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Local bifurcation analysis of a four-dimensional hyperchaotic system*

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Local bifurcation phenomena in a four-dimensional continuous hyperchaotic system, which has rich and complex dynamical behaviours, are analysed. The local bifurcations of the system are investigated by utilizing the bifurcation theory and the centre manifold theorem, and thus the conditions of the existence of pitchfork bifurcation and Hopf bifurcation are derived in detail. Numerical simulations are presented to verify the theoretical analysis, and they show some interesting dynamics, including stable periodic orbits emerging from the new fixed points generated by pitchfork bifurcation, coexistence of a stable limit cycle and a chaotic attractor, as well as chaos within quite a wide parameter region.

Keywords: hyperchaos, pitchfork bifurcation, Hopf bifurcation, centre manifold theorem
PACC: 0545, 0547

1. Introduction

There has been considerable research about chaos since the first remarkable chaotic model, i.e. Lorenz system,[1] was presented. Chaos, though it is extremely and intrinsically complex and disordered, can also be beneficial to many real-world applications, including fluid mixing, secure communication, heating and some other scientific and engineering applications. By contraries, under some circumstance, chaos is undesirable and should be weakened or eliminated. To sum up, chaos needs a thorough understanding of its physical essence, thereby making it more useful. During recent years, reports about chaos have concentrated on not only proposing new and interesting chaotic systems (Chua circuits,[2] Chen system,[3] Lü system,[4] Lorenz system family,[5] Qi system[6] and others[7−10]) or studying chaos control[11,12] and chaos synchronization, but also theoretically analysing their local and global characteristics which are essential for what is meant by chaos. For example, there have been some detailed bifurcation analyses about Chen system[13−19] and Qi system,[10] such as pitchfork bifurcation, Hopf bifurcation, homoclinic bifurcation, tangent bifurcation and some other complex dynamical behaviours, of which all show that these systems have rich nonlinear dynamics and are of significance in practical application. There are also some other papers devoted to the mathematical analysis of some other chaotic systems.[20−22]

Hyperchaotic system, which has more than one positive Lyapunov exponent, is expanded in two or more directions simultaneously, and thus it is more complex than a chaotic system. This property makes hyperchaos have better application prospect in secure communication and other chaos-based fields. Recent researches about hyperchaos are mainly concerned with the generation of hyperchaotic systems,[23−26] hyperchaos control and synchronization.[27−29] To our best knowledge, there are few investigations into the bifurcation phenomena of hyperchaotic systems because of their higher dimension and complexity.

Therefore, it is meaningful to study the bifurcations of hyperchaotic systems, and this work will take a step forward in investigating the local bifurcation of a new four-dimensional hyperchaotic system. The new system has not only rich dynamical behaviours, but also two relatively big positive Lyapunov exponents when it is hyperchaotic. Two types of bifurca-

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tions, i.e. pitchfork bifurcation and Hopf bifurcation, are found in this system, besides, the conditions of their existence are derived in detail by using the centre manifold theorem and the bifurcation theory. Numerical simulations show great consistence with mathematical analyses, moreover, some interesting dynamics, including stable periodic orbits emerging from the new fixed points generated from the pitchfork bifurcation, coexistence of a stable limit cycle and a chaotic attractor, and chaos within a large parameter region. All of these show that this new hyperchaotic system has rich and complex nonlinear dynamics.

The rest of the present paper is organized as follows: in Section 2, we give a new four-dimensional hyperchaotic system, and simply analyse its basic properties. In Section 3, pitchfork bifurcation and Hopf bifurcation of zero equilibrium are derived in detail by using the centre manifold theorem. Numerical simulation results are given in Section 4. Finally, the conclusion drawn from the present study is given in Section 5.

2. The hyperchaotic system and its equilibrium properties

In this paper, the hyperchaotic system we will analyse is a four-dimensional continuous system

$$\begin{aligned}\dot{x} &= a(y-x)+eyz-kw,\\ \dot{y} &= cx-dy-xz,\\ \dot{z} &= xy-bz,\\ \dot{w} &= ry+fyz,\end{aligned} \tag{1}$$

where $(x, y, z, w) \in R^4$, $r \in R$, and a, b, c, d, e, f, and k are all positive real constant parameters.

Note that system (1) is invariant under the coordinate transformation

$$(x, y, z, w) \to (-x, -y, z, -w),$$

i.e. the system is symmetrical about the z-axis, and if (x, y, z, w) is a solution of system (1), $(-x, -y, z, -w)$ is also a solution.

By solving the following equations:

$$\begin{aligned}&a\,(y-x)+eyz-kw=0,\\ &cx-dy-xz=0,\\ &xy-bz=0, \quad \text{and} \quad ry+fyz=0,\end{aligned}$$

we can easily obtain that system (1) has an equilibrium $S_1 = (0, 0, 0, 0)$, and if $-bdr(cf + r) > 0$, then it has the other two nonzero equilibria, i.e.

$$S_2 = \left(-\frac{bdr}{\sqrt{-bdr(cf+r)}}, \frac{\sqrt{-bdr(cf+r)}}{df}, -\frac{r}{f}, \frac{b(adf^2-acf^2+cefr-afr+er^2)r}{kf^2\sqrt{-bdr(cf+r)}}\right),$$
$$S_3 = \left(\frac{bdr}{\sqrt{-bdr(cf+r)}}, -\frac{\sqrt{-bdr(cf+r)}}{df}, -\frac{r}{f}, -\frac{b(adf^2-acf^2+cefr-afr+er^2)r}{kf^2\sqrt{-bdr(cf+r)}}\right).$$

In the following, we will simply analyse the stability of zero equilibrium S_1. The Jacobian matrix at S_1 is

$$J = \begin{bmatrix} -a & a & 0 & -k \\ c & -d & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & r & 0 & 0 \end{bmatrix}, \tag{2}$$

and its characteristic equation is

$$f(\lambda) = (\lambda + b) \cdot f_1(\lambda) = 0, \tag{3}$$

where $f_1(\lambda) = \lambda^3 + (a + d)\lambda^2 + (ad - ac)\lambda + ckr$.

Obviously, $-b$ is one of the roots of Eq.(3), and the other three roots do not always have negative real parts according to the Routh–Hurwitz condition. In Section 3, we will discuss the bifurcation phenomena when the zero equilibrium is nonhyperbolic.

This system can evolve into periodic, quasi-periodic, chaotic and hyperchaotic attractors, furthermore, the two positive Lyapunov exponents of hyperchaotic attractors are relatively large as shown in Fig.1. The largest Lyapunov exponent can reach about 10 and the second positive Lyapunov exponent can reach 7. Besides, system (1) has a rich dynamical evolution process, and we give some typical attractors of the system in Fig.2.

