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Controlling bifurcations and chaos in discrete small-world networks*

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We propose an impulsive hybrid control method to control the period-doubling bifurcations and stabilize unstable periodic orbits embedded in a chaotic attractor of a small-world network. Simulation results show that the bifurcations can be delayed or completely eliminated. A periodic orbit of the system can be controlled to any desired periodic orbit by using this method.

Keywords: bifurcation, chaos, small-world networks, impulsive hybrid control
PACC: 0545, 0547

1. Introduction

Watts and Strogatz[1] presented a model for a small-world network with a high degree of local clustering and a small average distance. Recently, there has been found the possibility that a number of systems in nature may have the same topological structures as small-world networks, which are intermediate of the local regular networks and the fully random networks. Such small-world networks are usually characterized by two interesting features: high clustering, which is a characteristic of regular networks, and the short path length, which is typically observed in random networks. There are many studies on applications of small-world networks.[2−6] Most models are linear models and there is no time delay in their models. However, there are always time delays in spreading and response in an actual system. Recently, to simulate more realistic networks, Yang[6] presented a more general nonlinear small-world network model (see Eq.(1)), the nonlinear interaction and time delay are all considered. The nonlinear interaction due to competition and congestion of vertices can cause bifurcations and chaos in network systems.

Since the pioneering work by Ott *et al*,[7] there have been many other ideas and methods of controlling bifurcations and chaotic systems.[8−26] It has been realized that delaying bifurcations and suppressing chaos can be useful for many practical problems. In this paper, we develop an impulsive hybrid control strategy to control the period-doubling bifurcations and chaos in small-world networks.

The rest of the present paper is organized as follows. The dynamics of the small-world model is analysed in Section 2. An impulsive hybrid control strategy for a discrete nonlinear system is presented and the asymptotical stability of impulsive control system is studied in Section 3. The dynamics of the impulsive controlled model is analysed and the simulation results are given in Section 4. The brief conclusion are presented in the last section.

2. Small-world network model

The nonlinear small-world network model was presented by Yang,[6] and it is expressed as

$$\frac{\mathrm{d}^d V}{\mathrm{d}t^d} = \xi^d + V(t-\delta) - \mu\xi^d V^2(t-\delta), \qquad (1)$$

where $V(t)$ is the total influenced volume, d is the dimension of the network, ξ is the Newman–Watts length scale,[2] δ is the delay time, μ is a measure of nonlinear interactions in the network.

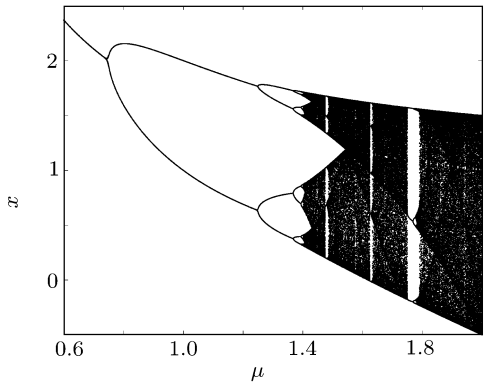
From the theory of dynamic systems, the dynamic features can be shown more clearly by using the representation in the Poincaré plane, which usually transforms a nonlinear differential equation into a nonlinear

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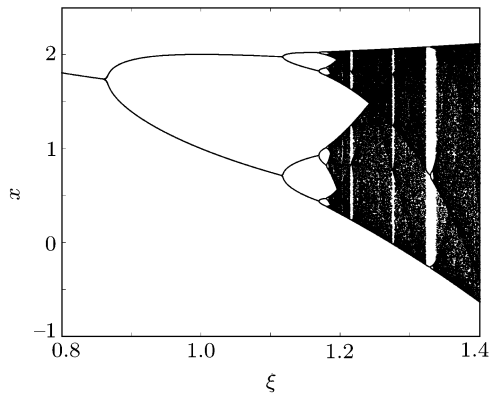
iterated map. For simplicity, we take $\delta = 1$, and $d = 1$ (in one dimension) and then transform Eq.(1) into a map

$$x_{n+1} = \xi + 2x_n - \mu\xi x_n^2 = f(\mu, x_n), \tag{2}$$

where μ and ξ are real parameters. In this paper, we view $\mu(0 \le \mu \le 2)$ as the bifurcation parameter and fix $\xi = 1$ (we also can fix $\mu = 1$ and view ξ as the bifurcation parameter). The bifurcation diagrams are shown in Fig.1. The Lyapunov exponent diagram is shown in Fig.2.



