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Explicit solutions to some nonlinear physical models by a two-step ansatz*

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Explicit solutions are derived for some nonlinear physical model equations by using a delicate way of two-step ansatz method.

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1. Introduction

The nonlinear differential equations (NDEs) in mathematical physics are a major subject in many fields of physics and nonlinear sciences, and seeking exact solutions to NDEs is now playing a more and more important role in understanding these fields. To deal with these problems, various methods have been proposed, such as inverse scattering theory,[1] Hirota bilinear transformations,[2] truncated Painlevé expansion,[3] homogeneous method[4,5] and direct ansatz method.[6−10] For the last method many attempts such as Jacobi elliptic function expansion[11,12] and sine–cosine method[13] have been used but there appears no general rule.

In this paper, we intend to develop a two-step ansatz and apply this method to some nonlinear differential equations and coupled ones for obtaining new types of solutions.

Roughly speaking, the general idea of this ansatz method can be described below. For example, if a given nonlinear partial differential governing equation of physical interest is written as

$$F(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0, \tag{1}$$

where F is a polynomial in its variables, and its travelling wave solution $u(x,t) = u(\xi)$, where $\xi = x - \lambda t$ with λ denoting the wave speed, is considered, then Eq.(1) will become an ordinary differential equation

$$G(u, u', u'', ...) = 0. \tag{2}$$

First, our method assumes that the solution to Eq.(2) can be obtained by using the following two-step ansatz:

$$u(\xi) = \sum_{i=0}^{p} h_i z^{2i}(\xi), \tag{3}$$

and

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\xi}\right)^2 = lz^2(\xi) + mz^4(\xi) + nz^6(\xi), \tag{4}$$

in which $h_i (i = 0, 1, \ldots, p)$, l, m, n are all real constants, and p is a positive integer to be determined. Some particular solutions of Eq.(4) are listed in Table 1. It should be noted that a similar ansatz is employed in Refs.[14,15].

Then, we substitute expression(3) and Eq.(4) into Eq.(2) and equate the coefficients of powers of $z(\xi)$ to zero. Thus, we obtain a group of algebraic equations for unknowns $l, m, n, \lambda, h_i (i = 0, 1, \ldots, p)$.

Finally, we use Mathematica to solve the above algebraic equations and know l, m, n, λ, h_i $(i = 0, 1, \ldots, p)$. Hence, we have the solutions of Eqs.(2) and (1).

In what follows, we present some applications of this method to single and coupled NDEs.

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Table 1. Solutions of Eq. (4) with $\Delta = m^2 - 4ln$ and $\epsilon = \pm 1$.

l	m	n	Δ	$z(\xi)$
>0				$\mathrm{sech}(\sqrt{l}\xi)\Big(-\frac{lm}{m^2-ln(1+\epsilon\ \tanh(\sqrt{l}\xi))^2}\Big)^{\frac{1}{2}}$
>0				$\mathrm{csch}(\sqrt{l}\xi)\Big(-\frac{lm}{m^2-ln(1+\epsilon\ \coth(\sqrt{l}\xi))^2}\Big)^{\frac{1}{2}}$
>0			>0	$\Big(\frac{2l}{\epsilon\sqrt{\Delta}\ \cosh(2\sqrt{l}\xi)-m}\Big)^{\frac{1}{2}}$
<0			>0	$\Big(\frac{2l}{\epsilon\sqrt{\Delta}\ \cos(2\sqrt{-l}\xi)-m}\Big)^{\frac{1}{2}}$
>0			<0	$\Big(\frac{2l}{\epsilon\sqrt{-\Delta}\ \sinh(2\sqrt{l}\xi)-m}\Big)^{\frac{1}{2}}$
<0			>0	$\Big(\frac{2l}{\epsilon\sqrt{\Delta}\ \sin(2\sqrt{-l}\xi)-m}\Big)^{\frac{1}{2}}$
>0		>0		$\mathrm{sech}(\sqrt{l}\xi)\Big(-\frac{l}{m+2\epsilon\sqrt{ln}\ \tanh(\sqrt{l}\xi)}\Big)^{\frac{1}{2}}$
<0		>0		$\sec(\sqrt{-l}\xi)\Big(-\frac{l}{m+2\epsilon\sqrt{-ln}\ \tan(\sqrt{-l}\xi)}\Big)^{\frac{1}{2}}$
>0		>0		$\mathrm{csch}(\sqrt{l}\xi)\Big(\frac{l}{m+2\epsilon\sqrt{ln}\ \coth(\sqrt{l}\xi)}\Big)^{\frac{1}{2}}$
<0		>0		$\csc(\sqrt{-l}\xi)\Big(-\frac{l}{m+2\epsilon\sqrt{-ln}\ \tan(\sqrt{-l}\xi)}\Big)^{\frac{1}{2}}$
>0			0	$\Big(-\frac{l}{m}(1+\epsilon\ \tanh(\frac{\sqrt{l}}{2}\xi))\Big)^{\frac{1}{2}}$
>0			0	$\Big(-\frac{l}{m}(1+\epsilon\ \coth(\frac{\sqrt{l}}{2}\xi))\Big)^{\frac{1}{2}}$
>0				$4\Big(\frac{l\mathrm{e}^{2\epsilon\sqrt{l}\xi}}{(\mathrm{e}^{2\epsilon\sqrt{l}\xi}-4m)^2-64ln}\Big)^{\frac{1}{2}}$
>0	0			$4\Big(\frac{\pm l\mathrm{e}^{2\epsilon\sqrt{l}\xi}}{1-64ln\mathrm{e}^{4\epsilon\sqrt{l}\xi}}\Big)^{\frac{1}{2}}$
>0		0		$-\frac{2\sqrt{l}\mathrm{e}^{\sqrt{l}\xi}}{\mathrm{e}^{2\sqrt{l}\xi}-m}$
<0	>0	0		$\pm\sqrt{\frac{-l}{m}}\ \sec(\sqrt{-l}\xi)$

2. Application of the method to some coupled equations

Here we will use the above-mentioned method to construct our simple ansatz between the unknown functions to obtain explicit solutions to the following three coupled equations:

(a) The coupled Korteweg–de Vries KdV equations[16]

$$u_t + 6\alpha u u_x - 2bvv_x + \alpha u_{xxx} = 0, \tag{5}$$

and

$$v_t + 3\beta u v_x + \beta v_{xxx} = 0, \tag{6}$$

which are derived to model the interaction of water waves. Here α and β are constants.