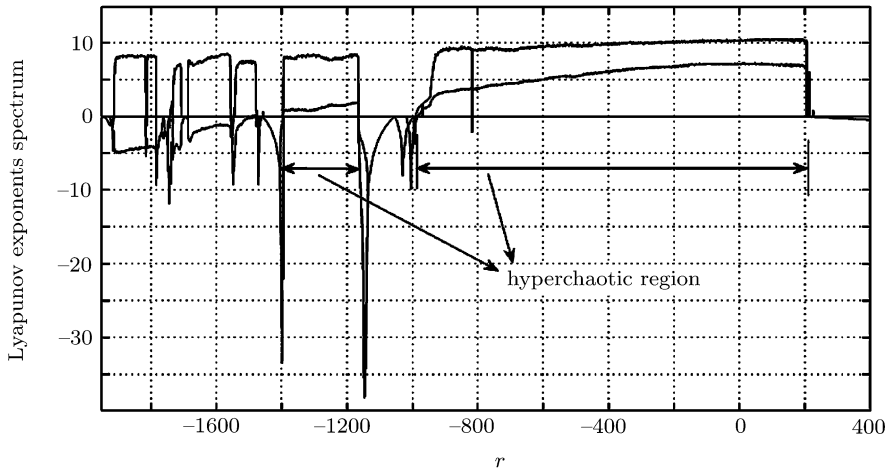
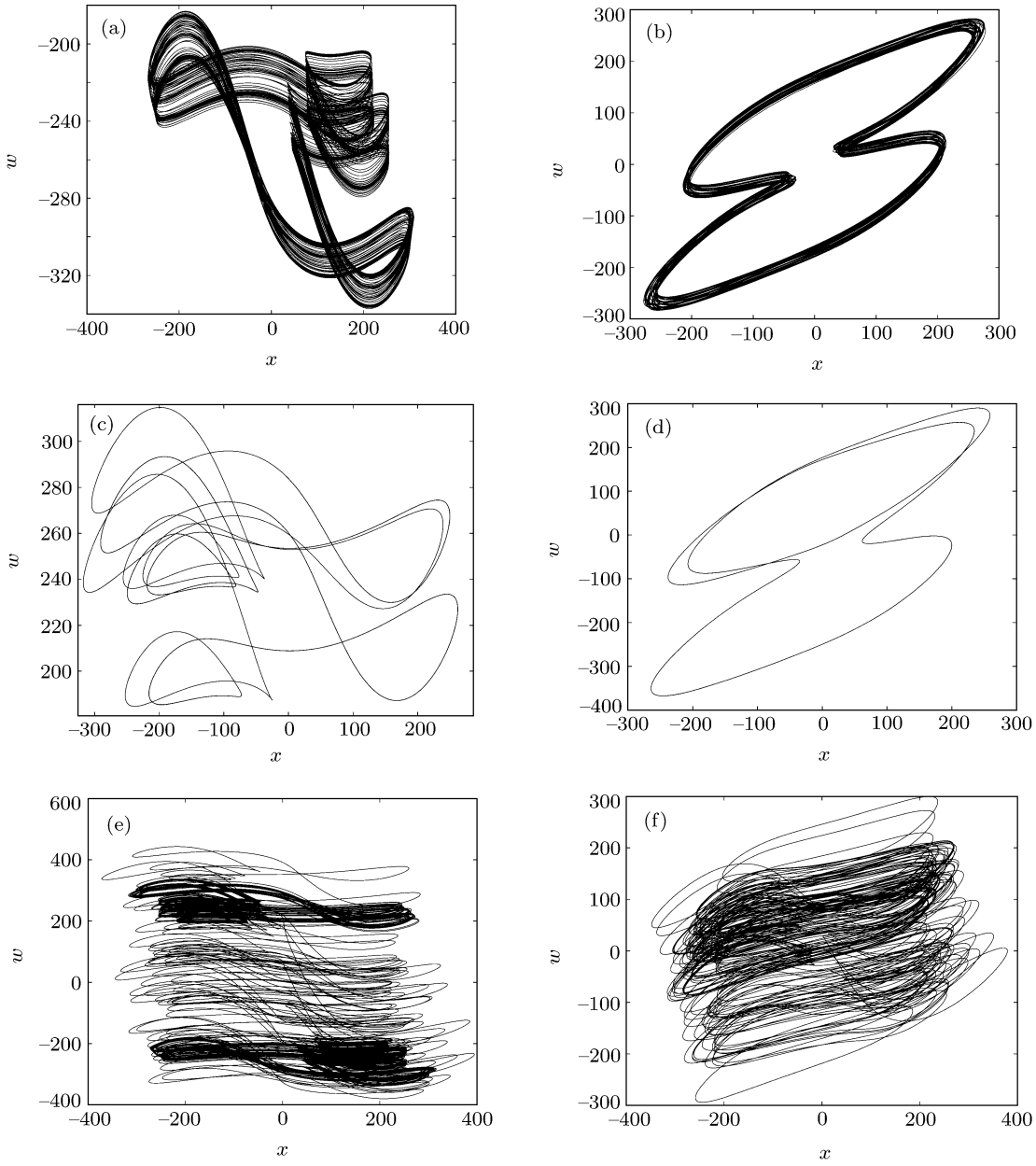


Fig.1. The three largest Lyapunov exponents, with r varying and a=56, b=16, c=49, d=9, k=8, e=30, and f=40 fixed.



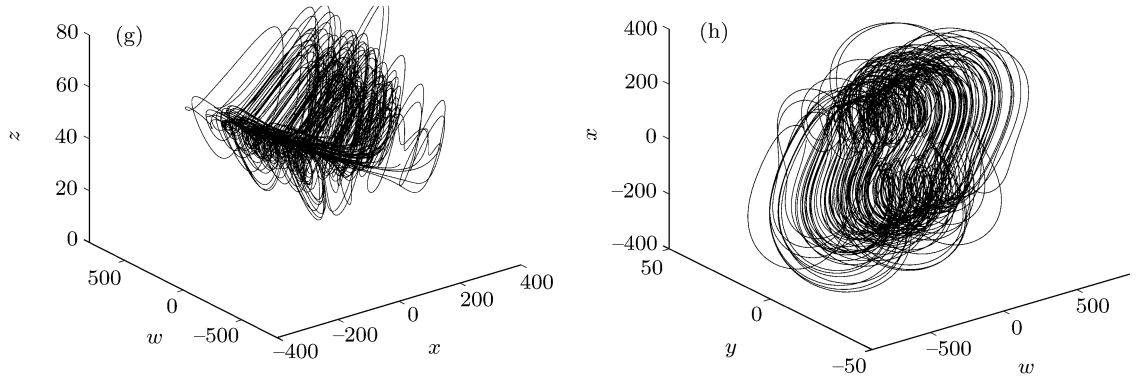


Fig.2. Various attractors in system (1) with a=56, b=16, c=49, d=9, k=8, e=30, and f=40; (a) $r=-1925$, quasi-periodic attractor; (b) $r=-987$, quasi-periodic attractor; (c) $r=-1815$, periodic attractor; (d) $r=-818$, periodic attractor; (e) $r=-1908$, chaotic attractor; (f) $r=-1430$, chaotic attractor; (g) $r=-400$, hyperchaos in $x-z-w$ space; (h) $r=-400$, hyperchaos in $x-y-w$ space.

