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Bound entangled states invariant under U_x

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This paper obtains an entangled condition for isotropic-like states by using an atomic map. It constructs a class of bound entangled states from the entangled condition and shows that the partial transposition of the state from the constructed bound entangled class is an edge bound entangled state by using range criterion.

Keywords: Werner-like state, isotropic-like state, bound entangled state
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Quantum entangled states have played very important role in quantum information processing and communication, such as quantum teleportation,[1] cryptography,[2] dense coding[3] and parallel computation.[4] Although there is a clear definition of what an entangled state is,[5] it is still difficult to discriminate entanglement from separability for high dimension and multipartite mixed states.

A significant necessary and sufficient criterion used to testify the separability of a bipartite state was developed by Horodecki *et al.*[6] It applies the positive but not completely positive maps. Now we define $\mathcal{B}(\mathcal{H})$ to be the set of operators on a Hilbert space $\mathcal{H}$. Let $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map. Φ is said to be a positive map if, for all Hermitian $\sigma \in \mathcal{H}$ such that $\sigma \geq 0$, then $\Phi(\sigma)$ is Hermitian and $\Phi(\sigma) \geq 0$. Φ is completely positive if, for all Hilbert spaces $\mathcal{K}$, the map $I \otimes \Phi: \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ (where I is the identity map) is also positive. We say Φ is completely copositive if $\Phi \circ T$ is completely positive, where T is transposition operator. The simplest example of positive but not completely positive map is the transposition operator T. The action of transposition on one of the subsystem is called partial transposition. It provides a strong condition for separability, namely, ρ is entangled if it has a nonpositive partial transposition. This criterion is the well-known Peres criterion.[7] Horodecki *et al* show that it is also a sufficient condition for separability of a state in $2 \otimes 2$ and $2 \otimes 3$ systems in Ref.[6]. However, in higher dimensions, there exists entangled states which are positive partial transposition (PPT). The latter states have the interesting property that their entanglement is undistillable and belong to the class of bound entangled states. On physical ground, Horodecki proposed the first example of bound entangled state,[8] following Woronowicz construction.[9] Since then, many examples of bound entangled states have been introduced in Refs.[10–15]. It has been shown that most of these states are part of a same family of PPT states in Ref.[16]. A first systematic method to construct PPT bound entangled states was provided[17] and based on the concept of unextendable product basis.

The main result of this paper is to construct a class of bound entangled states. Our first step is to find a class of PPT states. After taking PPT states, we only need to give the entangled condition for these PPT states by use of some available separability criterion, e.g. range criterion,[8] realignment criterion[18,19] and indecomposable positive maps.[9,20] Thus we obtain a class of bound entangled states according to the entangled condition. In general, PPT structure is unknown, that is, one may check whether a given state is PPT but we do not know how to construct a general quantum state with PPT property. Fortunately, Chruściński and Kossakowski proposed a method of constructing PPT symmetric states in Ref.[16]. Recall that these states are invariant under the maximal commutative subgroup of $U(d)$, i.e. the d-dimension torus $T^d = U(1) \times \ldots \times U(1)$. This commutative subgroup is generated by d mutually commuting operators

$$t_k = |k\rangle\langle k|, \ k = 1, \ \ldots, \ d,$$

where $|k\rangle$ denotes an orthonormal basis in $\mathcal{C}^d$. The following element in T^d is generated by any vector

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$x \in \mathcal{R}^d$:

$$U_x = \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{t}},$$

where $\boldsymbol{t} = (t_1, \ldots, t_\mathrm{d})$. Obviously, $U_xU_y = U_{x+y}$. The new family of symmetric states includes two classes of bipartite states invariant under U_x. One is $U_x \otimes U_x$-invariant state which is named Werner-like state

$$(U_x \otimes U_x)\rho = \rho(U_x \otimes U_x),$$

for all $x \in \mathcal{R}^d$. It can be written in the following form:

$$\rho = \sum_{i,j=1}^{d} b_{ij}|ij\rangle\langle ji| + \sum_{i\neq j} c_{ij}|ij\rangle\langle ij|, \tag{1}$$

where $b_{ij} = b_{ji}^*,\ c_{ij} \geq 0,\ b_{ii} \geq 0,\ c_{ij}c_{ji} - |b_{ji}|^2 \geq 0$. Another is $U_x \otimes U_x^*$-invariant state which is called isotropic-like state

$$(U_x \otimes U_x^*)\rho = \rho(U_x \otimes U_x^*), \tag{2}$$

for all $x \in \mathcal{R}^d$, where U_x^* denotes complex conjugation of U_x. It has the following form:

$$\rho = \sum_{i,j=1}^{d} a_{ij}|ii\rangle\langle jj| + \sum_{i\neq j} c_{ij}|ij\rangle\langle ij|,$$

where $A = (a_{ij})$ is Hermitian and positive, $c_{ij} \geq 0$. Here it still needs to satisfy

$$\mathrm{Tr}A + \sum_{i\neq j} c_{ij} = \sum_{i=1}^{d} a_{ii} + \sum_{i\neq j} c_{ij} = 1. \tag{3}$$

