + \lambda T_1 + 1/2 = 0, \end{aligned} \tag{3}$$

where T_i $(i = 1, 2)$ is the MFPT corresponding to the probability density P_i. Eq.(3) can be written in the following matrix form

$$\partial_x \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{\lambda}{E-a} & \dfrac{\lambda}{E-a} \\ \dfrac{\lambda}{E-b} & -\dfrac{\lambda}{E-b} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} \dfrac{1}{2(E-a)} \\ \dfrac{1}{2(E-b)} \end{pmatrix}. \tag{4}$$

The MFPT for the particle over the non-fluctuating potential barrier that starts at $x = 0$ is $T = \sum_{i=1}^{2} T_i(0)$. The boundary condition for Eq.(4) is more complicated. In the following section we shall analyse the boundary condition of Eq.(4) for the particle to escape over the non-fluctuating potential barrier.

3. Analysis for the escape of particles over the potential barrier and the MFPT's boundary condition

First we assume $E > 0$ and $a > b$ with $a > 0$ and $b < 0$. In this assumption, there are two cases, which are $a > E > b$ and $E > a > b$.

Case 1 $a > E > b$. In this case, the two velocities of the particle corresponding to $P_1 = P(x,t,a,E)$ and $P_2 = P(x,t,b,E)$ are, respectively, $\dot{x} = a - E > 0$ and $\dot{x} = b - E < 0$. The reflecting boundary condition $\partial_x T_2 \mid_{x=0} = 0$ is tenable, this is due to the fact that the particle at the boundary $x = 0$ with the negative velocities $\dot{x} = b - E < 0$ can reflect and jump back into the domain [0,1]; while the condition $\partial_x T_1 \mid_{x=0} = 0$ is untenable, since the velocity of the

particle $\dot{x} = a - E > 0$ can not make the particle reflect at the boundary $x = 0$. Because the particle at the boundary $x = 1$ with the positive velocity $\dot{x} = a - E > 0$ can escape from the domain [0,1], the absorbing boundary condition $T_1 \mid_{x=1} = 0$ is tenable; while the condition $T_2 \mid_{x=1} = 0$ is untenable, since the velocity of the particle $\dot{x} = b - E < 0$ can not make the particle escape from the domain [0,1].

Case 2 $E > a > b$. Now the two velocities corresponding to $P_1 = P(x, t, a, E)$ and $P_2 = P(x, t, b, E)$ are, respectively, $\dot{x} = a - E < 0$ and $\dot{x} = b - E < 0$. Because the two velocities of the particle are all negative, now the particle cannot escape over the potential barrier.

Above we analysed the boundary condition of Eq.(4) in the assumption of $E > 0$ and $a > b$ with $a > 0$ and $b < 0$. But there are other assumptions. For example, if we assume $E > 0$ and $a > b$ with $a, b > 0$, there are three cases, which are $a > b > E$, $a > E > b$ and $E > a > b$. Now using the similar analysis as above, for this assumption, we can obtain the boundary condition. For this example, it should be mentioned that, in the case of $E > a > b$, a particle obeys Eq.(1) can not escape over the non-fluctuating potential barrier since now the velocities of the particle are all negative.

In conclusion, when $E > a > b$, a particle which obeys Eq.(1) can not escape over the non-fluctuating potential barrier. But, in other circumstances, a particle which obeys Eq.(1) can escape over the non-fluctuating potential barrier. We can get the boundary condition for Eq.(1) by using the above method.

For Eq.(1), we can get the exact expression of the MFPT over the non-fluctuating potential barrier. In the following we shall give the derivation of the expression for the MFPT.

4. Expression of the MFPT

When $2E - a - b \neq 0$, the solution of Eq.(4) is

$$T_i = A_i + B_i x + C_i \mathrm{e}^{rx}, \tag{5}$$

where $i = 1$ and 2, r is the nonzero eigenvalue of the matrix in Eq.(4), which is $r = -\lambda[1/(E - a) + 1/(E - b)]$. Substituting Eq.(5) into Eq.(4), we can get $B_2 = B_1 = 1/(2E - a - b)$, $A_2 = A_1 - B_1/[\lambda(a - E)]$ and $C_2 = C_1[1 - (a - E)r/\lambda]$. Then, using the boundary condition (here we only take the case $a > E > b$ for the assumption $E > 0$ and $a > b$ with $a > 0$ and $b < 0$), we obtain $C_1 = B_1/[(a - E)r^2/\lambda - 1]$ and $A_1 = C_1[r - (a - E)r^2/\lambda - e^r]$. The MFPT for the particle over the non-fluctuating potential barrier is

$$\begin{aligned} T = T_1(0) + T_2(0) &= A_1 + C_1 \\ &+ \left[A_1 - \frac{1}{2\lambda} - \frac{B_1}{\lambda(a - E)}\right] \\ &+ C_1\left(1 - \frac{a - E}{\lambda} r\right]. \end{aligned} \tag{6}$$

The condition for validity of Eq.(6) is $a > E > b$ for the assumption $E > 0$ and $a > b$ with $a > 0$ and $b < 0$.

