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Hawking radiation from Kerr–Newman–Kasuya black hole via quantum anomalies*

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We have studied the Hawking radiation of the Kerr–Newman–Kasuya black hole via gauge and gravitational anomaly in the dragging coordinates. The fluxes of the electromagnetic current and the energy momentum tensor for each partial wave in two-dimensional field are obtained.

Keywords: Hawking radiation, Kerr–Newman–Kasuya black hole, anomaly
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1. Introduction

The study of Hawking radiation has attracted more and more attention in theoretical physics since Hawking[1] first proved the existence of black hole radiation, especially in recent years. There are several derivations of Hawking radiation in the literature. One of them was proposed by Parikh and Wilczek several years ago.[2] They treated Hawking radiation as a tunnelling process, by using the Wentzel–Kramers–Brillouin (WKB) method. Following their framework, Hawking radiations from several types of black holes have been studied.[3−16]

Very recently, Robinson and Wilczek[17] have presented a new partial derivation of Hawking radiation, which ties its existence to the cancellation of gravitational anomaly at the horizon of the Schwarzschild-type black hole. The anomaly in field theory is defined if the symmetry of the action or the corresponding conservation law, valid in the classical theory, is violated in the quantum theory. The basic idea of Robinson and Wilczek's work is as follows: by reducing dimensions, the physics near the horizon can be described with an infinite collection of free (1+1)-dimensional fields because the mass and the interaction terms of quantum field in the background are suppressed. If only the effective field theory near the horizon is considered, it becomes chiral since outgoing modes there are eliminated. And quantum mechanically, the effective action for the metric due to matter fields becomes anomalous with respect to general coordinate transformations. The Hawking flux of energy momentum tensor is determined so that it cancels the gravitational anomaly in the consistent form at the horizon. Robinson and Wilczek's approach is effectively equivalent to dilaton-coupled trace anomaly in two dimensions. With regard to this, many efforts have been devoted to extending their proposal to many other cases.[18−25] However, much less attention was paid to the case of Hawking radiation of the Kerr–Newman–Kasuya black hole, which is a type of rotating black hole with electric charge and magnetic charge. Hawking flux from this type of black hole can also be determined in terms of the values of anomalies at the horizon by demanding gauge invariance and general coordinate covariance at the quantum level.

The remainder of this paper is organized as follows. In Section 2, we investigate the Hawking radiation from the Kerr–Newman–Kasuya black hole in the Boyer–Lindquist coordinates and in the dragging coordinates. In Section 3, we give some discussion and conclusions.

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2. Hawking radiation from Kerr–Newman–Kasuya black hole via quantum anomalies

According to an exact solution of a rotating dyon black hole, the metric of the four-dimensional Kerr–Newman–Kasuya black hole in the Boyer–Lindquist coordinates can be expressed as[26,27]

$$\begin{aligned}\mathrm{d}s^2 =&-\left(1-\frac{2Mr-e^2-q^2}{\Sigma}\right)\mathrm{d}t^2+\frac{\Delta}{\Sigma}\mathrm{d}r^2\\&+\Sigma\mathrm{d}\theta^2+\left(\frac{(2Mr-e^2-q^2)a^2\sin^2\theta}{\Sigma}\right.\\&\left.+r^2+a^2\right)\sin^2\theta\mathrm{d}\phi^2\\&-\frac{2a\sin\theta}{\Sigma}(2Mr-e^2-q^2)\mathrm{d}t\mathrm{d}\phi,\end{aligned}\tag{1}$$

where $\Sigma=r^2+a^2\cos^2\theta$,

$$\begin{aligned}\Delta=&r^2+a^2+e^2+q^2-2Mr\\=&(r-r_+)(r-r_-),\end{aligned}$$

$$a=\frac{J}{M}.$$

The parameters M, e, q represent the mass, the electric charge and the magnetic charge of the black hole, respectively, and a is the angular momentum per unit mass of the black hole. r_+, r_- are the outer and the inner horizons, $r_\pm=M\pm\sqrt{M^2-a^2-e^2-q^2}$.