System (1) has other complex and rich dynamics which will not be discussed in this paper. In the following sections, we will lay emphasis on the local bifurcations of the system.

3. Bifurcations of $S_1=(0,0,0,0)$

3.1. Pitchfork bifurcation of $S_1=(0,0,0,0)$

In the following, we consider r as a bifurcation parameter.

From Jacobian matrix (2), its eigenvalues, when $r=0$, are given by $\lambda_1=0$, $\lambda_2=-b$, and $\lambda_{3,4}=\dfrac{-(a+d)\pm\sqrt{a^2-2ad+d^2+4ac}}{2}$, with the corresponding eigenvectors being

$$\begin{pmatrix} 1 \\ \dfrac{c}{d} \\ 0 \\ \dfrac{a(c-d)}{dk} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \dfrac{d-a-\sqrt{a^2-2ad+d^2+4ac}}{2c} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \dfrac{d-a+\sqrt{a^2-2ad+d^2+4ac}}{2c} \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

Hence, $(0,0,0,0)$ is nonhyperbolic, and we will investigate the essence of stability near $(0,0,0,0)$ by using the centre manifold theorem.[30]

Firstly, we must transform Jacobian matrix (2) into the standard form. Using eigenbasis (4), we obtain the transformation

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{bmatrix} 1 & 0 & \dfrac{d-a-\sqrt{a^2-2ad+d^2+4ac}}{2c} & \dfrac{d-a+\sqrt{a^2-2ad+d^2+4ac}}{2c} \\ \dfrac{c}{d} & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \dfrac{a(c-d)}{dk} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{5}$$

which transforms system (1) into

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & -b & 0 & 0\\ 0 & 0 & \dfrac{-(a+d)-\sqrt{a^2-2ad+d^2+4ac}}{2} & 0\\ 0 & 0 & 0 & \dfrac{-(a+d)+\sqrt{a^2-2ad+d^2+4ac}}{2}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}+\begin{pmatrix}g_1\\ g_2\\ g_3\\ g_4\end{pmatrix}, \quad (6)$$

where

$$\begin{aligned}
g_1 =& f\left(\frac{c}{d}x_1x_2+x_2x_3+x_3x_4\right)+r\left(\frac{c}{d}x_1+x_3+x_4\right),\\
g_2 =& \frac{c}{d}x_1^2+\frac{3d-a-\sqrt{a^2-2ad+d^2+4ac}}{2d}x_1x_3+\frac{3d-a+\sqrt{a^2-2ad+d^2+4ac}}{2d}x_1x_4\\
&+\frac{d-a-\sqrt{a^2-2ad+d^2+4ac}}{2c}x_3^2+\frac{d-a}{c}x_3x_4+\frac{d-a+\sqrt{a^2-2ad+d^2+4ac}}{2c}x_4^2,\\
g_3 =& -\frac{c}{\sqrt{a^2-2ad+d^2+4ac}}e\left(\frac{c}{d}x_1x_2+x_2x_3+x_3x_4\right)\\
&+\frac{-d+a-\sqrt{a^2-2ad+d^2+4ac}}{2\sqrt{a^2-2ad+d^2+4ac}}\left(x_1x_2+\frac{d-a-\sqrt{a^2-2ad+d^2+4ac}}{2c}x_2x_3\right.\\
&\left.+\frac{d-a+\sqrt{a^2-2ad+d^2+4ac}}{2c}x_2x_4\right)+\frac{(d+a-\sqrt{a^2-2ad+d^2+4ac})ck}{2a(c-d)\sqrt{a^2-2ad+d^2+4ac}}\\
&\times\left[f\left(\frac{c}{d}x_1x_2+x_2x_3+x_3x_4\right)+r\left(\frac{c}{d}x_1+x_3+x_4\right)\right],\\
g_4 =& +\frac{c}{\sqrt{a^2-2ad+d^2+4ac}}e\left(\frac{c}{d}x_1x_2+x_2x_3+x_3x_4\right)\\
&-\frac{-d+a-\sqrt{a^2-2ad+d^2+4ac}}{2\sqrt{a^2-2ad+d^2+4ac}}\left(x_1x_2+\frac{d-a-\sqrt{a^2-2ad+d^2+4ac}}{2c}x_2x_3\right.\\
&\left.+\frac{d-a+\sqrt{a^2-2ad+d^2+4ac}}{2c}x_2x_4\right)-\frac{(d+a-\sqrt{a^2-2ad+d^2+4ac})ck}{2a(c-d)\sqrt{a^2-2ad+d^2+4ac}}\\
&\times\left[f\left(\frac{c}{d}x_1x_2+x_2x_3+x_3x_4\right)+r\left(\frac{c}{d}x_1+x_3+x_4\right)\right].
\end{aligned}$$

Thus, from the centre manifold theory, the stability of $S_1=(0,0,0,0)$ near $r=0$ can be determined by studying a one-parameter family of first-order ordinary differential equations on a centre manifold, which can be represented as a graph over the variables x_1 and r as follows:

$$\begin{aligned}
W_c(0) =& \{(x_1,x_2,x_3,x_4,r)\in R^5\big|\, x_2=h_1(x_1,r),x_3=h_2(x_1,r),x_4=h_3(x_1,r),\\
&|x_1|<\delta,|r|<\bar{\delta},h_i(0,0)=0,Dh_i(0,0)=0,i=1,2,3\}
\end{aligned} \quad (7)$$

for δ and $\bar{\delta}$ sufficiently small.