(b) The generalized nonlinearly coupled Langmuir and dust-acoustic wave equations

$$3aE'' = \lambda E + GE + rG^2E, \tag{7}$$

and

$$G'' = a^2 fG + apG^2 + aqG^3 - aE^2, \tag{8}$$

where λ, r, a, f, p and q are specific physical parameters. When $r = 0$, Eqs.(7) and (8) are reduced to the case in Ref.[17].

(c) The generalized Hirota–Satsuma coupled equations[18]

$$u_t = \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \tag{9}$$

$$v_t = -v_{xxx} + 3uv_x, \tag{10}$$

and

$$w_t = -w_{xxx} + 3uw_x. \tag{11}$$

In the following subsections, explicit and new solutions of these coupled equations will be given.

2.1. Exact solutions to Eqs.(5) and (6)

Here we consider the travelling wave solution in the form of $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$, and $\xi =$

$x-\lambda t$.

Let

$$u=a_0+a_1v+a_2v^2, \tag{12}$$

in which a_0, a_1 and a_2 are undetermined real constants, then,

$$u''=2a_2v'^2+(2a_2v+a_1)v'', \tag{13}$$

In what follows we choose the nonzero integration constants in order to take into account more possible cases. Inserting Eq.(12) into Eq.(6) and integrating the resulting equation once yield

$$v''=e_1+\frac{\lambda-3a_0\beta}{\beta}v-\frac{3}{2}a_1v^2-a_2v^3, \tag{14}$$

and

$$v'^2=d+2e_1v+\frac{\lambda-3a_0\beta}{\beta}-a_1v^3-\frac{a_2}{2}v^4, \tag{15}$$

where e_1 and d are integration constants. Let

$$v(\xi)=\sum_{i=0}^{p}h_iz^{2i}(\xi),$$

where $z(\xi)$ satisfies Eq.(4).

Balancing v'' with v^3 yields $p=1$. Therefore, we have

$$v(\xi)=h_0+h_1z^2(\xi). \tag{16}$$

From Eq.(4), we know

$$zz''=lz^2+2mz^4+3nz^6. \tag{17}$$

Inserting Eqs.(4) and (17) into Eq.(16) yields

$$v''=4h_1lz^2+6h_1mz^4+8h_1nz^6. \tag{18}$$

From Eqs.(14), (16) and (18), we balance the coefficients of the same power of v and obtain the following algebraic equations:

$$-a_2h_1^3=8h_1n,$$

$$e_1+h_0\frac{\lambda-3a_2\beta}{\beta}-\frac{3}{2}a_1h_0^2-a_2h_0^3=0,$$

$$\frac{h_1}{\beta}(\lambda-3a_0\beta)-3a_1h_0h_1-3a_2h_0^2h_1=4h_1l,$$

and

$$-\frac{3}{2}a_1h_1^2-3a_2h_0h_1^2=6h_1m \tag{19}.$$

Keeping in mind that $u(x,t)=u(\xi), v(x,t)=v(\xi)$ and $\xi=x-\lambda t$, and integrating Eq.(5) once give

$$-\lambda u+3\alpha u^2-bv^2+\alpha u''=e_2, \tag{20}$$

where e_2 is an integration constant. Substituting Eqs.(12), (13), (14) and (15) into Eq.(20) arrives at

$$-\lambda a_0+3\alpha a_0^2+a_1\alpha e_1+2\alpha a_2d=e_2,$$

$$-\lambda a_1+6\alpha a_0a_1+6\alpha a_2e_1+a_1\alpha\frac{\lambda-3a_0\beta}{\beta}=0,$$

and

$$-\lambda a_2-b+\frac{3}{2}\alpha a_1^2+6\alpha a_0a_2+4\alpha a_2\frac{\lambda-3a_0\beta}{\beta}=0. \tag{21}$$

Now, we solve Eqs.(19) and (21) by using Mathematica to get the following cases:

Case a.1 $\alpha=\beta$, $a_1=0$, $h_0=0$, $b=a_2(-6a_0\beta+3\lambda)$, $d=\dfrac{a_0\left(-3a_0+\dfrac{\lambda}{\beta}\right)}{2a_2}$, $l=\dfrac{-3a_0\beta-3a_1h_0\beta+\lambda}{4\beta}$, $m=0$, $n=-\dfrac{a_2h_1^2}{8}$ and $a_2\beta\neq0$.

Substituting it with the solutions given by Table 1, we have:

Case a.1.1 when $l>0$,

$$u_{a1}=a_0+\frac{256a_2h_1^2l^2\ \mathrm{e}^{\pm4\sqrt{l}\xi}}{(1-64ln\ \mathrm{e}^{\pm4\sqrt{l}\xi})^2},$$

and

$$v_{a1}=\frac{\pm16h_1l\ \mathrm{e}^{\pm2\sqrt{l}\xi}}{1-64ln\ \mathrm{e}^{\pm4\sqrt{l}\xi}};$$