By using a dimensional reduction technique, the Kerr–Newman–Kasuya black hole in the Boyer–Lindquist coordinates (t, r, θ, ϕ) can be reduced to that in a two-dimensional spacetime. For a while we consider a scalar field for simplicity. The action for the scalar field in the Kerr–Newman–Kasuya spacetime is

$$\begin{aligned}S[\varphi]=&\frac{1}{2}\int\mathrm{d}^4x\sqrt{-g}\varphi\nabla^2\varphi+S_{\rm int}\\=&\frac{1}{2}\int\mathrm{d}t\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\sin\theta\varphi\left[\frac{\Delta a^2\sin^2\theta-(r^2+a^2)^2}{\Delta}\partial_t^2+\partial_r\Delta\partial_r+\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta\right.\\&\left.+\frac{\Delta-a^2\sin^2\theta}{\Delta\sin^2\theta}\partial_\phi^2+\frac{2a(\Delta-r^2-a^2)}{\Delta}\partial_t\partial_\phi\right]\varphi+S_{\rm int},\end{aligned}\tag{2}$$

where $S_{\rm int}$ includes a mass term, potential terms and interaction terms. Taking the limit $r\to r_+$, and considering that the kinetic term makes a dominant contribution to the action, we can ignore the mass and interaction terms $S_{\rm int}$. Keeping the dominant terms, the action becomes

$$\begin{aligned}S[\varphi]=&\frac{1}{2}\int\mathrm{d}t\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\sin\theta\varphi\left[-\frac{(r_+^2+a^2)^2}{\Delta}\partial_t^2+\partial_r\Delta\partial_r\right.\\&\left.-\frac{a^2}{\Delta}\partial_\phi^2+\frac{2a(r_+^2+a^2)}{\Delta}\partial_t\partial_\phi\right]\varphi.\end{aligned}\tag{3}$$

Then, we introduce the dragging coordinates (ξ, r, θ, ψ) through $\xi=t$ and $\psi=\phi-\Omega_+t$, where Ω_+ is the dragging speed at the outer horizon, and $\Omega_+\equiv-\left.\frac{g_{t\phi}}{g_{\phi\phi}}\right|_{r_+}=\frac{a}{r_+^2+a^2}$.[28,29] To consider the physics near the horizon, we take the new dragging coordinates and transform them into the tortoise coordinate defined by $\left.\frac{\mathrm{d}r_*}{\mathrm{d}r}\right|_{r_+}=\frac{r_+^2+a^2}{\Delta}$, then the action expression (3) can be rewritten as

$$\begin{aligned}S[\varphi]=&\frac{a}{2\Omega_+}\int\mathrm{d}\xi\mathrm{d}r\mathrm{d}\theta\mathrm{d}\psi\sin\theta\varphi\\&\times\left(-\frac{1}{f(r)}\partial_\xi^2+\partial_rf(r)\partial_r\right)\varphi.\end{aligned}\tag{4}$$

The expression shows that the angular derivative terms disappear completely from the action. Performing the partial wave decomposition of the scalar field in the Kerr–Newman–Kasuya black hole in terms of the spherical harmonics $\varphi=\sum\limits_{l,m}\varphi_{lm}(t,r)Y_{lm}(\theta,\phi)$ at the horizon, the action finally reads

$$\begin{aligned}S[\varphi]=&\frac{a}{\Omega_+}\sum_{l,m}\frac{1}{2}\int\mathrm{d}\xi\mathrm{d}r\varphi_{lm}\\&\times\left(-\frac{1}{f(r)}\partial_\xi^2+\partial_rf(r)\partial_r\right)\varphi_{lm},\end{aligned}\tag{5}$$

where φ_{lm} is considered as a (1+1)- dimensional complex scalar field in the background of the dilaton Φ, metric $g_{\mu\nu}$ and $U(1)$ gauge field A_μ. Because

of the static background, however, the contribution of the dilaton background to the total flux can be neglected. Then, we can obtain the effective two-dimensional metric from the above action expression as $\mathrm{d}s^2 = -f(r)\mathrm{d}\xi^2 + \dfrac{1}{f(r)}\mathrm{d}r^2$. So we have $g_{\xi\xi} = -f(r) = -\dfrac{\Delta}{r^2+a^2}$, $g_{rr} = \dfrac{1}{f(r)}$, $A_t = -\dfrac{\sqrt{e^2+q^2}r}{r^2+a^2}$, and $A_r = 0$. Making the tortoise coordinate transformation and then performing the potential wave decomposition, we find that the effective radial potential for partial wave modes of the field ($\sim l(l+1)/r^2$) contains the factor $f(r(r_*))$ and vanishes exponentially fast near the horizon. The same consideration may be applied to mass terms and other interactions.