5. Conclusion and discussion

In Fig.2, we plot the logarithm of the MFPT based on Eq.(6) as the function of the logarithm of λ. From the figure, we can find that, there is the RA phenomenon for the MFPT over the non-fluctuating potential barrier. We shall give the reason for the emergence of RA in Fig.2 below. The resonance occurs in Fig.2 when the crossing which takes place with

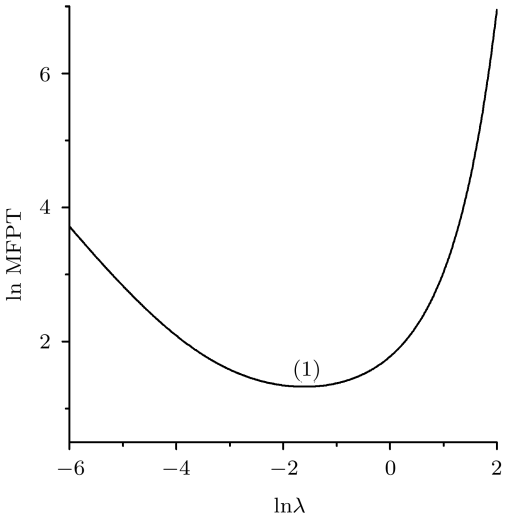


Fig.2. The ln of MFPT vs. the ln of λ for Eq.(1), with the parameters values $E = 1.5$, $a = 2.2$ and $b = -1$, in the case of $2E - a - b \neq 0$. The marked point is that where the transition time is equal to the MFPT over the potential barrier with the potential barrier in $U = \min(E - a, E - b)$ configuration.

the potential barrier must be likely in $U = \min(E - a, E - b)$ configuration (i.e., the "down" configuration). Now the MFPT presents a minimum for the transition rate of the dichotomous noise on the order of the reverse of the time required to cross the potential with the potential barrier in $U = \min(E - a, E - b)$ configuration. In Fig.2, a marked point [point (1)] is plot-

ted where the transition time equals to the MFPT over the potential barrier with the potential barrier in $U = \min(E-a, E-b)$ configuration. Clearly, this accords with the above reason for the appearance of RA in Fig.2.

Above we only considered the case when $2E-a-b \neq 0$. When $2E-a-b=0$, the matrix of Eq.(4) has two zero eigenvalues. The solution of Eq.(4) is

$$T_i = D_i + F_i x + G_i x^2, \tag{7}$$

where $i = 1, 2$. Substituting Eq.(7) into Eq.(4) and using the boundary condition (here we only take the case $a > E > b$ for the assumption $E > 0$ and $a > b$ with $a > 0$ and $b < 0$), we get $D_2 = D_1 - \dfrac{1}{2\lambda} + \dfrac{E-a}{\lambda}F_1$, $F_2 = F_1 + \dfrac{1}{E-b}$ and $G_2 = G_1$ with $D_1 = -G_1 + \dfrac{1}{E-b}$, $G_1 = -\dfrac{2\lambda}{(a-b)(E-b)}$ and $F_1 = \dfrac{1}{E-a}$. The MFPT for the particle over the non-fluctuating potential barrier is

$$\begin{aligned} T &= T_1(0) + T_2(0) \\ &= D_1 + D_2 \\ &= 2D_1 + \frac{1}{2\lambda}. \end{aligned} \tag{8}$$

Similarly, the condition for validity of Eq.(7) is also $a > E > b$ for the assumption $E > 0$ and $a > b$ with $a > 0$ and $b < 0$.

In Fig.3, we depict the logarithm of the MFPT based on Eq.(8) as the function of the logarithm of λ.

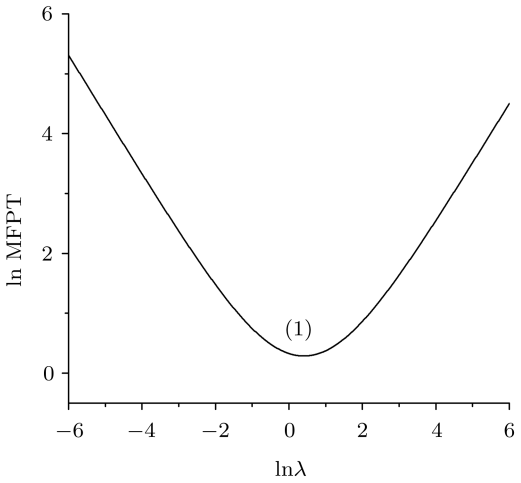


Fig.3. The ln of MFPT vs. the ln of λ for Eq.(1), with the parameters values $E = 2$, $a = 5$ and $b = -1$, in the case of $2E-a-b=0$. The marked point is that where the transition time is equal to the MFPT over the potential barrier with the potential barrier in $U = \min(E-a, E-b)$ configuration.