Case a.1.2 when $l>0$ and $n<0$,

$$u_{a2}=a_0-\frac{4a_2lh_1^2}{n}\ \mathrm{sech}^2(2\sqrt{l}\xi),$$

and

$$v_{a2}=\pm\frac{2h_1l}{\sqrt{-ln}}\ \mathrm{sech}(2\sqrt{l}\xi);$$

Case a.1.3 when $l>0$ and $n>0$,

$$u_{a3}=a_0-\frac{4a_2lh_1^2}{n}\ \mathrm{csch}^2(2\sqrt{l}\xi),$$

$$v_{a3}=\pm\frac{2h_1l}{\sqrt{-ln}}\ \mathrm{csch}(2\sqrt{l}\xi),$$

$$u_{a4}=a_0+\frac{a_2h_1^2l\ \mathrm{sech}^4(\sqrt{l}\xi)}{4n\ \tanh^2(\sqrt{l}\xi)},$$

$$v_{a4}=\frac{-h_1l\ \mathrm{sech}^2(\sqrt{l}\xi)}{\pm2\sqrt{ln}\ \tanh(\sqrt{l}\xi)},$$

$$u_{a5}=a_0+\frac{a_2h_1^2l\ \mathrm{csch}^4(\sqrt{l}\xi)}{4n\ \coth^2(\sqrt{l}\xi)},$$

and

$$v_{a5}=\frac{-h_1l\ \mathrm{csch}^2(\sqrt{l}\xi)}{\pm2\sqrt{ln}\ \coth(\sqrt{l}\xi)}.$$

Case a.2 $\alpha=\beta$, $a_0=0$, $h_0=0$, $2b=3a_1^2\beta+6a_2\lambda$, $d=0$, $l=\dfrac{-3a_1h_0\beta+\lambda}{4\beta}>0$, $m=-\dfrac{a_1h_1}{4}$, $n=-\dfrac{a_2h_1^2}{8}$ and $a_1a_2\beta\neq0$,

$$u_{a6}=\frac{2a_1h_1l}{\pm\sqrt{m^2-4ln}\ \cosh(2\sqrt{l}\xi)-m}+\frac{4a_2h_1^2l^2}{(\pm\sqrt{m^2-4ln}\ \cosh(2\sqrt{l}\xi)-m)^2},$$

and

$$v_{a6}=\frac{2h_1l}{\pm\sqrt{m^2-4ln}\ \cosh(2\sqrt{l}\xi)-m}.$$

Case a.3 $a_1=0$, $b=a_2(-6a_0\alpha+12a_2\alpha-3a_2\beta+4a_2h_0^2\alpha-a_2\beta h_0^2)$, $2d=a_0\left(-\dfrac{3a_0}{a_2}+\dfrac{3\beta}{\alpha}+\dfrac{h_0^2\beta}{\alpha}\right)$, $l=\dfrac{-3a_0-3a_1h_0+3a_2-2a_2h_0^2}{4}>0$, $m=-\dfrac{a_2h_0h_1}{2}$, $n=-\dfrac{a_2h_1^2}{8}$, $\lambda=a_2\beta(3+h_0^2)$ and $\alpha h_0a_2\beta\neq0$.

$$u_{a7}=a_0+a_2h_0^2-\frac{2a_2lmh_0h_1\ \mathrm{sech}^2(\sqrt{l}\xi)}{m^2-ln(1\pm\ \tanh(\sqrt{l}\xi))^2}+\frac{a_2l^2m^2h_1^2\ \mathrm{sech}^4(\sqrt{l}\xi)}{(m^2-ln(1\pm\ \tanh(\sqrt{l}\xi))^2)^2},$$

$$v_{a7}=h_0-\frac{lmh_1\ \mathrm{sech}^2(\sqrt{l}\xi)}{m^2-ln(1\pm\ \tanh(\sqrt{l}\xi))^2},$$

$$u_{a8}=a_0+a_2h_0^2-\frac{2a_2lmh_0h_1\ \mathrm{csch}^2(\sqrt{l}\xi)}{m^2-ln(1\pm\ \coth(\sqrt{l}\xi))^2}+\frac{a_2l^2m^2h_1^2\ \mathrm{csch}^4(\sqrt{l}\xi)}{(m^2-ln(1\pm\ \coth(\sqrt{l}\xi))^2)^2},$$

and

$$v_{a8}=h_0-\frac{lmh_1\ \mathrm{csch}^2(\sqrt{l}\xi)}{m^2-ln(1\pm\ \coth(\sqrt{l}\xi))^2}.$$

Case a.4 $a_1=0$, $b=a_2\left(-6a_0\alpha-\lambda+\dfrac{4\alpha\lambda}{\beta}\right)$, $d=\dfrac{a_0(-3a_0\alpha+\lambda)}{2a_2\alpha}$, $h_0=0$, $l=\dfrac{-3a_0\beta-3a_1h_0\beta+\lambda}{4\beta}>0$, $m=0$, $n=-\dfrac{a_2h_1^2}{8}$, $\alpha\beta a_2\neq0$ and $\alpha\neq\beta$.

$$u_{a9}=a_0+\frac{256a_2l^2h_1^2\ \mathrm{e}^{\pm4\sqrt{l}(\xi)}}{(1-64ln\ \mathrm{e}^{\pm4\sqrt{l}(\xi)})^2},$$

and

$$v_{a9}=\pm\frac{16lh_1\ \mathrm{e}^{\pm2\sqrt{l}(\xi)}}{1-64ln\ \mathrm{e}^{\pm4\sqrt{l}(\xi)}};$$

Case a.5 $\alpha=\beta$, $a_0=0$, $a_2=0$, $2b=3a_1^2\beta$, $h_0=0$, $l=\dfrac{-3a_1h_0\beta+\lambda}{4\beta}$, $m=-\dfrac{a_1h_1}{4}$, $n=0$ and $a_1\neq0$.