Near the horizon of the Kerr–Newman–Kasuya black hole, each partial wave of scalar fields behaves as a charged black hole. These two-dimensional fields are interacting with each other but because of the axial symmetry of the Kerr–Newman–Kasuya black hole metric in the ϕ-direction, the azimuthal quantum number, m of Y_{lm}, is conserved.

In the framework of Robinson and Wilczek's approach, once the ingoing modes fall into the black hole, they never come out classically and cannot affect the physics outside the black hole. Quantum mechanically, however, if we neglect the ingoing modes near the horizon, the effective two-dimensional theory becomes chiral near the horizon, and the gauge symmetry and the general coordinate covariance become anomalous due to the gauge and gravitational anomalies. In the case of Kerr–Newman–Kasuya black hole, by using the effective two-dimensional reduction, each partial wave exhibits an original gauge symmetry with respect to the electromagnetic field of the black hole and an induced $U(1)$ gauge symmetry originating from the isometry along the ϕ–direction. So the effective chiral for Kerr–Newman–Kasuya black hole contains two gauge anomalies and a gravitational anomaly in all after omitting the classically irrelevant ingoing modes at the horizon. If we introduce the dragging coordinate system, however, the $U(1)$ gauge anomaly associated with the induced symmetry originating from the isometry along the ϕ-direction vanishes. An observer rest in the dragging coordinates would not observe the $U(1)$ gauge current flux, for he is co-rotating with the black hole. Therefore, the effective theory contains only one gauge anomaly associated with the electromagnetic field of the black hole.

Firstly, we investigate the flux from the $U(1)$ gauge anomaly. We divide the two-dimensional region into two ones, i.e. the outside of the horizon region $r \geq r_+ + \varepsilon$ and the near-horizon region $r_+ \leq r \leq r_+ + \varepsilon$. After omitting the ingoing modes in the region $r_+ \leq r \leq r_+ + \varepsilon$, the current exhibits a gauge anomaly there. Take the $\varepsilon \to 0$ ultimately, the explicit form of the gauge anomaly will be given as[20]

$$\nabla_\mu J^\mu = \frac{m^2}{4\pi\sqrt{-g}}\varepsilon^{\mu\nu}\partial_\mu A_\nu, \tag{6}$$

where $\varepsilon^{\mu\nu}$ is the two-dimensional Levi-Civita tensor, the convention ε^{01}=+1 has been used and m has been treated as the gauge charge of the $U(1)$ gauge field A_ν.

Since the current J^μ transforms non-covariantly, we should define a new covariant current $\tilde{J}^\mu = J^\mu - \dfrac{m^2}{4\pi\sqrt{-g}}\varepsilon^{\lambda\nu}A_\lambda$, which satisfies $\nabla_\mu\tilde{J}^\mu = \dfrac{m^2}{4\pi\sqrt{-g}}\varepsilon_{\mu\nu}F^{\mu\nu}$, where the coefficient of the covariant anomaly is twice as large as that of the consistent anomaly.

In the outside of the horizon region $r \geq r_+ + \varepsilon$, since there exists no anomaly as shown by the fundamental theory, the electromagnetic current is conserved, and it satisfies the conserved equation

$$\partial_r J^r_{(O)} = 0. \tag{7}$$

While in the near-horizon region $r_+ \leq r \leq r_+ + \varepsilon$, since there exists only an outgoing field, the current satisfies the anomalous equation

$$\partial_r J^r_{(H)} = \frac{m^2}{4\pi}\partial_r A_t. \tag{8}$$

The subscripts O and H in Eqs.(7) and (8) represent the values in the regions $r \geq r_+ + \varepsilon$ and $r_+ \leq r \leq r_+ + \varepsilon$, respectively. Solve the above equations, and we will have

$$J^r_{(O)} = c_O, \tag{9}$$

and

$$J^r_{(H)} = c_H + \frac{m^2}{4\pi}(A_t(r) - A_t(r_+)), \tag{10}$$

where c_O and c_H are integration constants and represent the values of the consistent current at infinity and the horizon, respectively.

