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Orbital effect in the stationary axisymmetric field*

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This paper uses an elegant mathematical method to calculate the orbital effects in the axisymmetric field created by the spinning mass with electric charge and a large number of magnetic monopoles. In comparison with that in the Reissner–Nordström (R–N) field, the correction terms caused by the spinning mass decrease the advanced effect as the revolution direction of the test particle coincides with that of the Kerr field, however, the correction terms caused by the spinning charged mass increase the advance effect as the revolution direction of the test particle coincides with that of the Kerr–Newman–Kasnya (KNK) field. Generalizing the effect in the axisymmetric field, it obtains interesting results by discussing the parameters of the celestial body, these parameters provide a feasible experimental verification of the general relativity.

Keywords: general relativity, gravitational effect, space–time metric
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1. Introduction

The effect in the gravitational field is an intriguing problem, the effects in some symmetric fields (such as Schwarzschild, Reissner, Nordström, Kerr fields) have been researched rather completely.[1−4] We have studied the mass defect effects[5−8] and orbital effects[9,10] in spherically symmetric field. However, the celestial bodies all are moving, so it is more significative to study the orbital effect in the field with angular momentum.

The Kerr–Newman spacetime is a stationary axisymmetric solution[11,12] of the Einstein's equations in the presence of the Maxwell field.[13,14] In 1982, Kasuya discussed it in the case of the field source with magnetic charge. The form of the metric in Boyer–Lindquist coordinates (t, r, θ, φ) is given by[15](in this paper, we take $G = c = 1$.)

$$\begin{aligned} \mathrm{d}s^2 =& \frac{\Delta}{\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi)^2 \\ & - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 \\ & - \frac{\sin^2\theta}{\rho^2}[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi]^2, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Delta &= r^2 + a^2 - 2Mr + q_{\mathrm{e}}^2 + q_{\mathrm{m}}^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \\ a &= \frac{J}{M}, \end{aligned} \tag{2}$$

where M, J, q_{e} and q_{m} are, respectively, the total mass, the spinning angular momentum, the electric charge and the magnetic charge of the field source.

In this paper, we propose an elegant mathematical method to solve the general problem of the orbital effect in the gravitational field induced by the spinning mass with electric charge and many magnetic monopoles.[16] The geodesic motion of the test particle is analysed in the next section. We obtain the precession effect of the perihelion in the spinning fields in Section 3. Finally, we discuss the results applied to other kinds of conditions by substituting the parameters into the expressions.

2. The geodesic motion of the test particle

First, we discuss the orbital motion confined to the type of conic section, by the first integral of the equation of geodesic motion we obtain the proper energy ε and the proper vector area h; by its second

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integral we obtain the orbital equation. In the spherical coordinate system, we assume that the test particle moves in the plane $\theta = \frac{\pi}{2}$, the orbit has the form of a conical curve

$$u = \frac{1}{r} = \frac{1}{p}(1 + e\cos\psi),$$
$$\psi = \varphi + \alpha(\varphi) = \varphi + \Delta\varphi, \tag{3}$$

where p and e are, respectively, the focus parameter and eccentricity of the ellipse, ψ is real abnormal angular displacement, including (compared with the Newtonian theory) the additional angular displacement $\alpha(\varphi)$ (additional abnormal). When $e < 1$, $\alpha(\varphi)$ is the additional abnormal angular displacement of the perihelion which represents the precession. In the case of circular motion, the concept of period degenerates, but the general conception of additional abnormal is effective: $\lim\limits_{e\to 0}\alpha \neq 0$.

We integrate the equation of the motion, that is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\sigma\mu}\frac{dx^\sigma}{\mathrm{d}\lambda}\right) = \frac{1}{2}\frac{\partial g_{\tau\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\tau}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda},$$
$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 1. \tag{4}$$

In the spherical coordinate system $(x^0, r, \theta, \varphi)$, for stationary symmetric radiation field, $g_{\tau\sigma}$ does not include x^0 and φ, the components $(0, 3)$ are respectively,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{0\sigma}\dot{x}^\sigma) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{3\sigma}\dot{x}^\sigma) = 0. \tag{5}$$

Integrating Eq.(5) we obtain

$$g_{0\sigma}\dot{x}^\sigma = -\varepsilon, g_{3\sigma}\dot{x}^\sigma = h, \tag{6}$$

where ε and h are integration constants. Substituting Eq.(6) into $g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 1$, we obtain the orbital equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = F(u), \tag{7}$$

where $F(u)$ is an expression about $u = r^{-1}$. In different ways, we can obtain the solution of Eq.(7) in the form of Eq.(3).