Case a.5.1 when $l>0$,

$$u_{a10}=\frac{4a_1h_1l\ \mathrm{e}^{2\sqrt{l}\xi}}{(\mathrm{e}^{2\sqrt{l}\xi}-m)^2},$$

and

$$v_{a10}=\frac{4h_1l\ \mathrm{e}^{2\sqrt{l}\xi}}{(\mathrm{e}^{2\sqrt{l}\xi}-m)^2};$$

Case a.5.2 when $l<0$ and $m>0$,

$$u_{a11}=-\frac{a_1lh_1}{m}\ \sec^2(\sqrt{-l}\xi),$$

and

$$v_{a11}=-\frac{lh_1}{m}\ \sec^2(\sqrt{-l}\xi).$$

Case a.6 $a_0=0$, $a_2=0$, $\alpha=\beta$, $2b=3a_1^2\alpha$, $2\lambda=3a_1h_0\alpha$, $l=-\dfrac{3a_1h_0}{8}$, $m=-\dfrac{a_1h_1}{4}$, $n=0$ and $\alpha a_1h_0\neq0$.

Case a.6.1 when $l>0$,

$$u_{a12}=h_0a_1+\frac{4a_1h_1l\ \mathrm{e}^{2\sqrt{l}\xi}}{(\mathrm{e}^{2\sqrt{l}\xi}-m)^2},$$

and

$$v_{a12}=h_0+\frac{4h_1l\ \mathrm{e}^{2\sqrt{l}\xi}}{(\mathrm{e}^{2\sqrt{l}\xi}-m)^2};$$

Case a.6.2 when $l<0$ and $m>0$,

$$u_{a13}=a_1h_0-\frac{a_1lh_1}{m}\ \sec^2(\sqrt{-l}\xi),$$

and

$$v_{a13}=h_0-\frac{lh_1}{m}\ \sec^2(\sqrt{-l}\xi).$$

Case a.7 $a_0=0$, $a_2=0$, $2b=3a_1^2\alpha$, $h_0=0$, $\lambda=-\dfrac{3a_0\beta\alpha}{\alpha-\beta}$, $l=-\dfrac{3a_1h_0}{4}$, $m=-\dfrac{a_1h_1}{4}$, $n=0$, $\alpha\beta a_1\neq0$ and $\alpha\neq\beta$.

Case a.7.1 when $l>0$,

$$u_{a14}=\frac{4a_1h_1l\ \mathrm{e}^{2\sqrt{l}\xi}}{(\mathrm{e}^{2\sqrt{l}\xi}-m)^2},$$

and

$$v_{a14}=\frac{4h_1l\ \mathrm{e}^{2\sqrt{l}\xi}}{(\mathrm{e}^{2\sqrt{l}\xi}-m)^2};$$

Case a.7.2 when $l<0$ and $m>0$,

$$u_{a15}=-\frac{a_1lh_1}{m}\ \sec^2(\sqrt{-l}\xi),$$

and

$$v_{a15}=-\frac{lh_1}{m}\ \sec^2(\sqrt{-l}\xi).$$

The coupled KdV (a) with $\alpha=-a$ and $\beta=1$ was proposed by Hirota and Satsuma.[16] Since then it has attracted the attention of some authors. In Refs.[19,20] the authors obtained the Lax pair of the

CKdVE (a) with $\alpha = -a$ and $\beta = 1$ for $a = \frac{1}{2}$. Lu[21] proved that when $\alpha = -1$ and $\beta = 1$, Eqs.(3) and (4) have N-soliton solutions for all b. Feng[22] established the global well-posedness of the initial value problems for the CKdVE. Recently, the bell-shaped soliton solutions to the CKdVE with $\alpha = \beta$, $b = 3$ and $\alpha = -a$, $\beta = 1$ were obtained in Ref.[4, 5, 23] by using the so-called homogeneous balance method and the ansatz $u = A + Bv, v = My/(1+y)^2$ and $y = \mathrm{e}^{k\xi}$, respectively. In Ref.[24] the bell-shaped solution for u and the kink-type solution for v were found by using the ansatz method. Our ansatz differs from those used formerly, and can produce the existing types of solutions as well as some new ones such as u_{a1}, v_{a1}, u_{a10}, v_{a10}, u_{a11} and v_{a11}.

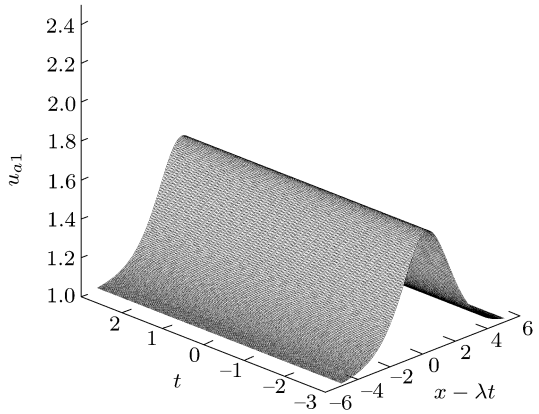


Fig.1. Travelling wave solution u_{a1} of Case (a.1.1) with $\lambda = -3$, $l = 0.25$, $a_0 = 1$, $n = -0.05$, $h_1 = 0.75$ and $a_2 = 2/9$.

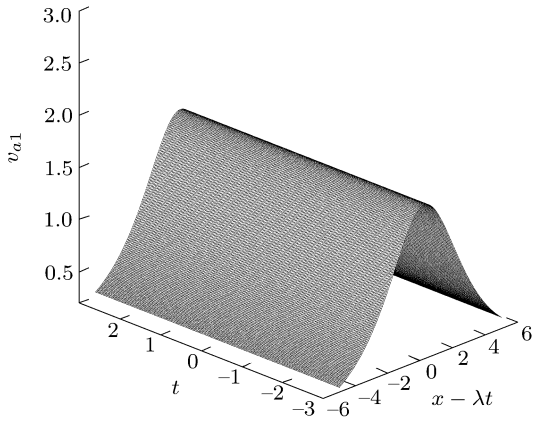


Fig.2. Travelling wave solution v_{a1} of Case (a.1.1) with $\lambda = -3$, $l = 0.25$, $a_0 = 1$, $n = -0.05$, $h_1 = 0.75$ and $a_2 = 2/9$.

