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Conformal invariance and integration of first-order differential equations*

He Guang(何 光)[a)†] and Mei Feng-Xiang(梅凤翔)[b)]

[a)] *School of Aerospace Science and Engineering, Beijing Institute of Technology, Beijing* 100081, *China*
[b)] *School of Science, Beijing Institute of Technology, Beijing* 100081, *China*

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This paper studies a conformal invariance and an integration of first-order differential equations. It obtains the corresponding infinitesimal generators of conformal invariance by using the symmetry of the differential equations, and expresses the differential equations by the equations of a Birkhoff system or a generalized Birkhoff system. If the infinitesimal generators are those of a Noether symmetry, the conserved quantity can be obtained by using the Noether theory of the Birkhoff system or the generalized Birkhoff system.

Keywords: differential equation, conformal invariance, Noether theory, integration
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1. Introduction

The research of symmetry, mainly including Noether symmetry,[1−5] Lie symmetry[2−4,6−11] and form invariance[4,12−17] and so on, and conserved quantity of mechanical system is very important both in theory and application. The conformal invariance of a Birkhoff system has been studied in Ref.[18]. In fact, the conformal invariance means that the equation influenced the infinitesimal operators equals to a conformal multiplier matrix multiplies itself. This paper studies the conformal invariance of first-order differential equations. If the corresponding infinitesimal generators of conformal invariance are those of the Noether symmetry, the integration of the equations would be gotten by finding out the conserved quantity using the Noether theory. In order to judge if the symmetry belongs to Noether's, differential equations should be transformed to those of a mechanical system since the Noether theory of the mechanical system has been already set up.[1−4]

2. Conformal invariance

We study the following first-order differential equations:

$$\dot{a}^{\mu} = \sigma_{\mu}\left(t, \boldsymbol{a}\right) \quad (\mu = 1, \ldots, 2n)\,. \tag{1}$$

The infinitesimal transformations of time t and variable $\boldsymbol{a}$ are introduced in the following form:

$$\begin{aligned} t^{*} &= t + \varepsilon\xi_{0}\left(t\right), \\ a^{\mu *}\left(t^{*}\right) &= a^{\mu}\left(t\right) + \varepsilon\xi_{\mu}\left(t, \boldsymbol{a}\right), \end{aligned} \tag{2}$$

where ε is an infinitesimal parameter, ξ_{0}, ξ_{μ} are infinitesimal generators.

Definition If there is a nonsingular matrix $\mathscr{L}_{\mu}^{\nu}$ which makes the following equation come into existence

$$X^{(1)}\left(\dot{a}^{\mu} - \sigma_{\mu}\right) = \mathscr{L}_{\mu}^{\nu}\left(\dot{a}^{\nu} - \sigma_{\nu}\right), \tag{3}$$

where

$$X^{(1)} = \xi_{0}\frac{\partial}{\partial t} + \xi_{\mu}\frac{\partial}{\partial a^{\mu}} + \left(\dot{\xi}_{\mu} - \dot{a}^{\mu}\dot{\xi}_{0}\right)\frac{\partial}{\partial \dot{a}^{\mu}}, \tag{4}$$

then Eq.(1) has conformal invariance under the infinitesimal transformations Eq.(2).

To study the conformal invariance, firstly the nonsingular matrix $\mathscr{L}_{\mu}^{\nu}$ must be found out, secondly the equations should be integrated.

We Calculate the difference value of the following expression:

$$X^{(1)}\left(\dot{a}^{\mu} - \sigma_{\mu}\right) - X^{(1)}\left(\dot{a}^{\mu} - \sigma_{\mu}\right)|_{\dot{a}^{\mu} - \sigma_{\mu} = 0}, \tag{5}$$

we get

$$\begin{aligned} & X^{(1)}\left(\dot{a}^{\mu} - \sigma_{\mu}\right) \\ &= \dot{\xi}_{\mu} - \dot{a}^{\mu}\dot{\xi}_{0} - X^{(0)}\left(\sigma_{\mu}\right) \\ &= \frac{\partial \xi_{\mu}}{\partial t} + \frac{\partial \xi_{\mu}}{\partial a^{\nu}}\dot{a}^{\nu} - \dot{a}^{\mu}\frac{\partial \xi_{0}}{\partial t} - X^{(0)}\left(\sigma_{\mu}\right), \end{aligned} \tag{6}$$