(a) $\xi = 1,\ \mu \in [0, 2]$



(b) $\mu = 1,\ \xi \in [0, 1.4]$

Fig.1. The bifurcation diagrams of system (2).

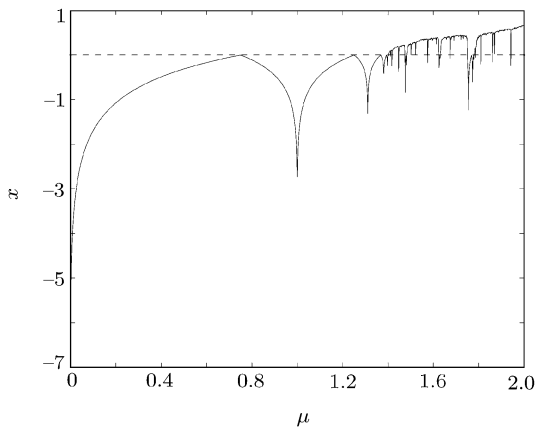


Fig.2. The Lyapunov exponent diagram with $\xi = 1$, and $\mu \in [0, 2]$.

Figures 1 and 2 show that there is a period-doubling cascade turning to chaos in the map. This period-doubling cascade turning to chaos is the focus of our control design.

3. Impulsive hybrid control of discrete nonlinear systems

Consider a discrete nonlinear system

$$x_{n+1} = Ax_n + \varphi(x_n) = f(x_n), \tag{3}$$

then the impulsive hybrid control for the discrete nonlinear system (3) is described as follows:

$$\begin{aligned} &x_{n+1} = f(x_n) && n \ne n_k, \\ &x_{n+1} = af(x_n) + (1-a)x_n, && n = n_k, k = 1, 2, \cdots, \end{aligned} \tag{4}$$

where x_n is the state variable of system (3), $\varphi(x_n) \in R^m$ is an m-dimensional nonlinear map satisfying $|\varphi(x_n)| < L|x_n|$, with $L > 0$ being constant, A is a $m \times m$ matrix, and n is a discrete time.

From Eqs.(3) and (4), we can obtain

$$\begin{aligned} &x_{n+1} = Ax_n + \varphi(x_n) && n \ne n_k, \\ &\Delta x_{n+1} = Bx_n + a\varphi(x_n), && n = n_k, k = 1, 2, \cdots, \end{aligned} \tag{5}$$

where $\Delta x_{n_k+1} = x_{n_k+1} - x_{n_k}$, B is an $m \times m$ matrix, $B = a(A - I)$, $0 < a < 1$, the discrete time set $\{n_k\}$ is a subset of the discrete time set $\{n\}$ and satisfies $n_k \to \infty$ as $k \to \infty$, for all $k \in \{1, 2, \cdots\}$, the impulsive interval is $\Delta = n_{k+1} - n_k \ge 2$, $0 < 1 = n_1 < n_2 < \cdots < n_k \cdots$.

We are to find such an impulsive controller $\{n_k, B\}$ that impulsive controlled system (5) is asymptotically stable.

Theorem 1 If there exists a positive define matrix $P \in R^{n\times n}$, such that

$$\begin{aligned} &\sum_{i=0}^{+\infty} \ln r_i = -\infty, \\ &r_i = \begin{cases} \sqrt{\alpha}, & \text{if } \ i \ne n_k, \\ \sqrt{\beta}, & \text{if } \ i = n_k, k \in \{1, 2, \cdots\}, \end{cases} \end{aligned} \tag{6}$$

where

$$\alpha = \{\lambda_{\max}(P^{-1}A^{\mathrm{T}}PA)^{1/2} + L\lambda_{\max}(P)/\lambda_{\min}(P)\}^2,$$

and

$$\begin{aligned} \beta = &\{\lambda_{\max}[P^{-1}(B+I)^{\mathrm{T}}P(B+I)]^{1/2} \\ &+ aL[\lambda_{\max}(P)/\lambda_{\min}(P)]^{1/2}\}^2, \end{aligned}$$

then system (5) is asymptotically stable.