In order to display the property of the new solutions, (u_{a1}, v_{a1}) in Case (a.1.1) is given in Figs.1 and 2. Here, $\lambda = -3$, $l = 0.25$, $a_0 = 1$, $n = -0.05$, $h_1 = 0.75$ and $a_2 = 2/9$. From these figures, it can be seen that (u_{a1}, v_{a1}) behaves like a standard bell-shaped solitary wave.

2.2. Exact solutions to Eqs.(7) and (8)

In order to solve Eqs.(7) and (8), we assume

$$E^2 = bG + cG^2 + dG^3, \tag{22}$$

where b, c and d are real constants to be determined. Thus

$$2EE'' + 2(E')^2 = bG'' + 2c(G')^2 + 2cGG'' + 6dG(G')^2 + 3dG^2G''. \tag{23}$$

Inserting Eq.(22) into Eq.(8) yields

$$G'' = (a^2 f - ab)G + (ap - ac)G^2 + (ad - aq)G^3, \tag{24}$$

and

$$(G')^2 = (a^2 f - ab)G^2 + \frac{2}{3}(ap - ac)G^3 + \frac{1}{2}(ad - aq)G^4. \tag{25}$$

We use

$$G(\xi) = \sum_{i=0}^{p} h_i z^{2i}(\xi),$$

where $z(\xi)$ satisfies Eq.(4), and h_i, l, m and n are undetermined. Balancing G'' and G^3, we have $p = 1$, i.e.

$$G(\xi) = h_0 + h_1 z^2(\xi), \tag{26}$$

and

$$G'' = 2h_1(z')^2 + 2h_1 zz''. \tag{27}$$

Inserting Eqs.(4) and (17) into Eq.(27) yields

$$G'' = 4h_1 lz^2 + 6h_1 mz^4 + 8h_1 nz^6. \tag{28}$$

From Eqs.(24), (26) and (28), we balance the coefficients of the same power of $z(\xi)$ and obtain the following algebraic equations:

$$(a^2 f - ab)h_0 + (ap - ac)h_0^2 + (ad - aq)h_0^3 = 0,$$

$$(a^2 f - ab)h_1 + 2(ap - ac)h_0 h_1 + 3(ad - aq)h_0^2 h_1 = 4h_1 l,$$

$$(ap - ac)h_1^2 + 3(ad - aq)h_0 h_1^2 = 6h_1 m,$$

and

$$(ad - aq)h_1^3 = 8h_1 n. \tag{29}$$

From Eq.(22) it follows that

$$E = \pm\sqrt{bG + cG^2 + dG^3}. \tag{30}$$

Then

$$(E')^2 = \frac{(b + 2cG + 3dG^2)^2 (G')^2}{4(bG + cG^2 + dG^3)}. \tag{31}$$

From Eq.(7) we have

$$3aEE'' = (\lambda + G + rG^2)E^2. \tag{32}$$

Insert Eqs.(22), (24), (25), (31) and (32) into Eq.(23) and balance the coefficients of the same power of G, then we will obtain the following algebraic system:

$$\frac{4\lambda}{3a}b^2 = b^2(a^2 f - ab),$$

$$-6bc(a^2 f - ab) - \frac{4}{3}b^2(ap - ac) + \frac{4b}{3a}(\lambda c + b) + \frac{4\lambda}{3a}bc = 0,$$

$$\begin{aligned}&(-4c^2 - 14bd)(a^2 f - ab) - 6bc(ap - ac)\\ &- \frac{3}{2}b^2(aq - ad) + \frac{4b}{3a}(\lambda d + c + rb)\\ &+ \frac{4c}{3a}(\lambda c + b) + \frac{4\lambda}{3a}bd = 0,\end{aligned}$$

$$\begin{aligned}&- 14cd(a^2 f - ab) - (4c^2 + 12bd)(ap - ac)\\ &- 6bc(aq - ad) + \frac{4b}{3a}(d + rc)\\ &+ \frac{4c}{3a}(\lambda d + c + rb) + \frac{4d}{3a}(\lambda c + b) = 0,\end{aligned}$$

$$\begin{aligned}&- 9d^2(a^2 f - ab) - \frac{38}{3}cd(ap - ac)\\ &- (4c^2 + 11bd)(aq - ad) + \frac{4}{3a}bdr\\ &+ \frac{4c}{3a}(d + rc) + \frac{4d}{3a}(\lambda d + c + rb) = 0,\end{aligned}$$

$$-8d^2(ap - ac) - 12cd(aq - ad) + \frac{4}{3a}rcd + \frac{4d}{3a}(d + rc) = 0,$$

and

$$\frac{4}{3a}rd^2 = \frac{15}{2}d^2(aq - ad). \tag{33}$$

By solving Eqs.(29) and (33), we can have lots of solutions, for instance,

$$l_{b1} = \frac{h_0(1 - 6a^2 h_0 q)}{12a}, \quad m_{b1} = \frac{h_1(1 - 9a^2 h_0 q)}{18a},$$
$$n_{b1} = -\frac{a h_1^2 q}{8};$$

$$l_{b2} = \frac{h_0(15 - 32h_0 r)}{360a}, \quad m_{b2} = \frac{h_1(5 - 16h_0 r)}{180a},$$
$$n_{b2} = -\frac{r h_1^2}{45a};$$

$$l_{b3} = \frac{h_0(9 - 16h_0 r)}{36a}, \quad m_{b3} = \frac{h_1(3 - 8h_0 r)}{18a},$$
$$n_{b3} = -\frac{r h_1^2}{9a}.$$

According to Table 1, we can obtain many groups of exact solutions of Eqs.(7) and (8) related to these three groups of solutions of coefficients l, m and n.