Substituting Eq.(3) into Eq.(7), we have

$$\frac{e^2}{p^2}\sin^2\psi\left(\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right)^2$$
$$= A(\varepsilon, h, p, e) + B(\varepsilon, h, p, e)\frac{e}{p}\cos\psi$$
$$+ C(\varepsilon, h, p, e)\frac{e^2}{p^2}\sin^2\psi. \tag{8}$$

Comparing coefficients of each term of the equation, we obtain

$$A(\varepsilon, h, p, e) = 0,$$
$$B(\varepsilon, h, p, e) = 0,$$
$$\psi'^2 = C(\varepsilon, h, p, e), \tag{9}$$

or we can obtain another expression of Eq.(3)

$$\alpha'^2 + 2\alpha' - C(\varepsilon, h, p, e) + 1 = 0, \tag{10}$$

where $\alpha' = \frac{\mathrm{d}\alpha}{\mathrm{d}\varphi}$, so that we obtain the expression of the periodic additional abnormal

$$\frac{\Delta\alpha}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} C^{-\frac{1}{2}}\mathrm{d}\psi - 1,$$
$$\int \mathrm{d}\alpha = -\Delta\alpha. \tag{11}$$

Substituting the metric into Eqs.(9), (10) and (11), we can obtain the expression of additional displacement at the perihelion. Substituting the values of the parameters of some planet into the expression we will obtain the precession effect of the planet.

3. The precession effect in KNK field

3.1. For Kerr spacetime

The metric is

$$\begin{aligned}\mathrm{d}s^2 =& \left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)\mathrm{d}t^2 \\ &- \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2}\mathrm{d}r^2 \\ &- (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 \\ &- \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta\mathrm{d}\varphi^2 \\ &+ \frac{4Mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\mathrm{d}t\mathrm{d}\varphi,\end{aligned} \tag{12}$$

when $\theta = \frac{\pi}{2}$, the nonzero components of $g_{\mu\nu}$ of the Kerr metric are found as follows:

$$g_{00} = 1 - \frac{2M}{r},$$
$$g_{11} = -\left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)^{-1},$$
$$g_{22} = -r^2,$$
$$g_{33} = -\left(r^2 + a^2 + \frac{2Ma^2}{r}\right),$$
$$g_{03} = g_{30} = \frac{2Ma}{r}. \tag{13}$$

Substituting Eq.(13) into Eq.(7), we obtain

$$
\begin{aligned}
F_{M,a}(u) =&\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 \\
=&\frac{\varepsilon^2-1}{h^2}+\frac{2M}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]u \\
&-\left(1+\frac{8M^2a}{h^3}\varepsilon^3\right)u^2+2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]u^3. \qquad (14)
\end{aligned}
$$

Substituting Eq.(3) into Eq.(14), and comparing Eq.(14) with Eq.(8) we obtain

$$
\begin{aligned}
A =&\frac{\varepsilon^2-1}{h^2}+\frac{2M}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]\frac{1}{p}-\left(1+\frac{8M^2a}{h^3}\varepsilon^3\right)\frac{1+e^2}{p^2} \\
&+2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]\frac{1+3e^2}{p^3}, \\
B =&\frac{2M}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]-\left(1+\frac{8M^2a}{h^3}\varepsilon^3\right)\frac{2}{p} \\
&+2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]\frac{3+e^2}{p^2}, \\
C =&1+\frac{8M^2a}{h^3}\varepsilon^3-2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]\frac{3+e\cos\psi}{p}. \qquad (15)
\end{aligned}
$$

Substituting the above results into Eq.(11) and integrating it, we can obtain the effect of test particle at the perihelion in the Kerr field which is induced by the spinning mass. We have

$$
\left(\frac{\Delta\alpha}{2\pi}\right)_a=-\frac{4a\sqrt{M}}{p\sqrt{p}}\left(1+\frac{9M}{p}\right). \qquad (16)
$$

Equation (16) shows, when the direction of the spinning field source reverses that of the particle rounding it, the correction terms make the Einstein's effect increase, vice versa, this effect will decrease.