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†E-mail: heguang@bit.edu.cn

where

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_\mu \frac{\partial}{\partial a^\mu}, \tag{7}$$

since

$$\begin{aligned} & X^{(1)} \left(\dot{a}^\mu - \sigma_\mu\right) |_{\dot{a}^\mu - \sigma_\mu = 0} \\ = & \frac{\partial \xi_\mu}{\partial t} + \frac{\partial \xi_\mu}{\partial a^\nu} \sigma_\nu - \sigma_\nu \frac{\partial \xi_0}{\partial t} - X^{(0)} \left(\sigma_\mu\right), \end{aligned} \tag{8}$$

then we get

$$\begin{aligned} & X^{(1)} \left(\dot{a}^\mu - \sigma_\mu\right) - X^{(1)} \left(\dot{a}^\mu - \sigma_\mu\right) |_{\dot{a}^\mu - \sigma_\mu = 0} \\ = & \frac{\partial \xi_\mu}{\partial a^\nu} \left(\dot{a}^\nu - \sigma_\nu\right) - \frac{\partial \xi_0}{\partial t} \left(\dot{a}^\mu - \sigma_\mu\right) \\ = & \beta_\mu^\nu \left(\dot{a}^\nu - \sigma_\nu\right), \end{aligned} \tag{9}$$

where

$$\beta_\mu^\nu = \frac{\partial \xi_\mu}{\partial a^\nu} - \frac{\partial \xi_0}{\partial t} \delta_\mu^\nu. \tag{10}$$

If the transformations Eq.(2) are the Lie symmetry of Eq.(1), then the following expression is true:

$$\mathscr{L}_\mu^\nu = \beta_\mu^\nu,$$

contrarily, if the transformations Eq.(2) are conformal symmetry of Eq.(1), then we get the following equation via Eq.(9):

$$\left(\mathscr{L}_\mu^\nu - \beta_\mu^\nu\right) \left(\dot{a}^\nu - \sigma_\nu\right) = X^{(1)} \left(\dot{a}^\mu - \sigma_\mu\right) |_{\dot{a}^\mu - \sigma_\mu = 0},$$

then we get the following proposition.

Proposition 1 When the sufficient and necessary condition is that the transformations Eq.(2) are the Lie symmetry of Eq.(1) as well as conformal symmetry, the following expression is true:

$$\mathscr{L}_\mu^\nu = \frac{\partial \xi_\mu}{\partial a^\nu} - \frac{\partial \xi_0}{\partial t} \delta_\mu^\nu. \tag{11}$$

3. Mechanization of differential equations and their integration

In order to integrate differential equations by using conformal symmetry, it is useful to transform them to those of a mechanical system first, and then the analytical mechanics method is feasible. First-order differential equations can be changed into those of Hamilton system under strict restriction conditions. It is very easy to change equations into those of a Birkhoff system or a generalized Birkhoff system. The latter method is easier than the former.

Let Eq.(1) have Birkhoff form, that is to say

$$\begin{aligned} & \dot{a}^\mu - \sigma_\mu \\ = & \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = 0, \end{aligned} \tag{12}$$

from which we get

$$\begin{aligned} \frac{\partial R_\mu}{\partial t} = & \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \sigma_\nu - \frac{\partial B}{\partial a^\mu} \\ & (\mu, \nu = 1, \ldots, 2n). \end{aligned} \tag{13}$$

For a given Birkhoffian B, Eq.(13) has a Cauchy–KovaLevskaya form, and whose solutions always exist.[19,20] But there are still many difficulties in solving R_μ from Eq.(13).

We can change Eq.(1) into those of the generalized Birkhoff system. The following equations of the generalized Birkhoff system are given in Ref.[21].

$$\begin{aligned} & \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = -\Lambda_\mu \\ & (\mu, \nu = 1, \ldots, 2n), \end{aligned} \tag{14}$$

where $\Lambda_\mu = \Lambda_\mu(t, \boldsymbol{a})$ is additional item. It is quite easier to express Eq.(1) into the generalized Birkhoff Eq.(14) than Birkhoff Eq.(12) since there is an adjustable function Λ_μ in Eq.(14).