It is obvious that the positivity of the partial transposition of an isotropic-like state in Eq.(2) is equivalent to the following condition

$$c_{ij}c_{ji} - |a_{ij}|^2 \geq 0. \tag{4}$$

One can easily find that PPT is not a necessary and sufficient condition for the above two families of invariant states. So there exist bound entangled states in both classes of invariant states. Thus we can construct bound entangled states from the above two families of symmetric states.

In this paper we construct a class of bound entangled isotropic-like states by using an atomic map at the beginning and then prove that the partial transposition of the state from the constructed bound entangled class is an edge-bound entangled state by using a range criterion.

From Ref.[6] we know that a bipartite state ρ is separable if and only if for any positive map Φ the operator $(I \otimes \Phi)\rho$ is positive. In other words, if $(I \otimes \Phi)\rho$ is not positive then ρ is entangled. Now we will give an entangled condition of isotropic-like state by use of an atomic map according to the above statement. An atomic map is a map which can not be decomposed into the sum of 2-positive linear maps and a 2-copositive maps.[21] From Ref.[22] we know that the map $\Phi \in P(M_n)$ is an atomic map for $n \geq 3$, where

$$\Phi(X) = \begin{pmatrix} (n-2)x_{11} + x_{nn} & -x_{12} & -x_{13} & \ldots & -x_{1n} \\ -x_{21} & (n-2)x_{22} + x_{11} & -x_{23} & \ldots & -x_{2n} \\ -x_{31} & -x_{32} & (n-2)x_{33} + x_{22} & \ldots & -x_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ -x_{n1} & -x_{n2} & -x_{n3} & \ldots & (n-2)x_{nn} + x_{n-1,n-1} \end{pmatrix},$$

$X = (x_{ij})$ is an $n \times n$ matrix. So for any isotropic-like state in Eq.(2) if $(I \otimes \Phi)\rho$ is not positive then ρ is entangled.

$$(I \otimes \Phi)\rho = \begin{pmatrix} \Phi X_{11} & \Phi X_{12} & \ldots & \Phi X_{1d} \\ \Phi X_{21} & \Phi X_{22} & \ldots & \Phi X_{2d} \\ \vdots & \vdots & & \vdots \\ \Phi X_{d1} & \Phi X_{d2} & \ldots & \Phi X_{dd} \end{pmatrix},$$

where the matrix X_{ii} is a $d \times d$ diagonal matrix whose (i,i)-th element is a_{ii} and (j,j)-th elements are $c_{ij},\ j \neq i$, X_{ij} is a $d \times d$ matrix whose (i,j)-th element is a_{ij} and the other elements are 0.

Thus we can get the following proposition.

Proposition For an isotropic-like state in Eq.(2), if

$$\mathcal{M}=\begin{pmatrix} (d-2)a_{11}+c_{1d} & -a_{12} & -a_{13} & \dots & -a_{1d} \\ -a_{21} & (d-2)a_{22}+c_{21} & -a_{23} & \dots & -a_{2d} \\ -a_{31} & -a_{32} & (d-2)a_{33}+c_{32} & \dots & -a_{3d} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{d1} & -a_{d2} & -a_{d3} & \dots & (d-2)a_{dd}+c_{d,d-1} \end{pmatrix}$$

is not positive and $c_{ij}c_{ji}-|a_{ij}|^2\geq 0$, then ρ is a bound entangled state.

Proof One can easily show that the positivity of $(I\otimes\Phi)\rho$ is determined by the positivity of the matrix $\mathcal{M}$ and $(d-2)c_{12}+a_{11},\ (d-2)c_{13}+c_{12},\dots,\ (d-2)c_{1d}+c_{1,d-1},\dots,\ (d-2)c_{d1}+a_{dd},\ (d-2)c_{d2}+c_{d1},\ \dots,\ (d-2)a_{dd}+c_{d,d-1}$. From the positivity of ρ we get that $a_{ii}>0,\ c_{ij}>0$. Since $\mathcal{M}$ is not positive ρ is entangled. Because ρ satisfies the condition $c_{ij}c_{ji}-|a_{ij}|^2\geq 0$, then ρ is a bound entangled state.