3.2. For KNK spacetime

The metric is given by

$$
\begin{aligned}
\mathrm{d}s^2 =&\left[1-\frac{2Mr-(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}{r^2+a^2\cos^2\theta}\right]\mathrm{d}t^2 \\
&-\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2+(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}\mathrm{d}r^2 \\
&-(r^2+a^2\cos^2\theta)\mathrm{d}\theta^2-\left\{r^2+a^2+\frac{[2Mr-(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)]a^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right\} \\
&\times\sin^2\theta\mathrm{d}\varphi^2+\frac{2a\sin^2\theta}{r^2+a^2\cos^2\theta}[2Mr-(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)]\mathrm{d}t\mathrm{d}\varphi, \qquad (17)
\end{aligned}
$$

when $\theta=\dfrac{\pi}{2}$, the nonzero components of $g_{\mu\nu}$ of the KNK metric are found as follows:

$$
\begin{aligned}
g_{00} &= 1-\frac{2M}{r}+\frac{(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}{r^2}, \\
g_{11} &= -\left[1-\frac{2M}{r}+\frac{a^2}{r^2}+\frac{(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}{r^2}\right]^{-1}, \\
g_{22} &= -r^2, \\
g_{33} &= -\left[r^2+a^2+\frac{2Ma^2}{r}-\frac{(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)a^2}{r^2}\right], \\
g_{03} &= g_{30}=\frac{2Ma}{r}-\frac{(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)a}{r^2}. \qquad (18)
\end{aligned}
$$

Substituting Eq.(18) into Eq.(7), we obtain

$$\begin{aligned}F(u)=&\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2,\\ =&\frac{\varepsilon^2-1}{h^2}+\frac{2M}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]u\\ &-\left\{1+\frac{8M^2a}{h^3}\varepsilon^3+\frac{q_{\mathrm{e}}^2+q_{\mathrm{m}}^2}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]\right\}u^2\\ &+\left\{2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]+\frac{8Ma(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}{h^3}\varepsilon^3\right\}u^3\\ &-(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)\left[1-\frac{24M^2a\varepsilon}{h^3}-\frac{32M^3a\varepsilon}{h^4}(\varepsilon^2-1)\right]u^4\\ &+o[(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)^2].\end{aligned}\tag{19}$$

Substituting Eq.(3) into Eq.(19), and comparing Eq.(19) with Eq.(8) we obtain

$$\begin{aligned}A=&\frac{\varepsilon^2-1}{h^2}+\frac{2M}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]\frac{1}{p}\\ &-\left\{1+\frac{8M^2a}{h^3}\varepsilon^3+\frac{q_{\mathrm{e}}^2+q_{\mathrm{m}}^2}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]\right\}\frac{1+e^2}{p^2}\\ &+\left\{2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]+\frac{8Ma(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}{h^3}\varepsilon^3\right\}\frac{1+3e^2}{p^3}\\ &-(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)\left[1-\frac{24M^2a\varepsilon}{h^3}-\frac{32M^3a\varepsilon}{h^4}(\varepsilon^2-1)\right]\frac{1+6e^2+e^4}{p^4}\\ &+o[(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)^2],\\ B=&\frac{2M}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]-\left\{1+\frac{8M^2a}{h^3}\varepsilon^3+\frac{q_{\mathrm{e}}^2+q_{\mathrm{m}}^2}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]\right\}\frac{2}{p}\\ &+\left\{2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]+\frac{8Ma(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}{h^3}\varepsilon^3\right\}\frac{3+e^2}{p^2}\\ &-(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)\left[1-\frac{24M^2a\varepsilon}{h^3}-\frac{32M^3a\varepsilon}{h^4}(\varepsilon^2-1)\right]\frac{4+4e^2}{p^3}+o[(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)^2],\\ C=&1+\frac{8M^2a}{h^3}\varepsilon^3+\frac{q_{\mathrm{e}}^2+q_{\mathrm{m}}^2}{h^2}\left[1-\frac{2a\varepsilon}{h}(\varepsilon^2-1)\right]\\ &-\left\{2M\left[1-\frac{8M^2a}{h^3}\varepsilon-\frac{8M^3a}{h^4}\varepsilon(\varepsilon^2-1)\right]+\frac{8Ma(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)}{h^3}\varepsilon^3\right\}\frac{3+e\cos\psi}{p}\\ &+(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)\left[1-\frac{24M^2a\varepsilon}{h^3}-\frac{32M^3a\varepsilon}{h^4}(\varepsilon^2-1)\right]\frac{6+4e\cos\psi+e^2(1+\cos^2\psi)}{p^2}\\ &+o[(q_{\mathrm{e}}^2+q_{\mathrm{m}}^2)^2].\end{aligned}\tag{20}$$