2.3. Exact travelling wave solutions to Eqs.(9), (10) and (11)

We apply the same method to Eqs.(9), (10) and (11) and obtain lots of exact solutions. For instance, when $l = \frac{ad^2h_0^2}{2}$, $m = \frac{ad^2h_0h_1}{6}$, $n = \frac{ad^2h_1^2}{8}$, $9c = -2ad^2h_0^2$, $5a^2d^3h_0^2 = 9$, $3\lambda = -ad^2h_0^2$, $a > 0$, $d > 0$, we have

$$w_{c1} = h_0\frac{17 \pm 18\tanh(\sqrt{l}\xi) - 3\tanh^2(\sqrt{l}\xi)}{(3\tanh(\sqrt{l}\xi) \pm 5)(3\tanh(\sqrt{l}\xi) \pm 1)},$$

$$v_{c1} = dh_0\frac{17 \pm 18\tanh(\sqrt{l}\xi) - 3\tanh^2(\sqrt{l}\xi)}{(3\tanh(\sqrt{l}\xi) \pm 5)(3\tanh(\sqrt{l}\xi) \pm 1)},$$

$$u_{c1} = -\frac{3dh_0^2(17 \pm 18\tanh(\sqrt{l}\xi) - 3\tanh^2(\sqrt{l}\xi))^2}{5\lambda(3\tanh(\sqrt{l}\xi) \pm 5)^2(3\tanh(\sqrt{l}\xi) \pm 1)^2} + \frac{2\lambda}{3};$$

$$w_{c2} = h_0\frac{-7 \pm 18\coth(\sqrt{l}\xi) + 21\coth^2(\sqrt{l}\xi)}{(3\coth(\sqrt{l}\xi) \pm 5)(3\coth(\sqrt{l}\xi) \pm 1)},$$

$$v_{c2} = dh_0\frac{-7 \pm 18\coth(\sqrt{l}\xi) + 21\coth^2(\sqrt{l}\xi)}{(3\coth(\sqrt{l}\xi) \pm 5)(3\coth(\sqrt{l}\xi) \pm 1)},$$

$$u_{c2} = -\frac{3dh_0^2(-7 \pm 18\coth(\sqrt{l}\xi) + 21\coth^2(\sqrt{l}\xi))^2}{5\lambda(3\coth(\sqrt{l}\xi) \pm 5)^2(3\coth(\sqrt{l}\xi) \pm 1)^2} + \frac{2\lambda}{3};$$

$$w_{c3} = h_0\frac{\pm 2\sqrt{2}\sinh(2\sqrt{l}\xi) + 5}{\pm 2\sqrt{2}\sinh(2\sqrt{l}\xi) - 1},$$

$$v_{c3} = dh_0\frac{\pm 2\sqrt{2}\sinh(2\sqrt{l}\xi) + 5}{\pm 2\sqrt{2}\sinh(2\sqrt{l}\xi) - 1},$$

$$u_{c3} = -\frac{3dh_0^2(\pm 2\sqrt{2}\sinh(2\sqrt{l}\xi) + 5)^2}{5\lambda(\pm 2\sqrt{2}\sinh(2\sqrt{l}\xi) - 1)^2} + \frac{2\lambda}{3};$$

$$w_{c4} = h_0\frac{3\tanh^2(\sqrt{l}\xi) \pm 3\tanh(\sqrt{l}\xi) - 2}{1 \pm 3\tanh(\sqrt{l}\xi)},$$

$$v_{c4} = dh_0\frac{3\tanh^2(\sqrt{l}\xi) \pm 3\tanh(\sqrt{l}\xi) - 2}{1 \pm 3\tanh(\sqrt{l}\xi)},$$

$$u_{c4} = -\frac{3dh_0^2(3\tanh^2(\sqrt{l}\xi) \pm 3\tanh(\sqrt{l}\xi) - 2)^2}{5\lambda(1 \pm 3\tanh(\sqrt{l}\xi))^2} + \frac{2\lambda}{3};$$

$$w_{c5} = h_0\frac{-3\coth^2(\sqrt{l}\xi) \pm 3\coth(\sqrt{l}\xi) + 4}{1 \pm 3\coth(\sqrt{l}\xi)},$$

$$v_{c5} = dh_0\frac{-3\coth^2(\sqrt{l}\xi) \pm 3\coth(\sqrt{l}\xi) + 4}{1 \pm 3\coth(\sqrt{l}\xi)},$$

$$u_{c5} = -\frac{3dh_0^2(-3\coth^2(\sqrt{l}\xi) \pm 3\coth(\sqrt{l}\xi) + 4)^2}{5\lambda(1 \pm 3\coth(\sqrt{l}\xi))^2} + \frac{2\lambda}{3};$$

$$w_{c6} = h_0\frac{9\mathrm{e}^{\pm 4\sqrt{l}\xi} + 60\mathrm{e}^{\pm 2\sqrt{l}\xi} - 32}{(3\mathrm{e}^{\pm 2\sqrt{l}\xi} - 8)(3\mathrm{e}^{\pm 2\sqrt{l}\xi} + 4)},$$

$$v_{c6} = dh_0\frac{9\mathrm{e}^{\pm 4\sqrt{l}\xi} + 60\mathrm{e}^{\pm 2\sqrt{l}\xi} - 32}{(3\mathrm{e}^{\pm 2\sqrt{l}\xi} - 8)(3\mathrm{e}^{\pm 2\sqrt{l}\xi} + 4)},$$

$$u_{c6} = -\frac{3dh_0^2(9\mathrm{e}^{\pm 4\sqrt{l}\xi} + 60\mathrm{e}^{\pm 2\sqrt{l}\xi} - 32)^2}{5\lambda(3\mathrm{e}^{\pm 2\sqrt{l}\xi} - 8)^2(3\mathrm{e}^{\pm 2\sqrt{l}\xi)} + 4)^2} + \frac{2\lambda}{3}.$$

Here u_{c6}, v_{c6} and w_{c6} are new solutions and the others are similar to those in Ref.[7].