Example Consider the following one-parameter state of a $d\otimes d$ system:

$$\begin{aligned}\rho=&\frac{2}{4d-5}P_d^+ +\frac{(4d-7)-(d-2)a}{4d-5}\sigma_+\\ &+\frac{(d-2)a}{4d-5}\sigma_-,\\ &2\leq a\leq 4+\frac{1}{d-2},\end{aligned}$$

where P_d^+ denotes a projector onto the canonical maximally entangled state and

$$\begin{aligned}\sigma_+ =&\frac{1}{d}\sum_{i=1}^{d}|i\oplus 1\rangle\langle i\oplus 1|\otimes|i\rangle\langle i|,\\ \sigma_- =&\frac{1}{d(d-2)}\sum_{i=1}^{d}\sum_{j=2}^{d-1}|i\rangle\langle i|\otimes|i\oplus(j-1)\rangle\\ &\times\langle i\oplus(j-1)|.\end{aligned}$$

In the above formulas $\oplus$ denotes addition modulo d. Obviously, ρ is an isotropic-like state with $a_{ij}=\frac{2}{d(4d-5)}$, $c_{1d}=c_{21}=c_{32}=\dots=c_{d,d-1}=\frac{(4d-7)-(d-2)a}{d(4d-5)}$ and other c_{ij}s are $\frac{a}{d(4d-5)}$. According to Eq.(4), we know that ρ is PPT for $a\leq 4$. By a direct calculation the eigenvalues of $\mathcal{M}$ are $\frac{(6d-9)-(d-2)a}{d(4d-5)}$ (with multiplicity $d-1$) and $\frac{(4d-9)-(d-2)a}{d(4d-5)}$. So ρ is entangled for $a>4-\frac{1}{d-2}$. Thus ρ is a bound entangled state for $4-\frac{1}{d-2}<a\leq 4$.

Remark The state ρ in the above example is separable for $2\leq a\leq 4-\frac{1}{d-2}$. For $d=3$ it is the Horodecki state[23] (constructed on the basis of Størmer matrices).[24] Recently, the Horodecki state is generalized for $d\otimes d$ systems:

$$\begin{aligned}\rho=&\frac{a_1}{d}P_+ +\sum_{i=1}^{d}\sum_{j=2}^{d}\frac{a_j}{d}|i,\ i+j-1\rangle\\ &\times\langle i,\ i+j-1|,\end{aligned}\tag{5}$$

where positive numbers a_i satisfy $\sum_i a_i=1$ in Ref.[15]. Notice that our example is a one-parameter state which belongs to the class of states in Eq.(5). And it is a direct generalization of the Horodecki state.

Now we give a class of unnormalized bound entangled states satisfying the conditions of the above proposition. Set

$$\rho_a=\sum_{i,j=1}^{d}a|ii\rangle\langle jj|+\sum_{i\neq j}c_{ij}|ij\rangle\langle ij|,$$

where $a>1,\ c_{21}=c_{32}=c_{43}=\dots=c_{d,d-1}=c_{1d}=1,\ c_{12}=c_{23}=c_{34}=\dots=c_{d-1,d}=c_{d1}=a^2$ and the other c_{ij}s satisfy either of the following conditions

$$\begin{aligned}&\text{(a) } c_{ij}=c_{ji}=a,\\ &\text{(b) } c_{ij}=1 \text{ and } c_{ji}=a^2.\end{aligned}\tag{6}$$

In fact, by a direct computation $\det\mathcal{M}=-[(d-1)a+1]^{d-1}(a-1)<0$, and $c_{ij}c_{ji}-|a_{ij}|^2=0$ shows that ρ_a is PPT. Notice that the bound entangled states constructed in Ref.[25] are included in our examples. For $d=3\rho_a$ is the edge PPT entangled state in Ref.[26]. It is known that the partial transposition of a PPT isotropic-like state is a Werner-like state. We conjecture that the partial transposition of ρ_a is an edge bound entangled state. In the following we denote the partial transposition of the bound entangled state ρ_a on the second subsystem as the state ρ_b, i.e $\rho_a^{T_B}=\rho_b$.