Comparing the coefficients of both sides of Eq.(8), we obtain the equations $A=0, B=0$. Solving it, we obtain the values of ε and h. Substituting them into the expression of C and integrating

$$\frac{\Delta\alpha}{2\pi}=\frac{1}{2\pi}\int_0^{2\pi}C^{-\frac{1}{2}}\mathrm{d}\psi-1,\tag{21}$$

we obtain the precession effect of the test particle at the perihelion. However, the equations include higher order terms of ε, h, so we cannot get the rigorous analytic solution, we only study the orbital effect of the test particle in the KNK field taking lower order approximation. Let the body with electric and magnetic charges spin with low speed, so the values ε, h of the test particle in the KNK field, respectively, equate approximatively to those of the values in the R–N field. For simple calculation, suppose the orbit of the test particle is a circle ($e=0$), and the radius of orbit p is large enough such that $\dfrac{M}{p}<1$. We obtain

$$\varepsilon^2 = \left(1 - \frac{2M}{p} + \frac{q_{\rm e}^2 + q_{\rm m}^2}{p^2}\right)^2 \left[1 - \frac{3M}{p} + \frac{2(q_{\rm e}^2 + q_{\rm m}^2)}{p^2}\right]^{-1},$$
$$h^2 = [Mp - (q_{\rm e}^2 + q_{\rm m}^2)]\left[1 - \frac{3M}{p} + \frac{2(q_{\rm e}^2 + q_{\rm m}^2)}{p^2}\right]^{-1}. \quad (22)$$

Substituting the above results into Eq.(21) and integrating it, we can obtain

$$\begin{aligned}\left(\frac{\Delta\alpha}{2\pi}\right) =& \frac{3M}{p} + \frac{27M^2}{p^2} + \frac{135M^3}{2p^3} - \frac{q_{\rm e}^2 + q_{\rm m}^2}{2pM} - \frac{6(q_{\rm e}^2 + q_{\rm m}^2)}{p^2} \\ & - \frac{1}{8}\left(\frac{q_{\rm e}^2 + q_{\rm m}^2}{pM}\right)^2 - \frac{189M(q_{\rm e}^2 + q_{\rm m}^2)}{4p^3} + \frac{33(q_{\rm e}^2 + q_{\rm m}^2)^2}{8p^3M} \\ & + \frac{3}{4}\left(\frac{q_{\rm e}^2 + q_{\rm m}^2}{pM}\right)^3 + \frac{24Ma^2}{p^3} - \frac{4a\sqrt{M}}{p\sqrt{p}}\left(1 + \frac{9M}{p} + \frac{75M^2}{2p^2}\right) \\ & + \frac{4a\sqrt{M}}{p\sqrt{p}}\left[\frac{11(q_{\rm e}^2 + q_{\rm m}^2)}{4pM} + \frac{161(q_{\rm e}^2 + q_{\rm m}^2)}{4p^2} + 2(\frac{q_{\rm e}^2 + q_{\rm m}^2}{pM})^2\right] \\ & + o\left(\frac{1}{p^4}\right) \end{aligned} \quad (23)$$

with the precession effect caused by spinning electric and magnetic charges as follows

$$\left(\frac{\Delta\alpha}{2\pi}\right)_{q_{\rm e},q_{\rm m},a} = \frac{4a\sqrt{M}}{p\sqrt{p}}\left[\frac{11(q_{\rm e}^2 + q_{\rm m}^2)}{4pM} + \frac{161(q_{\rm e}^2 + q_{\rm m}^2)}{4p^2} + 2\left(\frac{q_{\rm e}^2 + q_{\rm m}^2}{pM}\right)^2\right]. \quad (24)$$

Equation (24) shows, when the direction of the spinning field source is the same as that of the particle rounding it, the correction terms make the Einstein's effect increase, vice versa, this effect will decrease.

4. Discussion

The KNK field, which describes a body with charges, mass, and spinning angular momentum, as an axisymmetric field, is richer than the Schwarzschild field which is a spherical field in structure. Specializing it without the charges and the angular momentum of spinning, we find that the Eq.(23) takes the form

$$\left(\frac{\Delta\alpha}{2\pi}\right)_M = \frac{3M}{p} + \frac{27M^2}{p^2} + \frac{3M^2e^2}{2p^2}, \quad (25)$$

this result we have obtained in Ref.[9], on the right side of Eq.(25) the third term can be obtained in the condition that the test particle takes elliptical movement.

We can obtain the precession effects of other kinds of planets by substituting the parameters into the expressions, which is an interesting problem we will discuss in our next paper.

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