Since the current J^μ should be the sum of the both parts $J^r_{(O)}$ and $J^r_{(H)}$, we have

$$J^\mu = J^\mu_{(O)}\Theta_H(r) + J^\mu_{(H)}H(r), \tag{11}$$

where $\Theta_H(r)=\Theta(r-r_+-\varepsilon)$ is a scalar step function and $H(r)=1-\Theta_H(r)$ is a scalar "top hat" function.

Under gauge transformations, the variation of the effective action near the horizon is given by $-\delta W=\int \mathrm{d}^2x\sqrt{-g_{(2)}}\lambda\nabla_\mu J^\mu_{(2)}$, where λ is a gauge parameter. Then by using the anomaly equation, the variation becomes

$$-\delta W=\int \mathrm{d}^2x\sqrt{-g_{(2)}}\lambda\Big[\delta(r-r_+-\varepsilon)\Big(J^r_{(O)}-J^r_{(H)}+\frac{m^2}{4\pi}A_t\Big)+\partial_r\Big(\frac{m^2}{4\pi}A_tH\Big)\Big]. \tag{12}$$

The total effective action must be gauge invariant, so we have $-\delta W=0$. Since the last term cannot be cancelled by delta-function terms, it should be cancelled by quantum effects of the classically irrelevant ingoing modes. Thus, we have$J^r_{(O)}-J^r_{(H)}+\frac{m^2}{4\pi}A_t=0$, and substituting the expressions (9) and (10) to it yields

$$c_O=c_H-\frac{m^2}{4\pi}A_t(r_+). \tag{13}$$

In order to determine the current flux, we need to fix the value of the current at the horizon c_H. Since the condition should be gauge covariant, we impose that the coefficient of the covariant current at the horizon should vanish. Since $\tilde{J}^r=J^r+\frac{m^2}{4\pi}A_tH(r)$, and $\tilde{J}^r(r_+)=0$, those conditions determine the value of the charge flux to be

$$c_O=-\frac{m^2}{2\pi}A_t(r_+)=\frac{m^2\sqrt{e^2+q^2r_+}}{2\pi(r_+^2+a^2)}, \tag{14}$$

which is exactly equal to that of blackbody radiation at the Hawking temperature with an appropriate chemical potential.

Then, we investigate the flux of the energy momentum tensor. In the region $r\geq r_++\varepsilon$, there exists an effective gauge potential. Thus the energy momentum tensor satisfies the modified conservation equation

$$\partial_rT^r_{t(O)}=J^r_{(O)}\partial_rA_t, \tag{15}$$

which is related to a componentT^r_t of a 4-dimensional energy momentum tensor. In the near-horizon region $r_+\leq r\leq r_++\varepsilon$, the energy momentum tensor exhibits a gravitational anomaly. The explicit form of the gravitational anomaly is

$$\partial_rT^r_{t(H)}=F_{rt}J^r_{(H)}+A_t\nabla_\mu J^\mu_{(H)}+\partial_rN^r_t, \tag{16}$$

where $F_{rt}=\partial_rA_t$, and $N^r_t=\frac{1}{192\pi}(f'^2+ff'')$. The second term comes from the gauge anomaly and the third is the gravitational anomaly for the consistent energy momentum tensor. The first and second terms can be combined in terms of the covariant current $\tilde{J}^r_{(H)}$ into $F_{rr}\tilde{J}^r_{(H)}$. Using the results we have obtained before and solving Eqs.(15) and (16), we have

$$T^r_{t(O)}=a_O+c_OA_t(r), \tag{17}$$

$$T^r_{t(H)}=a_H+\int_{r_+}^r\mathrm{d}r\partial_r\Big(c_OA_t+\frac{m^2}{4\pi}A_t^2+N^r_t\Big), \tag{18}$$

where a_O and a_H are integrated constants, the values of the energy flux at the infinity and the horizon, respectively. The energy momentum tensor should be the sum of energy momentum tensors in the two regions, i.e. $T^\mu_\nu=T^\mu_{\nu(O)}\Theta_H(r)+T^\mu_{\nu(H)}H(r)$.