The Noether identity of the generalized Birkhoff system has form as

$$\begin{aligned} & \left(\frac{\partial R_\mu}{\partial t} \dot{a}^\mu - \frac{\partial B}{\partial t}\right) \xi_0 + \left(\frac{\partial R_\nu}{\partial a^\mu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right) \xi_\mu - B \dot{\xi}_0 \\ & + R_\mu \dot{\xi}_\mu + \Lambda_\mu \left(\xi_\mu - \dot{a}^\mu \xi_0\right) + \dot{G}_N = 0. \end{aligned} \tag{15}$$

If the generators ξ_0, ξ_μ and the gauge function G_N fit Eq.(15), then the system has conserved quantity as

$$I_N = R_\mu \xi_\mu - B \xi_0 + G_N = \text{const}. \tag{16}$$

Then we get

Proposition 2 If the generators ξ_0, ξ_μ of conformal invariance and the gauge function G_N fit Eq.(15) of Noether, then the conformal invariance brings on conserved quantity expression Eq.(16).

4. Example

We study the conformal invariance and integration of Emden equation. The Emden equation as an example is

$$\ddot{q} + \frac{2}{t} \dot{q} + q^5 = 0. \tag{17}$$

Let

$$a^1 = q, \quad a^2 = \dot{q},$$

then we have

$$\dot{a}^1 = a^2, \quad \dot{a}^2 = -\frac{2}{t}a^2 - \left(a^1\right)^5, \tag{18}$$

which can be changed into those of a Birkhoff system.[18] Choosing

$$R_1 = \frac{1}{2}t^2a^2, \quad R_2 = -\frac{1}{2}t^2a^1, \quad B = 0,$$
$$\Lambda_1 = -ta^2 - t^2(a^1)^5, \quad \Lambda_2 = -t^2a^2 - ta^1, \tag{19}$$

We can change Eq.(18) into those of the generalized Birkhoff system. We can validate that the infinitesimal generators

$$\xi_0 = -2t, \xi_1 = a^1, \xi_2 = 3a^2 \tag{20}$$

are that of Lie symmetry of Eq.(18). We obtain the following results by Eq.(11):

$$\mathscr{L}_1^1 = 3, \quad \mathscr{L}_1^2 = \mathscr{L}_2^1 = 0, \quad \mathscr{L}_2^2 = 5. \tag{21}$$

We can educe the conserved quantity by the conformal invariance Eq.(20) as the following steps, and we obtain the following by Noether Eq.(15):

$$\left(ta^2\dot{a}^1 - ta^1\dot{a}^2\right)(-2t) + \frac{1}{2}t^2\dot{a}^1\left(3a^2\right) - \frac{1}{2}t^2\dot{a}^2\left(a^1\right)$$
$$+\frac{1}{2}t^2a^2\dot{a}^1 - \frac{1}{2}t^2a^1\left(3\dot{a}^2\right)$$
$$-\left[ta^2 + t^2\left(a^1\right)^5\right]\left(a^1 + 2t\dot{a}^1\right)$$
$$-\left(t^2a^2 + ta^1\right)\left(3a^2 + 2t\dot{a}^2\right) + \dot{G}_N = 0, \tag{22}$$

which has solution as

$$G_N = t^2a^1a^2 + t^3\left(a^2\right)^2 + \frac{1}{3}t^3(a^1)^6, \tag{23}$$

and we can obtain the conserved quantity by Eq.(16) as

$$I_N = t^2a^1a^2 + t^3(a^2)^2 + \frac{1}{3}t^3(a^1)^6 = \text{const.} \tag{24}$$

5. Summary

A conformal invariance of first-order differential equations is studied in this paper, and we get that Lie symmetry of the equations is the sufficient and necessary condition of the conformal invariance. After changing the equations into those of a generalized Birkhoff system, if the generators of conformal invariance fit equations of Noether of a generalized Birkhoff system, then the conserved quantity of Noether, that is to say the integration, are found.

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