We now calculate the centre manifold and derive the vector field on the centre manifold. We assume that

$$\begin{aligned}
x_2 &= h_1(x_1,r)=a_1x_1^2+a_2x_1r+a_3r^2+\cdots,\\
x_3 &= h_2(x_1,r)=b_1x_1^2+b_2x_1r+b_3r^2+\cdots,\\
x_4 &= h_3(x_1,r)=c_1x_1^2+c_2x_1r+c_3r^2+\cdots.
\end{aligned} \quad (8)$$

Recall from Ref.[30] that the centre manifold must satisfy

$$\mathrm{N}(h(x,r))=Dh\cdot g_1-Bh-g=0, \quad (9)$$

where

$$h=\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix},\ g=\begin{pmatrix}g_1\\g_2\\g_3\end{pmatrix},\ B=\begin{bmatrix}-b & 0 & 0\\ 0 & \dfrac{-(a+d)-\sqrt{a^2-2ad+d^2+4ac}}{2} & 0\\ 0 & 0 & \dfrac{-(a+d)+\sqrt{a^2-2ad+d^2+4ac}}{2}\end{bmatrix}. \tag{10}$$

Substituting expression (8) to Eq.(9) and using expressions (10) give

x_1^2: $ba_1-\dfrac{c}{d}=0\Rightarrow a_1=\dfrac{c}{bd},\ \ b_1=0,\ c_1=0,$

x_1r: $a_2=0,$

$$\begin{aligned}&\frac{-(a+d)-\sqrt{a^2-2ad+d^2+4ac}}{2}b_2+\frac{(d+a-\sqrt{a^2-2ad+d^2+4ac})ck}{2a(c-d)\sqrt{a^2-2ad+d^2+4ac}}\frac{c}{d}=0\\ &\Rightarrow b_2=\frac{(d+a-\sqrt{a^2-2ad+d^2+4ac})c^2k}{ad(c-d)(a+d+\sqrt{a^2-2ad+d^2+4ac})\sqrt{a^2-2ad+d^2+4ac}},\\ &\frac{-(a+d)+\sqrt{a^2-2ad+d^2+4ac}}{2}c_2-\frac{(d+a-\sqrt{a^2-2ad+d^2+4ac})ck}{2a(c-d)\sqrt{a^2-2ad+d^2+4ac}}\frac{c}{d}=0\\ &\Rightarrow c_2=\frac{-c^2k}{ad(c-d)\sqrt{a^2-2ad+d^2+4ac}},\end{aligned} \tag{11}$$

r^2: $a_3=0,\ b_3=0,\ c_3=0,$

and then we obtain that

$$\begin{aligned}h_1(x_1,r)&=a_1x_1^2+\cdots,\\ h_2(x_1,r)&=b_2x_1r+\cdots,\\ h_3(x_1,r)&=c_2x_1r+\cdots.\end{aligned} \tag{12}$$

Finally, substituting expression (12) into expression (8) and (6), we obtain the vector field reduced to the centre manifold

$$\begin{aligned}\dot{x}_1=&\frac{fc}{d}a_1x_1^3+fa_1b_2x_1^3r+b_2c_2x_1^2r^2+\frac{c}{d}x_1r\\ &+b_2x_1r^2+c_2x_1r^2+\cdots,\\ \dot{r}=&0.\end{aligned} \tag{13}$$

Note that we compute the centre manifold to the third order of accuracy, and it is sufficient to answer the questions of stability. By the bifurcation theory of equilibrium, we can ensure that the following conditions are satisfied, for the equilibrium $(x_1,r)=(0,0)$ of expression (13) to undergo a pitchfork bifurcation at $r=0$:

$$F(0,0)=0,\ \ \left.\frac{\partial F}{\partial x_1}\right|_{(0,0)}=0,\ \ \left.\frac{\partial F}{\partial r}\right|_{(0,0)}=0,$$

$$\left.\frac{\partial^2F}{\partial x_1^2}\right|_{(0,0)}=0,\ \ \left.\frac{\partial^2F}{\partial x_1\partial r}\right|_{(0,0)}=\frac{c}{d}\neq0,$$

and

$$\left.\frac{\partial^3F}{\partial x_1^3}\right|_{(0,0)}=\frac{fc^2}{bd^2}\neq0,$$

where

$$\begin{aligned}F(x_1,r)=&\frac{fc}{d}a_1x_1^3+fa_1b_2x_1^3r+b_2c_2x_1^2r^2\\ &+\frac{c}{d}x_1r+b_2x_1r^2+c_2x_1r^2,\end{aligned}$$

a_1, b_2, c_2 are given by (13). Hence, we have obtained the following theorem.

Theorem 1 For $r=0$, system (1) undergoes a pitchfork bifurcation at $S_1=(0,0,0,0)$. Moreover, there is a stable equilibrium $(0,0,0,0)$ for $r>0$; there are three equilibria:

$$\begin{aligned}&S_1=(0,0,0,0),\\ &S_2=\left(-\frac{bdr}{\sqrt{-bdr(cf+r)}},\frac{\sqrt{-bdr(cf+r)}}{df},-\frac{r}{f},\frac{b(adf^2-acf^2+cefr-afr+er^2)r}{kf^2\sqrt{-bdr(cf+r)}}\right),\\ &S_3=\left(\frac{bdr}{\sqrt{-bdr(cf+r)}},-\frac{\sqrt{-bdr(cf+r)}}{df},-\frac{r}{f},-\frac{b(adf^2-acf^2+cefr-afr+er^2)r}{kf^2\sqrt{-bdr(cf+r)}}\right),\end{aligned}$$

for $r<0$; S_1 is unstable and both S_2 and S_3 are stable near $r=0$.

Demonstration of the pitchfork bifurcation is shown in Fig.3.

3.2. Hopf bifurcation of $S_1=(0,0,0,0)$

In this section, we deal with a Hopf bifurcation about the equilibrium $(0,0,0,0)$ of system (1). Before analysing the system, we review the basic Hopf bifurcation theorem first. Hopf bifurcation is the birth of a limit cycle from an equilibrium in dynamical systems, when the equilibrium changes stability by means of a pair of purely imaginary eigenvalues. Consider a vector field

$$\dot{x}=f(x,\ \mu),\quad x\in R^n,\ \mu\in R,$$

then, in order to undergo a Hopf bifurcation, the vector field must satisfy the following three conditions:
(C1) $f(x_0,\mu_0)=0$, the Jacobian $Df(x_0,\mu_0)$ has two purely imaginary eigenvalues, i.e. $\varphi_1(\mu)$ and $\varphi_2(\mu)$, when $\mu=\mu_0$;
(C2) transversality condition: the real parts of eigenvalue satisfy $\dfrac{\mathrm{d}}{\mathrm{d}\mu}(\Re(\varphi_{1,2}(\mu)))\Big|_{\mu=\mu_c}\neq 0$;
(C3) index number is nonzero.