This figure also shows the appearance of the RA phenomenon. The reason for the emergence of the RA phenomenon in Fig.3 is the same as that in Fig.2. To avoid unnecessary repetition, here we do not give the reason which is the same as that in Fig.2.

Appendix A

In this appendix we shall derive the probability density and statistical properties of the dichotomous noise $\xi(t)$ in Eq.(1). The master equation of this noise is

$$\begin{aligned} \partial_t P(a,t \mid x,t_0) &= -\lambda P(a,t \mid x,t_0) + \lambda P(b,t \mid x,t_0), \\ \partial_t P(b,t \mid x,t_0) &= \lambda P(a,t \mid x,t_0) \\ &\quad - \lambda P(b,t \mid x,t_0), \end{aligned} \tag{A1}$$

where $P(a,t \mid x,t_0)$ denotes that the particle is at $\xi(t) = a$ configuration and at time t with the initial state $t = t_0$ and $\xi(t_0) = x$. There is the same understanding for $P(b,t \mid x,t_0)$. For Eq.(A1), the solution can be computed by noting that

$$P(a,t \mid x,t_0) + P(b,t \mid x,t_0) = 1, \tag{A2}$$

and the initial condition of Eq.(A1) is

$$P(x',t_0 \mid x,t_0) = \delta_{xx'}. \tag{A3}$$

The solution of Eq.(A1) is

$$\begin{aligned} P(a,t \mid x,t_0) &= \frac{1}{2} + \left(\frac{1}{2}\delta_{ax} - \frac{1}{2}\delta_{bx}\right) \mathrm{e}^{-2\lambda(t-t_0)}, \\ P(b,t \mid x,t_0) &= \frac{1}{2} - \left(\frac{1}{2}\delta_{ax} - \frac{1}{2}\delta_{bx}\right) \mathrm{e}^{-2\lambda(t-t_0)}. \end{aligned} \tag{A4}$$

From Eq.(A4), it is easy to find that the stationary solution of Eq.(A1) is

$$\begin{aligned} P_s(a) &= \frac{1}{2}, \\ P_s(b) &= \frac{1}{2}. \end{aligned}$$

The mean of $\xi(t)$ is

$$\begin{aligned} \langle \xi(t) \mid [x_0,t_0] \rangle &= \sum_{x=a,b} x P(x,t \mid x_0,t_0) \\ &= \frac{a+b}{2} + \left(x_0 - \frac{a+b}{2}\right) \mathrm{e}^{-2\lambda(t-t_0)}. \end{aligned} \tag{A5}$$

Eq.(A5) shows that the stationary mean is

$$\langle \xi(t) \rangle_s = \frac{a+b}{2}.$$

The stationary variance can be got by using the above method

$$\mathrm{var}\{\xi(t)\}_s = \frac{(a-b)^2}{4}.$$

The stationary time correlation function is ($t \geq s$)

$$\langle \xi(t)\xi(s) \rangle_s = \sum_{xx'} xx' P(x,t \mid x',s) P_s(x')$$
$$= \sum_{x'} x' < \xi(t) \mid [x',s] > P_s(x'). \quad \text{(A6)}$$

Substituting Eq.(A5) into Eq.(A6), we can get

$$\langle \xi(t)\xi(s) \rangle_s = \langle \xi(t) \rangle_s^2 + \mathrm{var}\{\xi(t)\}_s \mathrm{e}^{-2\lambda(t-s)}$$
$$= \left(\frac{a+b}{2}\right)^2 + \frac{(a-b)^2}{4}\mathrm{e}^{-2\lambda(t-s)}. \quad \text{(A7)}$$

Similarly, we can compute the case when $t \leq s$. Finally, we can obtain ($t \geq s$ and $t \leq s$)

$$\langle \xi(t)\xi(s) \rangle_s = \left(\frac{a+b}{2}\right)^2 + \frac{(a-b)^2}{4}\mathrm{e}^{-2\lambda|t-s|}. \quad \text{(A8)}$$

The noise strength is

$$D = \frac{1}{2}\int_{-\infty}^{\infty} (\langle \xi(\tau)\xi(0) \rangle_s - \langle \xi(\tau) \rangle_s^2) \mathrm{d}\tau$$
$$= \frac{(a-b)^2}{8\lambda}. \quad \text{(A9)}$$

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