Proof Let the Lyapunov function be as follows:

$$V(x_n) = x_n^{\mathrm{T}} P x_n, \tag{7}$$

then we will obtain

$$\lambda_{\min}(P) x_n^{\mathrm{T}} x_n \le V(x) \le \lambda_{\max}(P) x_n^{\mathrm{T}} x_n. \tag{8}$$

For $n \neq n_k$, according to Eqs.(5) and (7), we have

$$\begin{aligned} V(x_{x+1}) =& x_{n+1}^{\mathrm{T}} P x_{n+1} \\ =& (Ax_n + \varphi(x_n))^{\mathrm{T}} P (Ax_n + \varphi(x_n)) \\ \le& x_n^{\mathrm{T}} A^{\mathrm{T}} P A x_n + 2 x_n^{\mathrm{T}} A^{\mathrm{T}} P \varphi(x_n) \\ &+ \varphi^{\mathrm{T}}(x_n) P \varphi(x_n). \end{aligned} \tag{9}$$

The terms in Eq.(9) can be estimated as

$$\begin{aligned} & x_n^{\mathrm{T}} A^{\mathrm{T}} P \varphi(x_n) \le \left\| x_n^{\mathrm{T}} A^{\mathrm{T}} P^{1/2} \right\| \cdot \left\| P^{1/2} \varphi(x_n) \right\| \\ \le& L \sqrt{\lambda_{\max}(P^{-1} A^{\mathrm{T}} P A)} \cdot [\lambda_{\max}(P)/\lambda_{\min}(P)]^{1/2} x_n^{\mathrm{T}} P x_n. \end{aligned} \tag{10}$$

Substituting inequality (10) into inequality (9), we obtain

$$\begin{aligned} V(x_{n+1}) \le& \{\lambda_{\max}(P^{-1} A^{\mathrm{T}} P A) + L^2 \lambda_{\max}(P)/\lambda_{\min}(P) \\ &+ 2L[\lambda_{\max}(P^{-1} A^{\mathrm{T}} \cdot P A)]^{1/2} [\lambda_{\max}(P)/\lambda_{\min}(P)]^{1/2}\} V(x_n) \\ =& \{\lambda_{\max}(P^{-1} A^{\mathrm{T}} P A)^{1/2} + L \lambda_{\max}(P)/\lambda_{\min}(P)\}^2 V(x_n) \\ =& \alpha V(x_n). \end{aligned} \tag{11}$$

For $n = n_k$, $k = 1, 2, \cdots$, we can obtain the following inequality similar to inequality (11):

$$\begin{aligned} V(x_{n_k+1}) =& x_{n_k+1}^{\mathrm{T}} p x_{n_k+1} \\ =& [(B+1)x_{n_k} + a\varphi(x_{n_k})]^{\mathrm{T}} P [(B+1)x_{n_k} + a\varphi(x_{n_k})] \\ \le& \{\lambda_{\max}[P^{-1}(B+1)^{\mathrm{T}} P (B+1)]^{1/2} + aL[\lambda_{\max}(P)/\lambda_{\min}(P)]^{1/2}\}^2 V(x_{n_k}) \\ =& \beta V(x_{n_k}). \end{aligned} \tag{12}$$

For $n \in (n_k, n_{k+1}], k = 1, 2, \cdots$, from inequalities (11) and (12), we have

$$V(x_n) \le \left(\prod_{i=0}^{n-1} r^2 \right) V(x_0) = \exp\left(2 \sum_{i=0}^{n-1} \ln r \right) V(x_0). \tag{13}$$

From inequalities (8) and (13), we can deduce

$$\begin{aligned} \|x_n\| \le& [\lambda_{\max}(P)/\lambda_{\min}(P)]^{1/2} \\ &\times \exp\left(\sum_{i=0}^{n-1} \ln r_i \right) \|x_0\|. \end{aligned} \tag{14}$$

For any $x_0 \in R^{n \times n}$, if $\lim\limits_{n \to \infty} \sum\limits_{i=0}^{n-1} \ln r_i = -\infty$, then from inequality (14), we have $\lim\limits_{n \to \infty} \|x_n\| = 0$. Thus system (5) is a globally asymptotical stable.