For system (1), we consider characteristic equation (3) and let r be the bifurcation parameter. It is easy to obtain that when

$$r=r_0=\frac{a(a+d)(d-c)}{ck},\ ad-ac>0, \tag{14}$$

the Jacobian matrix of system (1) about zero has a pair of imaginary eigenvalues and two real eigenvalues:

$$\lambda_1=-b,\ \lambda_2=-(a+d),$$

and

$$\lambda_{3,4}=\alpha(r_0)\pm \mathrm{i}\omega(r_0)=\pm \mathrm{i}\omega_0,$$

where

$$\omega(r_0)=\omega_0=\sqrt{ad-ac}. \tag{15}$$

The condition (C1) is satisfied, and we will investigate the other two conditions that ensure that the Hopf bifurcation in system (1) exists, i.e.

$$\rho_1=\frac{\mathrm{d}\alpha(r)}{\mathrm{d}r}\Big|_{r_0}\neq 0, \tag{16}$$

and

$$\text{index number } \Lambda_1\neq 0. \tag{17}$$

we first check the transversality condition (16). Let $\lambda=\alpha(r)+\mathrm{i}\omega(r)$, and differentiate all the terms of $f_1(\lambda)$ in Eq.(3) with respect to r, then we will obtain

$$\rho_1=\frac{\mathrm{d}\alpha(r)}{\mathrm{d}r}\Big|_{r_0}=\frac{ck(ad-ac)}{2(ad-ac)^2+2(a+d)^2(ad-ac)^2}.$$

We assume that

$$ck\neq 0, \tag{18}$$

then $\rho_1\neq 0$. The condition (C2) is satisfied. Next, we should check the index number Λ_1 by using the centre manifold theorem.[30] Utilizing the same method we transform the Jacobian matrix in expression (2) to the standard form.
Let

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{bmatrix}d\omega_0 & -\omega_0^2 & 0 & -ak\\ c\omega_0 & 0 & 0 & ck\\ 0 & 0 & 1 & 0\\ 0 & cr_0 & 0 & -(ad-ac)\end{bmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}, \tag{19}$$

then system (1) can be changed into

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{bmatrix}=\begin{bmatrix}0 & -\omega_0 & 0 & 0\\ \omega_0 & 0 & 0 & 0\\ 0 & 0 & -b & 0\\ 0 & 0 & 0 & -(a+d)\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}+\begin{bmatrix}f_1\\f_2\\f_3\\f_4\end{bmatrix}, \tag{20}$$

where

$$\begin{aligned}
f_1=&\frac{ckr_0}{\beta\omega_0}e(c\omega_0y_1y_3+cky_3y_4)-\frac{a(d\omega_0^2-c\omega_0^2+ckr_0)}{c\beta\omega_0}(d\omega_0y_1y_3-\omega_0y_2y_3+aky_3y_4)+\frac{k\omega_0}{\beta}f(c\omega_0y_1y_3+cky_3y_4),\\
f_2=&\frac{a(c-d)}{\beta}e(c\omega_0y_1y_3+cky_3y_4)+\frac{ad(c-d)}{c\beta}(d\omega_0y_1y_3-\omega_0y_2y_3+aky_3y_4)+\frac{k(a+d)}{\beta}f(c\omega_0y_1y_3+cky_3y_4),\\
f_3=&cd\omega_0^2y_1^2+(d\omega_0ck-akc\omega_0)y_1y_4-c\omega_0^3y_1y_2-ck\omega_0^2y_2y_4+ack^2y_4^2,\\
f_4=&-\frac{cr_0}{\beta}e(c\omega_0y_1y_3+cky_3y_4)-\frac{dr_0}{\beta}(d\omega_0y_1y_3-\omega_0y_2y_3+aky_3y_4)-\frac{\omega_0^2}{\beta}f(c\omega_0y_1y_3+cky_3y_4),\\
\beta=&dcr_0k+ad\omega_0^2-ac\omega_0^2+ackr_0.
\end{aligned} \tag{21}$$

Thus, according to the centre manifold theorem, there must exist a centre manifold for Eq.(20), which can be represented locally as follows:

$$\begin{aligned} W^{\mathrm{c}}(0) =&\{(y_1,y_2,y_3,y_4)\in R^4 \mid\ y_3=h^1(y_1,y_2),\ y_4=h^2(y_1,y_2),\ \ |(y_1,y_2)|<\delta,\\ &h^1(0,0)=0,\ Dh^1(0,0)=0,\ h^2(0,0)=0,\ \text{and}\ Dh^2(0,0)=0\} \end{aligned} \tag{22}$$

for δ sufficiently small. We assume that $h^1(y_1,y_2)$ and $h^2(y_1,y_2)$ have the forms respectively

$$h^1(x_1,x_2)=d_1y_1^2+d_2y_1y_2+d_3y_2^2+\cdots$$

and

$$\begin{aligned} h^2(x_1,x_2) =&e_1y_1^2+e_2y_1y_2+e_3y_2^2+e_4y_1^3\\ &+e_5y_1^2y_2+e_6y_1y_2^2+e_7y_2^3+\cdots. \end{aligned} \tag{23}$$

Through the approximate computation for the centre manifold, we obtain

$$\begin{aligned} &d_1=\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)},\\ &d_2=\frac{2cd\omega_0^2-bc\omega_0^3}{4\omega_0^2+b^2},\\ &d_3=\frac{2cd\omega_0^3-bc\omega_0^4}{b(4\omega_0^2+b^2)},\\ &e_1=0,\quad e_2=0,\quad e_3=0, \end{aligned}$$

$$\begin{aligned} e_4 =&\frac{2\omega_0^2(a+d)}{10\omega_0^2(a+d)^2+9\omega_0^4+(a+d)^4}\Bigg(\Bigg(-\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{\beta}\\ &-\frac{3\omega_0^2+(a+d)^2}{2\omega_0^2(a+d)}\left(\frac{dr_0\omega_0^2}{\beta}+\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)(a+d)}{\beta}\right)\Bigg)\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\ &+\left(\frac{dr_0\omega_0}{\beta}+\frac{(3\omega_0^2+(a+d)^2)(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{2\beta\omega_0(a+d)}\right)\frac{2cd\omega_0^2-bc\omega_0^3}{4\omega_0^2+b^2}\\ &+\left(-\frac{ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3}{\beta}-\frac{3dr_0\omega_0^2}{\beta(a+d)}\right)\frac{2cd\omega_0^3-bc\omega_0^4}{b(4\omega_0^2+b^2)}),\\ e_5 =&-\frac{ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3}{\beta\omega_0}\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\ &-\frac{a+d}{\omega_0}\frac{2\omega_0^2(a+d)}{10\omega_0^2(a+d)^2+9\omega_0^4+(a+d)^4}\Bigg(\Bigg(-\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{\beta}\\ &-\frac{3\omega_0^2+(a+d)^2}{2\omega_0^2(a+d)}\left(\frac{dr_0\omega_0^2}{\beta}+\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)(a+d)}{\beta}\right)\Bigg)\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\ &+\left(\frac{dr_0\omega_0}{\beta}+\frac{(3\omega_0^2+(a+d)^2)(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{2\beta\omega_0(a+d)}\right)\frac{2cd\omega_0^2-bc\omega_0^3}{4\omega_0^2+b^2}\\ &+\left(-\frac{ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3}{\beta}-\frac{3dr_0\omega_0^2}{\beta(a+d)}\right)\frac{2cd\omega_0^3-bc\omega_0^4}{b(4\omega_0^2+b^2)}),\\ e_6 =&-\frac{ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3}{2\beta\omega_0}\frac{2cd\omega_0^2-bc\omega_0^3}{4\omega_0^2+b^2}+\frac{dr_0\omega_0}{2\beta\omega_0}\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\ &+\frac{(a+d)(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{2\beta\omega_0^2}\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\ &+\frac{3\omega_0^2+(a+d)^2}{2\omega_0^2}\frac{2\omega_0^2(a+d)}{10\omega_0^2(a+d)^2+9\omega_0^4+(a+d)^4}\Bigg(\Bigg(-\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{\beta}\\ &-\frac{3\omega_0^2+(a+d)^2}{2\omega_0^2(a+d)}\left(\frac{dr_0\omega_0^2}{\beta}+\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)(a+d)}{\beta}\right)\Bigg)\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\ &+\left(\frac{dr_0\omega_0}{\beta}+\frac{(3\omega_0^2+(a+d)^2)(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{2\beta\omega_0(a+d)}\right)\frac{2cd\omega_0^2-bc\omega_0^3}{4\omega_0^2+b^2}\\ &+\left(-\frac{ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3}{\beta}-\frac{3dr_0\omega_0^2}{\beta(a+d)}\right)\frac{2cd\omega_0^3-bc\omega_0^4}{b(4\omega_0^2+b^2)}\Bigg), \end{aligned}$$