3. Application of the method to some single equations

Here we will use this method to solve two kinds of single partial nonlinear differential equations:

(d) The combined KdV–MKdV equation:

$$u_t + \mu u u_x + \delta u^2 u_x + u_{xxx} = 0, \tag{34}$$

where μ and δ are specific parameters. This equation may describe the wave propagation of the bound particle, sound wave and thermal pulse.[25−27]

(e) The (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawda equation:

$$36u_t + u_{xxxxx} + 15(uu_{xx})_x + 45u^2u_x - 5u_{xxy} - 15uu_y - 15u_x\partial_x^{-1}(u_y) - 5\partial_x^{-1}(u_{yy}) = 0, \tag{35}$$

which was first proposed by Konopelchenk and Dubovsky. Much work about this equation has been done.[28−30]

3.1. Exact solutions to equation (34)

For Eq.(34), let $u(\xi) = u(x,t)$ and $\xi = x - \lambda t$, then we will have

$$-\lambda u + \frac{\mu}{2}u^2 + \frac{\delta}{3}u^3 + u'' = 0. \tag{36}$$

Let

$$u(\xi) = \sum_{i=0}^{p} h_i z^{2i}(\xi),$$

where $z(\xi)$ satisfies Eq.(4).

Balancing u'' with u^3 yields $p = 1$. Therefore, we have

$$u(\xi) = h_0 + h_1 z^2(\xi). \tag{37}$$

Inserting Eqs.(4) and (17) into Eq.(37) yields

$$u'' = 4h_1 l z^2 + 6h_1 m z^4 + 8h_1 n z^6. \tag{38}$$

From Eqs.(36), (37) and (38), we balance the coefficients of the same power of z and obtain the following algebraic equations:

$$-6\lambda h_0 + 3\mu h_0^2 + 2\delta h_0^3 = 0,$$

$$-\lambda h_1 + \mu h_0 h_1 + \delta h_0^2 h_1 + 4h_1 l = 0,$$

$$\mu h_1^2 + 2\delta h_0 h_1^2 + 12h_1 m = 0,$$

and

$$\delta h_1^3 + 24h_1 n = 0. \tag{39}$$

Solving the above equations by use of Mathematica gives

$$l = -\frac{1}{24}h_0(3\mu + 4\delta h_0),$$

$$m = -\frac{1}{12}(\mu + 2\delta h_0)h_1,$$

$$n = -\frac{\delta h_1^2}{24},$$

and

$$\lambda = \frac{h_0(3\mu + 2\delta h_0)}{6}.$$

Substituting it with the solutions given by Table 1, we have:

Case d.1 when $l > 0$,

$$u_{d1} = h_0 - \frac{lmh_1\ \mathrm{sech}^2(\sqrt{l}\xi)}{m^2 - ln(1 \pm \tanh(\sqrt{l}\xi))^2},$$

$$u_{d2} = h_0 - \frac{lmh_1\ \mathrm{csch}^2(\sqrt{l}\xi)}{m^2 - ln(1 \pm \coth(\sqrt{l}\xi))^2},$$

and

$$u_{d3} = h_0 + \frac{16lh_1 \mathrm{e}^{\pm 2\sqrt{l}\xi}}{(\mathrm{e}^{\pm 2\sqrt{l}\xi} - 4m)^2 - 64ln};$$

Case d.2 when $l > 0$ and $n > 0$,

$$u_{d4} = h_0 - \frac{lh_1\ \mathrm{sech}^2(\sqrt{l}\xi)}{m \pm 2\sqrt{ln}\ \tanh(\sqrt{l}\xi)},$$

and

$$u_{d5} = h_0 - \frac{lh_1\ \mathrm{csch}^2(\sqrt{l}\xi)}{m \pm 2\sqrt{ln}\ \coth(\sqrt{l}\xi)};$$

Case d.3 when $l > 0$ and $\mu + h_0\delta = 0$,

$$u_{d6} = h_0 - \frac{lh_1}{m}\left(1 \pm \tanh\left(\frac{\sqrt{l}\xi}{2}\right)\right),$$

and

$$u_{d7} = h_0 - \frac{lh_1}{m}(1 \pm \coth(\frac{\sqrt{l}\xi}{2}));$$

Case d.4 when $l > 0$ and $\mu + 2h_0\delta = 0$,

$$u_{d8} = h_0 \pm \frac{16lh_1\ \mathrm{e}^{\pm 2\sqrt{l}\xi}}{1 - 64ln\ \mathrm{e}^{\pm 4\sqrt{l}\xi}},$$

where u_{d3} and u_{d8} are new solutions for Eq.(34).

3.2. Exact solutions to equation (35)

For Eq.(35), let $\xi = x + ey + dt, b = 36d - 5\mathrm{e}^2$, then

$$bu + u'''' + 15uu'' + 15u^3 - 5eu'' - 15eu^2 = 0. \tag{40}$$

Assume that

$$u(\xi) = \sum_{i=0}^{p} h_i z^{2i}(\xi),$$

where $z(\xi)$ satisfies Eq.(4).

Balancing u'''', uu'' with u^3 yields $m = 2$. Therefore, we have

$$u(\xi) = h_0 + h_1 z^2(\xi) + h_2 z^4. \tag{41}$$

Then expand u'''', uu'' and u^3 to powers of z, insert them into Eq.(40), and balance the coefficients of the same power of z, and we will obtain the following algebraic equations:

$$bh_0 - 15eh_0^2 + 15h_0^3 = 0,$$

$$bh_1 - 30eh_0h_1 + 45h_0^2h_1 - 20eh_1l + 60h_0h_1l + 16h_1l^2 = 0,$$

$$\begin{aligned} &-15eh_1^2 + 45h_0h_1^2 + bh_2 - 30eh_0h_2 + 45h_0^2h_2 \\ &+ 60h_1^2l - 80eh_2l + 240h_0h_2l + 256h_2l^2 \\ &- 30eh_1m + 90h_0h_1m + 120h_1lm = 0, \end{aligned}$$