4. System analysis and experiment

4.1. Controlling bifurcations in small-world networks

According to the control strategy shown in Eq.(5), we design the impulsive hybrid control law to control the period-doubling bifurcation, given as

$$\begin{aligned} & x_{n+1} = \xi + 2x_n - \mu \xi x_n^2, \qquad n \neq n_k, \\ & \Delta x_{n+1} = a(\xi + x_n - \mu \xi x_n^2), \ n = n_k, \quad k = 1, 2, \cdots. \end{aligned} \tag{15}$$

The hybrid control parameter $a \in (0, 1)$. For $a = 1$, controlled system (15) reduces to original system (2).

Theorem 2[11] Consider a discrete system as expression (3), where $x_n \in R^n$ for each integer n, and μ is the real-valued parameter. Suppose that the map

$f(\mu, x_n)$ is smooth with respect to both x and μ and sufficiently smooth and if

1) $x^* = f(\mu, x^*)$, $\mu \in R$ has a fixed point x^* for all μ;

2) for $\mu = \mu^*$, $f'(\mu, x_n)|_{x=x^*,\mu=\mu^*} = -1$;

3) $2\dfrac{\partial^2 f}{\partial x \partial \mu} + \dfrac{\partial f \partial^2 f}{\partial \mu \partial x^2}\Big|_{x=x^*,\mu=\mu^*} \neq 0,$

$$\eta = \left(\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \frac{1}{3}\frac{\partial^3 f^2}{\partial x^3}\right)\Bigg|_{x=x^*,\mu=\mu^*} \neq 0,$$

then, there will exist a period-doubling bifurcation in the system at point (μ^*, x^*). The sign of η determines the stability of bifurcation of the periodic orbits. If η is positive, the orbits are stable; if η is negative they are unstable.

We now discuss the stability condition of fixed points of the system given as expression (15).

When $n \neq n_k$, the system (15) is equal to the system (2), which has two fixed points:

$$x_1^* = (1 + \sqrt{1 + 4\mu\xi^2})/2\mu\xi,$$

and

$$x_2^* = (1 - \sqrt{1 + 4\mu\xi^2})/2\mu\xi.$$

The Jacobian matrix of Eq.(2) at fixed point x_1^* is

$$J_1 = \frac{\partial f(\mu, x)}{\partial x}\Big|_{x=x^*} = [2 - 2\mu\xi x^*]. \tag{16}$$

The stability condition of Eq.(2) at point x_1^* is

$$|2 - 2\mu\xi x_1^*| < 1. \tag{17}$$

Then the stability range of 1-periodic orbit is

$$0 < \mu < 0.75\xi^{-2}. \tag{18}$$

It can reach the condition $f'(\mu, x^*) = -1$ at $\mu = 0.75\xi^{-2}$. From theorem 2, we know that, for the case $\mu = 0.75\xi^{-2}$, the first period-doubling bifurcation takes place.

Theorem 3[27] Consider a nonlinear system $f(\mu, x)$, the stability condition of an n-periodic orbit at its fixed point x^* is

$$s \equiv \left|\frac{\partial f^{(n)}(x, \mu)}{\partial x}\right|_{x=x^*} \leq 1 \text{ or } s \equiv \left|\prod_{i=1}^{n} \partial f'(x_i, \mu)\right| \leq 1.$$

We can calculate the 2-periodic orbit (x_{11}^*, x_{22}^*):

$$x_{11,22}^* = \frac{4\xi - 1 \pm \sqrt{4\mu + 16\xi^2 - 20\xi + 1}}{2\mu\xi}. \tag{19}$$

The 2-periodic orbit (x_{11}^*, x_{22}^*) satisfies the following equation:

$$x_{22}^* = f(\mu, x_{11}^*), \text{ and } x_{11}^* = f(\mu, x_{22}^*). \tag{20}$$