Thus

$$\rho_b = \sum_{i,j=1}^{d} a|ij\rangle\langle ji| + \sum_{i\neq j} c_{ij}|ij\rangle\langle ij|,$$

where $c_{21} = c_{32} = c_{43} = \ldots = c_{d,d-1} = c_{1d} = 1,\ c_{12} = c_{23} = c_{34} = \ldots = c_{d-1,d} = c_{d1} = a^2$ and the other c_{ij}s satisfy either of the conditions (6). Here we only consider the state ρ_b satisfying the condition (a). In the following we will show that ρ_b is an edge bound entangled state by using range criterion. Obviously, every vector in the range of ρ_b can be written as

$$\begin{aligned} V = (&A_{11},\ aA_{12},\ A_{13},\ \ldots,\ A_{1,d-1},\ A_{1d},\\ &A_{12},\ A_{22},\ aA_{23},\ \ldots,\ A_{2,d-1},\ A_{2d},\\ &A_{13},\ A_{23},\ A_{33},\ \ldots,\ A_{3,d-1},\ A_{3d},\\ &\ldots \quad \ldots\\ &A_{1,d-1},\ A_{2,d-1},\ A_{3,d-1},\ \ldots,\\ &A_{d-1,d-1},\ aA_{d-1,d},\ aA_{1d},\\ &A_{2d},\ A_{3d},\ \ldots,\ A_{d-1,d},\ A_{dd}), \end{aligned} \tag{7}$$

where $A_{ij} \in \mathcal{C}$ for any i and j. While every vector in the range of $\rho_b^{T_B}$ (i.e. ρ_a) takes the form

$$\begin{aligned} V = (&A,\ A_{12},\ A_{13},\ \ldots,\ A_{1,d-1},\ A_{1d},\\ &A_{21},\ A,\ A_{23},\ \ldots,\ A_{2,d-1},\ A_{2d},\\ &A_{31},\ A_{32},\ A,\ \ldots,\ A_{3,d-1},\ A_{3d},\\ &\ldots \quad \ldots\\ &A_{d-1,1},\ A_{d-1,2},\ A_{d-1,3},\ \ldots,\ A,\ A_{d-1,d},\\ &A_{1d},\ A_{2d},\ A_{d3},\ \ldots,\ A_{d,d-1},\ A) \end{aligned} \tag{8}$$

where $A,\ A_{ij} \in \mathcal{C}$ for any i and j. In the next step one will use the divide and conquer technique from Ref.[8] generally. However, the divide and conquer technique does not work so well as there are no zeros to start with. Fortunately Clarisse[27] gives us a good way to solve it. Now we are going to show directly that if the product vector

$$V = (\xi_1,\ \xi_2,\ \ldots,\ \xi_d) \otimes (\eta_1,\ \eta_2,\ \ldots,\ \eta_d)$$

belongs to the range of ρ_b, then $V' = (\xi_1,\ \xi_2,\ \ldots,\ \xi_d) \otimes (\eta_1^*,\ \eta_2^*,\ \ldots,\ \eta_d^*)$ does not belong to the range of $\rho_b^{T_B}$. Comparing Eqs.(7) and (8) with their product form we get the following equations:

$$\begin{aligned} &\xi_1\eta_2 = a\xi_2\eta_1,\ \xi_1\eta_3 = \xi_3\eta_1,\ \ldots,\ a\xi_1\eta_d = \xi_d\eta_1,\\ &\xi_2\eta_3 = a\xi_3\eta_2,\ \xi_2\eta_4 = \xi_4\eta_2,\ \ldots,\ \xi_2\eta_d = \xi_d\eta_2,\\ &\ldots \quad \ldots\\ &\xi_{d-2}\eta_{d-1} = a\xi_{d-1}\eta_{d-2},\ \xi_{d-2}\eta_d = \xi_d\eta_{d-2},\\ &\xi_{d-1}\eta_d = a\xi_d\eta_{d-1}, \end{aligned} \tag{9}$$

and

$$\xi_1\eta_1^* = \xi_2\eta_2^* = \xi_3\eta_3^* = \ldots = \xi_d\eta_d^*. \tag{10}$$

We consider the following subcases:

(1) $\xi_1\eta_1 \neq 0$, and it follows that all parameters are different from zero. Without loss of generality, set $\xi_2 = 1$ then Eq.(10) implies

$$\eta_2 = \xi_1^*\eta_1 = \xi_3^*\eta_3 = \ldots = \xi_d^*\eta_d. \tag{11}$$

On one hand, applying Eq.(11) and $\xi_2 = 1$ to $\xi_1\eta_2 = a\xi_2\eta_1,\ \xi_2\eta_3 = a\xi_3\eta_2,\ \xi_2\eta_4 = \xi_4\eta_2,\ \ldots,\ \xi_2\eta_d = \xi_d\eta_2$ in Eq.(9) we get

$$\begin{aligned} &|\xi_1| = \sqrt{a},\ |\xi_3|^2 = \frac{1}{\sqrt{a}},\\ &|\xi_4| = 1,\ \ldots, |\xi_d| = 1. \end{aligned} \tag{12}$$