$$\begin{aligned} &15h_1^3 - 30eh_1h_2 + 90h_0h_1h_2 + 300h_1h_2l + 90h_1^2m \\ &- 100eh_2m + 300h_0h_2m + 1040h_2lm + 120h_1m^2 \\ &+ 120h_0h_1n + 320h_1ln - 40eh_1n = 0, \end{aligned}$$

$$\begin{aligned} &45h_1^2h_2 - 15eh_2^2 + 45h_0h_2^2 + 240h_2^2l + 390h_1h_2m \\ &+ 840h_2m^2 + 120h_1^2n - 120eh_2n \\ &+ 360h_0h_2n + 1920h_2ln + 480h_1mn = 0, \end{aligned}$$

$$45h_1h_2^2 + 300h_2^2m + 480h_1h_2n + 2688h_2mn + 384h_1n^2 = 0,$$

and

$$15h_2^3 + 360h_2^2n + 1920h_2n^2 = 0. \tag{42}$$

Solving the above equations by use of Mathematica yields

$$\begin{aligned} &l = \frac{1}{8}(5e - 15h_0 \pm \sqrt{5}\sqrt{5e^2 - 18eh_0 + 21h_0^2}), \\ &m = -\frac{h_1}{2}, n = 0, h_2 = 0, \\ &b = 15(eh_0 - h_0^2), \quad \text{and} \quad 4e^2 - 9eh_0 + 6h_0^2 \neq 0. \end{aligned}$$

Substituting it with the solutions given by Table 1, we have

Case e.1 when $l > 0$ and $n = 0$,

$$z_{e1} = -\frac{2\sqrt{l}\ \mathrm{e}^{\sqrt{l}\xi}}{\mathrm{e}^{2\sqrt{l}\xi} + \frac{h_1}{2}},$$

and

$$u_{e1} = h_0 + \frac{4h_1l\ \mathrm{e}^{2\sqrt{l}\xi}}{\left(\mathrm{e}^{2\sqrt{l}\xi} + \frac{h_1}{2}\right)^2};$$

Case e.2 when $l < 0$, $n = 0$ and $h_1 < 0$,

$$z_{e2} = \pm\sqrt{\frac{-l}{m}}\ \sec(\sqrt{-l}\xi),$$

and

$$u_{e2} = h_0 - \frac{lh_1}{m}\ \sec^2(\sqrt{-l}\xi).$$

These two solutions are obtained here for the first time.

4. Remark and discussion

We have presented a direct method of obtaining many new and generalized exact travelling wave solutions for some nonlinear differential equations and coupled ones. It is expected that this method can be applied to many other equations or coupled equations. But in this two-step ansatz method, the second-step ansatz can be used only for even rank equations but not for odd ones.[31] For example, the well-known Burgers–Korteweg–de Vries equations

$$u_t + \alpha uu_x + \beta u_{xx} + \gamma u_{xxx} = 0$$

cannot be solved by using the present method, because its rank is odd. According to Ref.[31], we can assume $u' = f(u)$ or $u = g(w)$, and $w' = f(w)$ to solve this equation.

In summary, based on some simple ansatzes we have generated many explicit solutions to many physical equations. It should be noted that in order to ensure that the results are correct, the algebraic equations here are solved and verified by using Mathematica.

References

[1] Gardner C S, Kruskal J M and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095

[2] Matsuno Y 1984 *Bilinear Transform Method* (London: Academic)

[3] Cariello F and Tabor M 1991 *Physica* D **53** 59

[4] Fan E G and Zhang H Q 1998 *Phys. Lett.* A **246** 403

[5] Wang M L, Zhou Y B and Li Z B 1996 *Phys. Lett.* A **216** 67

[6] Feng X 1996 *Phys. Lett.* A **213** 167
[7] Hu J L 2001 *Phys. Lett.* A **287** 81
[8] Hu J L 2004 *Chin. Phys.* **13** 297
[9] Hu J L 2005 *Chin. Phys.* **14** 455
[10] Hu J L 2007 *Chin. Phys.* **16** 3192
[11] Parkes E J, Duffy B R and Abbott P C 2002 *Phys. Lett.* A **295** 280
[12] Chen Y, Wang Q and Li B 2005 *Chaos, Solitons and Fractals* **26** 231
[13] Yan C T 1996 *Phys. Lett.* A **224** 77
[14] Sirendaoreji 2006 *Phys. Lett.* A **356** 124
[15] Sirendaoreji 2007 *Phys. Lett.* A **363** 440
[16] Hirota R and Satsuma J 1981 *Phys. Lett.* A **85** 407
[17] Singh S V, Rao N N and Shukla P K 1998 *J. Plasma Physics* **60** 551
[18] Fan E G 2001 *Phys. Lett.* A **282** 18
[19] Dodd R and Fordy A 1982 *Phys. Lett.* A **89** 168
[20] Wilson G 1982 *Phys. Lett.* A **89** 332
[21] Lu B Q 1994 *Phys. Lett.* A **189** 25
[22] Feng X 1994 *Math. Manus.* **84** 361
[23] Fan E G and Zhang H Q 1998 *Phys. Lett.* A **245** 389
[24] Lu H J and Wang M X 1999 *Phys. Lett.* A **255** 249
[25] Fan E G and Hon Y C 2003 *Chaos, Solitons and Fractals* **15** 559
[26] Wadati M 1975 *J. Phys. Soc. Jpn.* **38** 673
[27] Mohamad M N B 1992 *Math. Meth. Appl. Sci.* **15** 73
[28] Sawada K and Kotera J 1974 *Prog. Theor. Phys.* **51** 1355
[29] Gibbon J D, Dodd R K and Caudrey P J 1976 *Proc. Roy. Soc. London* A **351** 407
[30] Gibbon J D and Dodd R K 1977 *Proc. Roy. Soc. London* A **358** 287
[31] Feng X 2000 *Int. J. Theor. Phys.* **39** 207