and

$$\begin{aligned}e_7 =&\frac{dr_0\omega_0}{\beta(a+d)}\frac{2cd\omega_0^3-bc\omega_0^4}{b(4\omega_0^2+b^2)}\\&+\frac{\omega_0}{a+d}\bigg(-\frac{ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3}{2\beta\omega_0}\frac{2cd\omega_0^2-bc\omega_0^3}{4\omega_0^2+b^2}+\frac{dr_0\omega_0}{2\beta\omega_0}\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\&+\frac{(a+d)(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{2\beta\omega_0^2}\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\&+\frac{3\omega_0^2+(a+d)^2}{2\omega_0^2}\frac{2\omega_0^2(a+d)}{10\omega_0^2(a+d)^2+9\omega_0^4+(a+d)^4}\bigg(\bigg(-\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{\beta}\\&-\frac{3\omega_0^2+(a+d)^2}{2\omega_0^2(a+d)}\bigg(\frac{dr_0\omega_0^2}{\beta}+\frac{2(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)(a+d)}{\beta}\bigg)\bigg)\frac{2cd\omega_0^3+b^2cd\omega_0+bc\omega_0^4}{b(4\omega_0^2+b^2)}\\&+\bigg(\frac{dr_0\omega_0}{\beta}+\frac{(3\omega_0^2+(a+d)^2)(ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3)}{2\beta\omega_0(a+d)}\bigg)\frac{2cd\omega_0^2-bc\omega_0^3}{4\omega_0^2+b^2}\\&+\bigg(-\frac{ec^2r_0\omega_0+d^2r_0\omega_0+fc\omega_0^3}{\beta}-\frac{3dr_0\omega_0^2}{\beta(a+d)}\bigg)\frac{2cd\omega_0^3-bc\omega_0^4}{b(4\omega_0^2+b^2)}\bigg)\bigg).\end{aligned}\tag{24}$$

We thus obtain the vector field reduced to the centre manifold as follows:

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}0&-\omega_0\\ \omega_0&0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}f^1(y_1,y_2)\\ f^2(y_1,y_2)\end{bmatrix},\tag{25}$$

where

$$f^1(y_1,y_2)=f_1(y_1,y_2,h^1(y_1,y_2),h^2(y_1,y_2)),$$

and

$$f^2(y_1,y_2)=f_2(y_1,y_2,h^1(y_1,y_2),h^2(y_1,y_2)).\tag{26}$$

Then we compute the index number Λ_1 as follows:

$$\begin{aligned}\Lambda_1=&\frac{1}{16}[f^1_{y_1y_1y_1}+f^1_{y_1y_2y_2}+f^2_{y_1y_1y_2}+f^2_{y_2y_2y_2}]\\&+\frac{1}{16\omega_0}[f^1_{y_1y_2}(f^1_{y_1y_1}+f^1_{y_2y_2})-f^2_{y_1y_2}(f^2_{y_1y_1}+f^2_{y_2y_2})-f^1_{y_1y_1}f^2_{y_1y_1}+f^1_{y_2y_2}f^2_{y_2y_2}]\\=&\frac{1}{16}\bigg[\frac{16cd\omega_0^3+6b^2cd\omega_0+4bc\omega_0^4}{bc\beta(4\omega_0^2+b^2)}(c^3ker_0-ad(d\omega_0^2-c\omega_0^2+ckr_0)+kfc^2\omega_0^2)\\&-\frac{a(4cd\omega_0^2-2bc\omega_0^3)(d\omega_0^2-c\omega_0^2+ckr_0)}{c\beta(4\omega_0^2+b^2)}\\&+\frac{4cd\omega_0^2-2bc\omega_0^3}{c\beta(4\omega_0^2+b^2)}(ac^2(c-d)e\omega_0+ad^2(c-d)\omega_0+kfc^2(a+d)\omega_0)\\&-\frac{16cd\omega_0^3+2b^2cd\omega_0-4bc\omega_0^4}{bc\beta(4\omega_0^2+b^2)}ad(c-d)\omega_0\bigg].\end{aligned}\tag{27}$$

Theorem 2 If the conditions (14) and (18) hold and $\Lambda_1\neq 0$ in expression (27), then system (1) undergoes a Poincaré–Andronov–Hopf bifurcation (Hopf bifurcation) at equilibrium $S_1=(0,0,0,0)$. Moreover, the periodic solution, emanating from the origin for a near r_0, is stable if $\Lambda_1<0$, and unstable if $\Lambda_1>0$. The direction of bifurcation is above (below) r_0 for $\rho_1\Lambda_1<0$ ($\rho_1\Lambda_1>0$).

4. Numerical simulations

In this section, we present some numerical simulations to verify the mathematical analysis above. The stability of equilibrium is studied near the pitchfork bifurcation point. The existence and the stability of the periodic orbits emerging from the Hopf bifurcation accord with Theorem 2.