From theorem 3, we obtain the stability condition of 2-periodic orbit at its fixed point as

$$|f'(\mu, x_{11}^*) f'(\mu, x_{22}^*)| \leq 1, \tag{21}$$

that is

$$|8 - 4\xi - 4\mu| \leq 1. \tag{22}$$

Then the stability range of 2-periodic orbit is

$$1.75 - \xi < \mu < 2.25 - \xi, \tag{23}$$

$$s = \frac{\mathrm{d}}{\mathrm{d}x} f^{(2)}(\mu, x)\Big|_{x=x^*} = 8 - 4\xi - 4\mu, \tag{24}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{(2)}(\mu, x^*) = -1 \text{ at } \mu = 2.25 - \xi. \tag{25}$$

From theorem 2, we know that for the case $\mu = 2.25 - \xi$, the second period doubling bifurcation takes place.

Remark 1 We fix $\xi = 1$. It can be proved that if $0 < \mu < 0.75$ then the fixed point x_1^* is stable, and x_2^* is always unstable in a range of $0 < \mu < 2$. As $\mu = 0.75$, system (2) appears as period-doubling bifurcations for the first time, then a stable 2-periodic orbit is generated. Next, as $0.75 < \mu < 1.25$, then the 2-period orbit is stable. As $\mu = 1.25$, then system (2) has period-doubling bifurcations appearing for the second time, and a stable 4-periodic orbit also appears, followed by stable $8, 16, \cdots, 2^n-$periodic orbits until the onset of chaos, as the parameter μ gradually increases.

When $n = n_k$, system (15) is

$$\begin{aligned} x_{n+1} =& a(\xi + 2x_n - \mu\xi x_n^2) + (1 - a)x_n \\ =& f_s(\mu, x_n). \end{aligned} \tag{26}$$

The Jacobian matrix of expression (26) at fixed point x_1^* is

$$J_2 = \frac{\partial f_s(\mu, x)}{\partial x}\Big|_{x=x^*} = [1 + a - 2a\mu\xi x^*]. \tag{27}$$

The stability condition of Eq.(25) at fixed point x_1^* is

$$|1 + a - 2a\mu\xi x^*| < 1. \tag{28}$$

Then the stability range of 1-periodic orbit is

$$0 < \mu < 0.75\xi^{-2} + \xi^{-2}(a^{-2} - 1), \tag{29}$$

$$f_1'(\mu, x^*) = -1 \text{ at } \mu_1 = 0.75\xi^{-2} + \xi^{-2}(a^{-2} - 1). \tag{30}$$

From theorem 2, we know that for the case $\mu = \mu_1$, the first period-doubling bifurcation of controlled system (15) takes place.

Similarly, when the 2-period point (x_{11}^*, x_{22}^*) of controlled system (15) is stabilized, we can calculate the stability range of controlled system (15) for 2-periodic orbits, given as

$$\frac{3a^2+4a-8a\xi+4a^2\xi}{4a^2} \le \mu \le \frac{2+3a^2+4a-8a\xi+4a^2\xi}{4a^2} \tag{31}$$

and controlled system (15) has period-doubling bifurcations for the second time at $\mu = (2+3a^2+4a-8a\xi+4a^2\xi)/4a^2$. For the 4, 8, 16, 2^n period orbits, we can use the same methods to obtain the stability range of parameter μ for which period orbits are stable.

The bifurcation control results are shown in Fig.3. And we can see that the period-doubling bifurcations of the small-world network are delayed or eliminated completely under this control method (in Figs.3(a) and 3(b)); Fig.3(c) shows that the range of parameter μ in controlled system (15) is enlarged.