Obviously, $\xi_{d-2}\eta_d = \xi_d\eta_{d-2}$ implies that $|\xi_{d-2}| \cdot |\eta_d| = |\xi_d| \cdot |\eta_{d-2}|$. Thus from Eq.(12) we get $|\eta_{d-2}| = |\eta_d|$. On the other hand, from the last three equations of Eq.(9) one can obtain $\xi_{d-2}\eta_{d-1} = a\xi_{d-1}\eta_{d-2},\ \xi_{d-2}^*\eta_d^* = \xi_d^*\eta_{d-2}^*,\ \xi_{d-1}\eta_d = a\xi_d\eta_{d-1}$. Consider the products of the three equations we get the relation $|\xi_{d-2}|^2|\eta_d|^2\xi_{d-1}\eta_{d-1} = a^2|\xi_d|^2|\eta_{d-2}|^2\xi_{d-1}\eta_{d-1}$. It follows that $|\eta_d| = a|\eta_{d-2}|$ from the assumption and $|\xi_{d-2}| = |\xi_d| = 1$. Thus $a = 1$ which contradicts $a > 1$.

(2) $\xi_1 = \eta_1 = 0$, in this case it is straightforward to obtain that $\xi_i = 0$ or $\eta_i = 0$ from Eq.(9) for any i.

(3) $\xi_1 \neq 0,\ \eta_1 = 0$, following from the first d equations of Eq.(9) we can get $\eta_i = 0$ for any i.

(4) $\xi_1 = 0,\ \eta_1 \neq 0$, similar to the Subcase (3) we obtain $\xi_i = 0$ for any i.

Thus ρ_b is a bound entangled state. Similarly, one can show that ρ_b satisfying the condition (b) of Eq.(6) is an edge bound entangled state. Thus we construct a class of edge bound entangled Werner-like states.

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References

[1] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895

[2] Fuchs C A, Gisin N, Griffiths R B, Niu C S and Peres A 1997 *Phys. Rev.* A **56** 1163

[3] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881

[4] DiVincenzo D P 1995 *Science* **270** 255

[5] Werner R F 1989 *Phys. Rev.* A **40** 4277

[6] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett.* A **223** 1

[7] Peres A 1996 *Phys. Rev. Lett.* **77** 1413

[8] Horodecki P 1997 *Phys. Lett.* A **232** 333

[9] Woronowicz S L 1976 *Rep. Math. Phys.* **10** 165

[10] Horodecki P and Lewenstein M 2000 *Phys. Rev. Lett.* **85** 2657

[11] Bruss D and Peres A 2000 *Phys. Rev.* A **61** 030301(R)

[12] Piani M 2006 *Phys. Rev.* A **73** 012345

[13] Fei S M, Li Jost X Q and Sun B Z 2006 *Phys. Lett.* A **352** 321

[14] Piani M and Mora C E 2007 *Phys. Rev.* A **75** 012305

[15] Yu S and Liu N L 2005 *Phys. Rev. Lett.* **95** 150504

[16] Chruściński D and Kossakowski A 2006 *Phys. Rev.* A **74** 022308

[17] Bennett C H, DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 1999 *Phys. Rev. Lett.* **82** 5385

[18] Rudolph O 2005 *Quantum Inf. Proc.* **4** 219

[19] Chen K and Wu L A 2003 *Quantum Inf. Comput.* **3** 193

[20] Choi M D 1975 *Linear Algebra Appl.* **12** 95

[21] Tanahashi K and Tomiyama J 1988 *Canad. Math. Bull.* **31** 308

[22] Osaka H 1993 *Linear Algebra Appl.* **186** 45

[23] Horodecki M, Horodecki P and Horodecki R 2001 *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments, Springer Tracts in Modern Physics* (Berlin: Springer) p167

[24] Størmer E 1963 *Acta Math.* **110** 233

[25] Chruściński D and Kossakowski A 2007 *Open Sys. Inf. Dyn.* **14** 275

[26] Ha K C and Kye S H 2005 *J. Phys.* A: *Math. Gen.* **38** 9039

[27] Clarisse L 2006 *Phys. Lett.* A **359** 603