Let $a = 40$, $b = 16$, $c = 7$, $d = 9$, $k = 20$, $e = 30$, and $f = 40$. According to Theorem 1, there should be a stable zero equilibrium for $r > 0$ and three equilibria for $r < 0$, moreover, the nonzero equilibria should be unstable. Under the above parameter conditions, the y-coordinate values of fixed points near $r = 0$ are plotted in Fig.3, where the arrows along the vertical lines represent the flow generated by system (1) along the y-direction. Obviously, for $r > 0$, there is only one equilibrium $S_1 = (0, 0, 0, 0)$, which is stable. For $r < 0$, S_1 is still an equilibrium, but it becomes unstable and other two new stable equilibria, S_2 and S_3, are created at $r = 0$. Thus, the simulation result accords with Theorem 1 well.

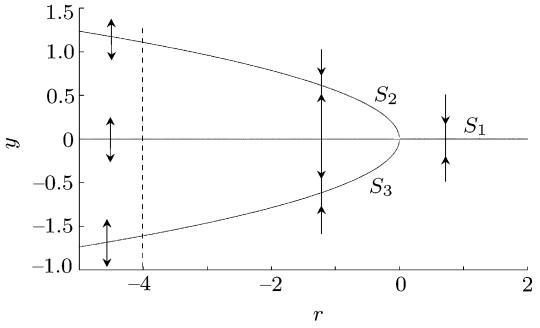


Fig.3. Pitchfork bifurcation diagram near $r = 0$.

However, it is easy to see that the stability of the nonzero equilibrium cannot hold as r decreases, which just implies that the result in Theorem 1 is local but not global. Theorem 1 is applicable to r in a limited neighbourhood of $r = 0$. For example, we find that when $r = -4$, the two nonzero equilibria are $S_2^{-4} = (1.4446,\ 1.1075,\ 0.1,\ -0.5080)$ and $S_3^{-4} = (-1.4446,\ -1.1075,\ 0.1,\ 0.5080)$, which are symmetric about the z-axis. If we choose two different initial points both near zero equilibrium, the system orbits would be away from zero and attracted to the stable equilibria as shown in Fig.4(a), which demonstrates that both S_2^{-4} and S_3^{-4} are stable. However, when $r = -5$, the nonzero equilibria continuously become $S_2^{-5} = (1.6181,\ 1.2360,\ 0.125,\ -0.5323)$ and $S_3^{-5} = (-1.6181,\ -1.2360,\ 0.125,\ 0.5323)$, which are not stable any more. No matter where the initial point is (near zero equilibrium or near the nonzero equilibria), the system orbit is always attracted to a limit cycle as depicted in Fig.4(b). That is to say, stable limit cycles emerge from the nonzero equilibria S_2^{-5} and S_3^{-5}. By analysing the eigenvalues of the Jacobian matrix at S_2, we obtain that when $r = -4$, the eigenvalues corresponding to S_2^{-4} are $\{-0.0175 \pm 4.7349\mathrm{i},\ -16.1357,\ -48.8293\}$, and when $r = -5$, the eigenvalues corresponding to S_2^{-5} are $\{0.1711 + 5.2588\mathrm{i},\ -16.1569,\ -49.1853\}$. Thus, there must be some value between -4 and -5, which is denoted as r', whose corresponding Jacobian matrix at S_2 has a pair of purely imaginary eigenvalues. Through trial computations, we obtain that $r' \approx -4.0915$. That is, the equilibrium $S_2(S_3)$ is stable when $r' < r < 0$ and becomes unstable when $r < r'$, at the same time, a local stable limit cycle emerges from $S_2(S_3)$, thus it seems that a Hopf bifurcation occurs at the equilibrium $S_2(S_3)$ near $r = r'$.

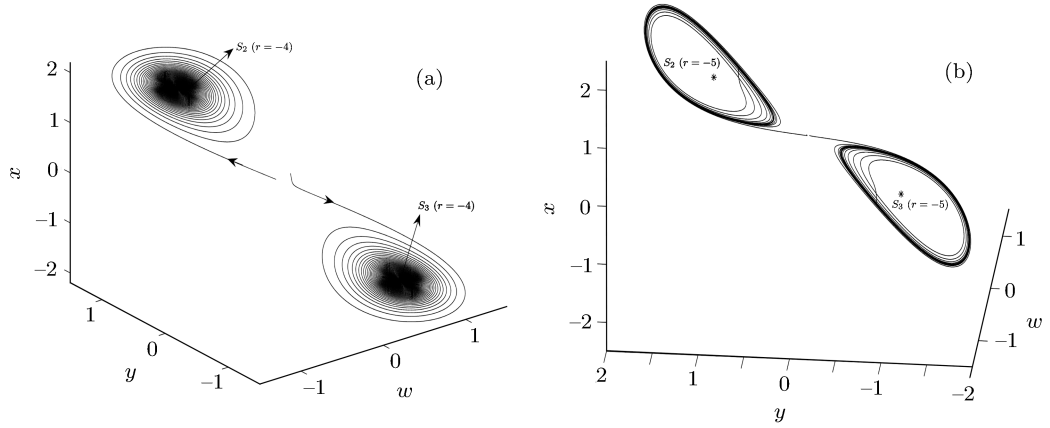


Fig.4. Different dynamical behaviour near $r' = -4.0915$. (a) stable at fixed points S_2 and S_3; (b) stable limit cycles emerging from S_2 and S_3.

In addition, there is an interesting dynamical behaviour: when $r = -5$, we can observe the coexistence of a limit cycle and a chaotic attractor in system (1), which is shown in Fig.5. In particular, the inset of Fig.5 clearly shows the limit cycle, which is in fact located in the hole of the chaotic attractor. When $r = -6$, the limit cycle disappears and the system totally evolves into a chaotic attractor, of which the Lyapunov exponents are {1.9232, 0, –7.1979, –59.7266}. We depict this chaotic attractor in Fig.6.

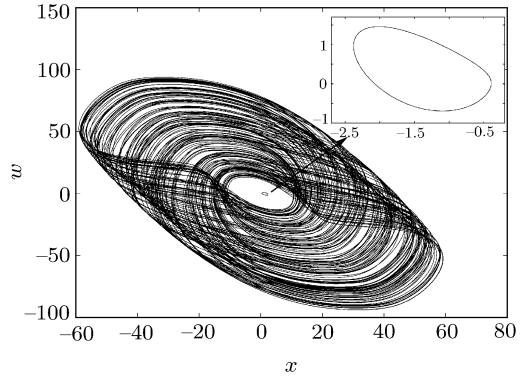


Fig.5. Coexistence of a limit cycle and a chaotic attractor in $x - w$ plane.