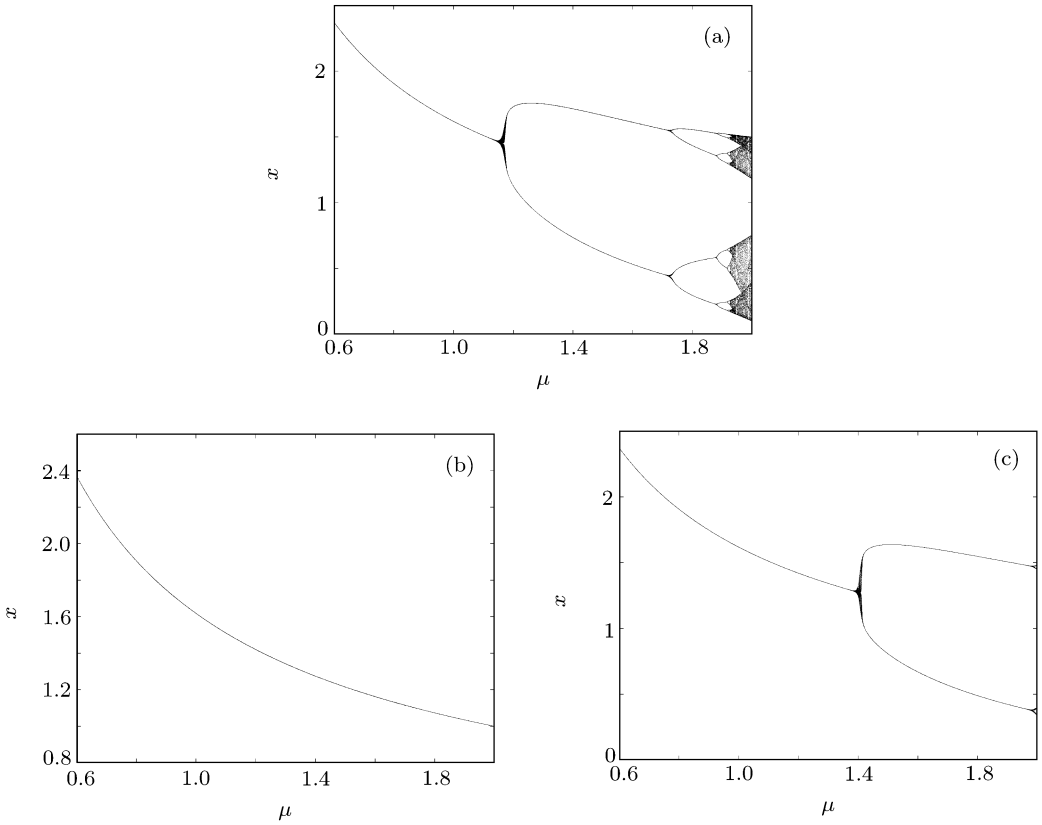


Fig.3. The results of bifurcations control. (a) The result of controlling period-doubling bifurcation for the first time with a= 0.7, and Δ=2. (b) The result of controlling period-doubling bifurcation for the first time with a=0.4, and Δ=2. (c) The result of controlling period-doubling bifurcation for the second time with a=0.6, and Δ=2.

The above research indicates that the controlled system can maintain its stable dynamical behaviours in a large range of parameter value variation and the degree of stability can be made sufficiently large. Thus further period-doubling bifurcation is prevented in the parameter range of interest. The period-doubling bifurcations of the small world network system can be delayed or completely eliminated. The period-doubling route to chaos is therefore at least delayed to greater parameter values.

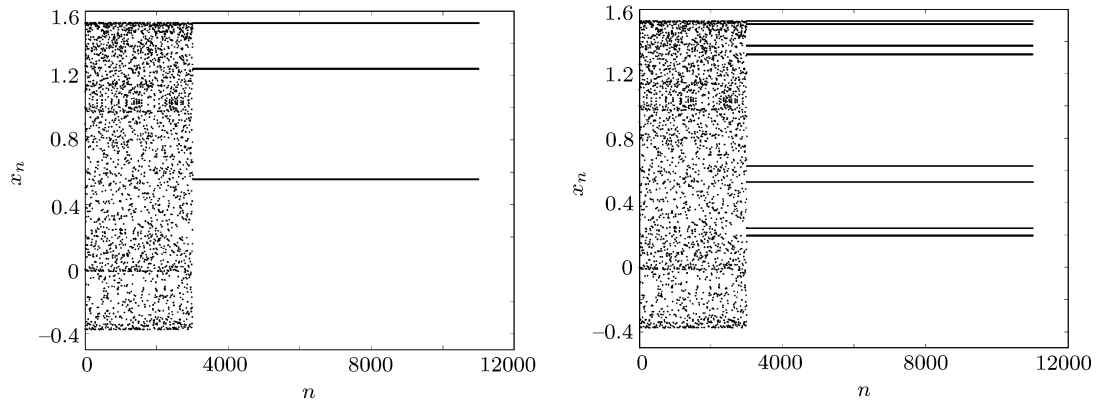
4.2. Controlling chaos in small-world networks