Under the general coordinate transformation $x^\mu\to x^\mu-\lambda^\mu$, the variation of the effective action is induced as

$$\begin{aligned}-\delta_\lambda W&=\int \mathrm{d}^2x\sqrt{-g}\lambda^t\nabla_\mu T^\mu_t\\&=\int \mathrm{d}^2x\lambda^t\Big[\delta(r-r_+-\varepsilon)\Big(T^r_{t(O)}-T^r_{t(H)}+\frac{m^2}{4\pi}A_t^2+N^r_t\Big)+c_O\partial_rA_t(r)\\&\quad+\partial_r\Big(\frac{m^2}{4\pi}A_t^2+N^r_t\Big)\Big],\end{aligned} \tag{19}$$

where the second term comes from the classical effect of the background electromagnetic field for constant flow, and the third should be cancelled by the quantum effect of the ingoing modes. In order to restore the diffeomorphism covariance at the horizon, the coefficient of the first term should vanish. So we have

$$a_O=a_H+\frac{m^2}{4\pi}A_t^2(r_+)-N^r_t(r_+), \tag{20}$$

in which $a_O=T^r_{t(O)}-c_OA_t$. Imposing a condition[19,20,30] under which the covariant energy momentum tensor $\tilde{T}^r_t=T^r_t+N^r_t$ is required to vanish at the horizon, we can obtain

$$a_H=2N^r_t(r_+)=\frac{\kappa^2}{24\pi}, \tag{21}$$

where $\kappa=\frac{1}{2}\partial_rf|_{r_+}$ is the surface gravity of the black hole.

The flux of the energy momentum tensor, which is required to restore the general coordinate covariance at a quantum level in the effective field theory, is

$$\begin{aligned}a_O=&\frac{m^2(e^2+q^2)}{4\pi r_+^2}+N^r_t(r_+)\\=&\frac{m^2(e^2+q^2)}{4\pi r_+^2}+\frac{\pi}{12}T_+^2,\end{aligned} \tag{22}$$

where $T_+ = \dfrac{\kappa}{2\pi}$ is the Hawking temperature of the black hole at the horizon.

3. Discussion and conclusions

The framework of this paper, however, is a little different from that of Robinson and Wilczek, in which the outgoing modes near the horizon are eliminated and thus effective theory is chiral there, and the Hawking flux of energy-momentum tensor is determined such that it cancels the gravitational anomaly in the consistent form at the horizon. In this paper, ingoing modes, which are classically irrelevant to physics outside the horizon, are integrated out near the horizon. The Hawking fluxes are determined by requiring the covariant current and the energy momentum tensor to vanish at the horizon.

We now compare our results with the fluxes from the blackbody radiation moving in the positive r-direction at an inverse temperature $\beta = \dfrac{1}{T_+}$ with a chemical potential.

In the dragging coordinates, the Hawking distribution is given by Planck distribution $N_{Q,M}(\omega') = \dfrac{1}{\mathrm{e}^{\beta(\omega' - \sqrt{e^2+q^2}A_t(r_+))}+1}$, where $\omega' = \omega - m\Omega_+$ is the energy carried by the observer in the dragging coordinate system. Here we have used the Planck distribution for fermions in order to avoid the problem of superradiance which is related to scatterings away from the horizon. So the electromagnetic current and energy momentum tensor fluxes of Hawking radiation are given by

$$\begin{aligned} F_Q =& m\int_0^\infty \frac{1}{2\pi}[N_{+Q}(\omega') - N_{-Q}(\omega')]\mathrm{d}\omega' \\ =& \frac{m^2\sqrt{e^2+q^2}r_+}{2\pi(r_+^2+a^2)}, \end{aligned} \tag{23}$$

$$\begin{aligned} F_M =& \int_0^\infty \frac{\omega'}{2\pi}[N_{+Q}(\omega') + N_{-Q}(\omega')]\mathrm{d}\omega' \\ =& \frac{m^2(e^2+q^2)}{4\pi r_+^2} + \frac{\pi}{12}T_+^2. \end{aligned} \tag{24}$$

The results show that in the dragging coordinates the electromagnetic current and the energy tensor fluxes of Hawking radiation from Kerr–Newman–Kasuya black hole have the equivalent forms to those obtained by requiring the gauge invariance and the general coordinate covariance at a quantum level to hold in the effective theory.

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