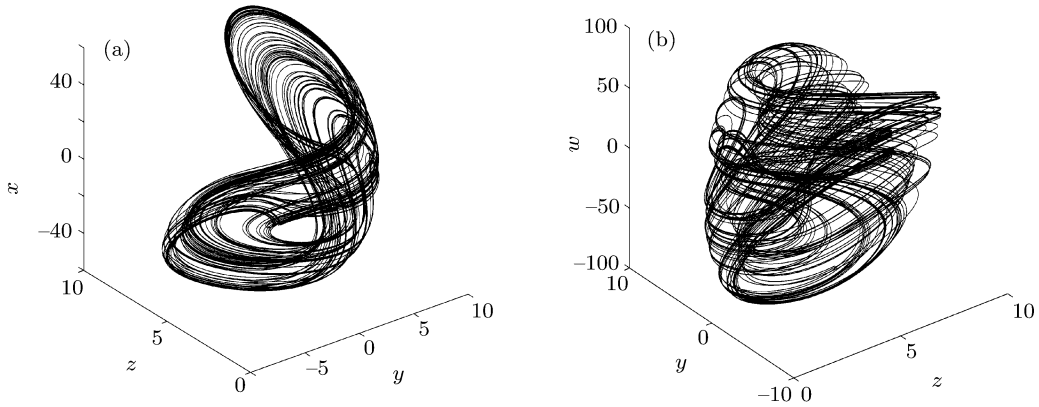


Fig.6. One chaotic attractor when $a = 40$, $b = 16$, $c = 7$, $d = 9$, $k = 20$, $e = 30$, $f = 40$, $r = -6$. $x - y - z$ subspace; (b) $y - z - w$ subspace.

According to Theorem 2, with $a = 40$, $b = 16$, $c = 7$, $d = 9$, $k = 20$, $e = 30$, and $f = 40$ adopted, then we will have $r_0 = 28$, $\rho_1 = \dfrac{ck(ad-ac)}{2(ad-ac)^2 + 2(a+d)^2(ad-ac)^2} > 0$ and $\Lambda_1 = -12.6107$. Due to $\Lambda_1 < 0$, there should be a stable periodic orbit near $r_0 = 28$, and the negative sign of $\rho_1\Lambda_1$ leads to the fact that the direction of bifurcation is above r_0. In other words, a stable periodic orbit emerges from zero when r exceeds r_0. We choose [0.01, 0.01, 0.01, 0.01], which is in the neighbourhood of zero, as an initial point, thereby obtaining Fig.7. By observing the simulation result, we can see that it is exactly in accordance with the conclusion obtained in Theorem 2.

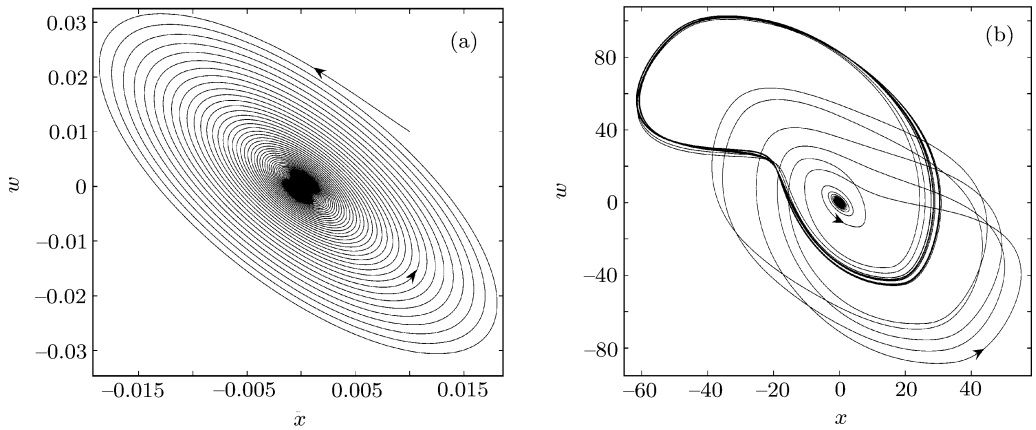


Fig.7. Hopf bifurcation in system (1) according to Theorem 2. (a) $r = 25 < r_0$, the zero equilibrium is stable, and thus the system orbit is attracted to zero; (b) $r = 31 > r_0$, the system orbit is attracted to a stable periodic orbit, at the same time, the zero equilibrium is unstable apparently.

As r increases, the periodic orbit emerging from zero disappears and system (1) undergoes a transition from periodic to chaotic. Moreover, the system keeps the chaotic dynamic in quite a wide parameter region as shown in Fig.8, from which it can be seen that the largest Lyapunov exponent keeps positive as r increases. Note that the chaotic region is too wide to be computed completely, and when r reaches 10000, system (1) is still chaotic. This property could make system (1) have more promising and wide applications.

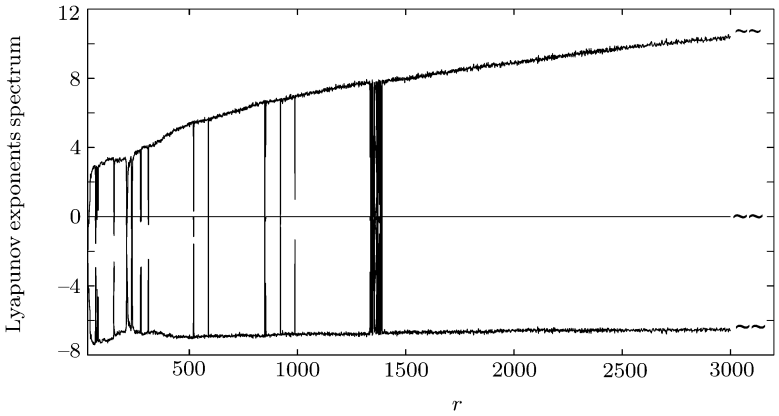


Fig.8. The three largest Lyapunov exponents when r varies in quite a wide region, with $a = 40$, $b = 16$, $c = 7$, $d = 9$, $k = 20$, $e = 30$, and $f = 40$ fixed.

5. Conclusion

In this paper, the pitchfork bifurcation and Hopf bifurcation of a new hyperchaotic system have been investigated. We have analysed some basic properties of the system, such as symmetry, equilibria, and stability of zero equilibrium, and we also have given various attractors in this system. Based on the rigorous theoretical analysis, the stability of fixed points, when the bifurcation parameter passes through the pitchfork bifurcation point, has been discussed, and both the conditions of the stability and the bifurcation direction of the periodic cycle emerging from the zero fixed point at Hopf bifurcation point have also been obtained. Numerical simulation results not only have verified the mathematical analysis, but also have displayed some other interesting dynamics. For example, two new fixed points emanating from zero lose stability and stable periodic orbits emerge when the bifurcation parameter decreases to some critical value, which looks as if a Hopf bifurcation occurred. We also have detected the coexistence of a stable limit cycle and a chaotic attractor, and chaotic behaviours within a very large parameter region. All of these have shown that this new hyperchaotic system has rich and complex dynamics.

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