This impulsive hybrid control method can control some unstable periodic orbits that are embedded in a chaotic attractor of original system. For a range of values of parameter μ, a chaotic trajectory in the small-world networks system will settle to a periodic orbit upon activation of the control law.

Figures 1(a) and 2 show that as $\mu = 1.9$, the

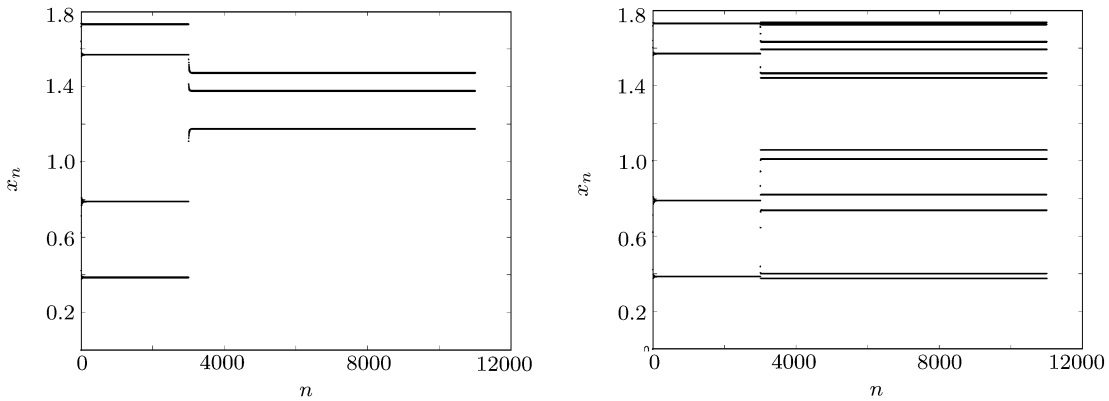
original system is chaotic. Figure 4 shows the results of controlling the chaotic system to the 3, 8 periodic orbits. Figure 5 shows the results of controlling a 4-periodic orbit onto 3, 12-periodic orbit with $\mu = 1.36$.

The simulation results show that we can control the system under any condition into different desired periodic orbits by using the appropriate parameters a and Δ. Also found in simulated process is that the control field of this control method is very wide, by appropriately choosing the control parameters, the system can be stabilized in $2^n \times 3^m$ periodic orbits, and system can be controlled in the same periodic orbit by different groups control parameters, which show that control parameter of obtaining the same periodic orbits is not unique.



(a) $\mu = 1.9$, $a = 0.15$, and $\Delta = 3$. (Chaos to $3p$) (b) $\mu = 1.9$, $a = 0.7$, and $\Delta = 2$. (Chaos to $8p$)

Fig.4. The results of controlling the chaotic system to the 3, and 8 periodic orbits.



(a) $\mu = 1.36$, $a = 0.2$, and $\Delta = 3$. $(4p \to 3p)$ (b) $\mu = 1.36$, $a = 0.9$, and $\Delta = 3$. $(4p \to 12p)$

Fig.5. The results of controlling a 4-periodic orbit onto 3, 2-periodic orbit with $\mu = 1.36$.

5. Conclusion

In this paper, we have presented an impulsive hybrid control strategy to control the period-doubling bifurcations and chaos of small-world networks. The period-doubling bifurcations in the small-world networks can be delayed or eliminated completely, the period-doubling route to chaos is therefore at least delayed to greater parameter values. The unstable periodic orbits embedded in a chaotic attractor of a discrete time small-world network can be controlled to any desired periodic orbit by adjusting two control parameters.

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