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Fractional Fourier transform of Cantor sets: further numerical study

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This paper is a further work of the authors' paper published previously (Liao T H and Gao Q 2005 *Chin. Phys. Lett.* **22** 2316). The amplitudes of fractional Fourier transform of Cantor sets are analysed from the viewpoint of multifractal by wavelet transform maxima method (WTMM). An integral operation is carried out before the application of WTMM, such that the function obtained can be considered as the perturbed devil staircase. Also, wavelets with large number of vanishing moments are used, which makes the complete singularity spectrum more accessible. The validity of multifractal formalism is guaranteed by restricting parameter q to a proper range, so that the phenomenon of multifractal phase transition can be explained reasonably. Particularly, the method of determining the range of parameter q in the above paper is developed to be more operational and rigorous.

Keywords: multifractal, fractional Fourier transform, wavelet transform, perturbed devil staircase
PACC: 0555, 4230K

1. Why a new paper?

In the authors' previous paper,[1] the amplitudes of fractional Fourier transform (FRFT) of Cantor sets are analysed from the viewpoint of multifractal by wavelet transform maxima method (WTMM). According to our current understanding, Ref.[1] is not mature in some aspects. Firstly, only one wavelet (first derivative of Gaussian function) is used in the multifractal analysis, which is not enough for a function with unknown singularity spectrum. Secondly, the method of determining the range of parameter q in Ref.[1] is not concise enough for its application.

In the current paper, these drawbacks will be corrected and some new ideas will also be included. This paper is organized as follows. In Section 2 the possibility of analytic computation of FRFT of Cantor sets is considered. Because of the tedious form of FRFT of rectangular function and the shift rule of FRFT, we have to resort to numerical method in the computation. Section 3 is the heart of this paper. We calculate the infinite integral of the amplitudes of FRFT of Cantor sets, and the obtained function can be considered as a perturbed version of devil staircase in common sense. The singularity spectrum of the obtained function is calculated by WTMM with wavelet of large number of vanishing moments. Section 4 discusses the results obtained and some important ideas are stressed.

2. Computation of FRFT of Cantor sets

The mathematical definition of FRFT can be found in Refs.[2–4] which is omitted here, and FRFT of order a is denoted as F^a throughout this paper.

Cantor sets can be constructed by expanding or dividing method. The expanding method is started with a rectangular function $f^0\ (x) = \text{rect}\ (1,0)$, which is of width 1 and centred at $x = 0$. Then Cantor set at nth stage can be generated recursively as[5]

$$f^n(x) = f^{n-1}(x) * \delta(x - 3^{n-1}) + f^{n-1}(x) * \delta(x + 3^{n-1}), \quad (1)$$

where * denotes convolution operator and $\delta(x - x_0)$ is a Dirac function at x_0. Taking ath order FRFT of the two sides of Eq.(1)

$$f_a^n(\xi) = F^a[f^{n-1}(x - 3^{n-1})](\xi) + F^a[f^{n-1}(x + 3^{n-1})](\xi), \quad (2)$$

where $f_a^n(\xi)$ is the FRFT of $f^n(x)$. According to the

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shift rule of FRFT,[4] we get

$$\begin{aligned} f_a^n(\xi) = & \exp(\mathrm{i}3^{2(n-1)}\sin\alpha\cos\alpha/2) \\ & \times[\exp\left(-\mathrm{i}3^{n-1}\xi\sin\alpha\right) f_a^{n-1}(\xi - 3^{n-1}\cos\alpha) \\ & + \exp(\mathrm{i}3^{n-1}\xi\sin\alpha) f_a^{n-1}(\xi + 3^{n-1}\cos\alpha)], \end{aligned} \tag{3}$$

where $\alpha = a\pi/2$. As indicated in Ref.[4], the general expression of FRFT of rectangular function is rather complex. Therefore, it is not a practical idea to write the expression of $f_a^n(\xi)$ recursively and compute it analytically, and numerical computation seems to be the only possible choice. However, there exist exceptional cases. If $\cos\alpha = 0$, it can be easily proved that

$$f_a^n(\xi) = 2^n \left[\prod_{j=0}^{n-1} \cos\left(3^j\xi\right)\right] f_a^0(\xi), \tag{4}$$

where

$$f_a^0(\xi) = \sin\pi\xi/\pi\xi.$$

For the convenience of numerical computation, this paper considers Cantor sets constructed by division method.[6] If we denote Cantor set at nth stage constructed in this way as $\tilde{f}^n(x)$, it can be easily found that

$$\tilde{f}^n(x) = f^n(x/3^n), \tag{5}$$

and

$$\tilde{f}_1^n(\xi) = 3^n f_1^n(3^n\xi). \tag{6}$$

For Cantor set at fifth stage studied in this paper, its Fourier amplitude is not larger than 5‰ for $\xi \geq 4000\pi$, and we take $\xi_c = 4000\pi$ as cut-off frequency. So the sampling interval in numerical computation should not be larger than 1/4000.

There are many works concerning the fast computation of FRFT,[7–9] and we use the program written by Bultheel and Sulbaran.[10]

3. Multifractal analysis

3.1. Model and method

This paper also studies the amplitudes of FRFT of Cantor sets from the viewpoint of multifractal, and the only difference with Ref.[1] is that an integral operation is carried out before the calculation of multifractal parameters, which is a common strategy in relevant studies,[11,12] i.e., the object of multifractal analysis is function

$$g_a^n(x) = \int_{-\infty}^{x} |\tilde{f}_a^n(\xi)| \mathrm{d}\xi. \tag{7}$$

An example of the pair of $\tilde{f}_a^n(\xi)$ and $g_a^n(x)$ is shown in Fig.1 by solid line, where the relevant Cantor set is at fifth stage with sampling interval 2^{-12}, and the order of FRFT is 1/20. Also, the related devil staircase in common sense[12] is shown in Fig.1(b) by dashed line (a constant is multiplied to match the former). We find that $g_a^n(x)$ can be considered as a perturbed version of devil staircase. In fact, devil staircase perturbed by polynomial or infinite differentiable function has been studied by Refs.[11,13,14], and the perturbation here is generated by FRFT of small orders.

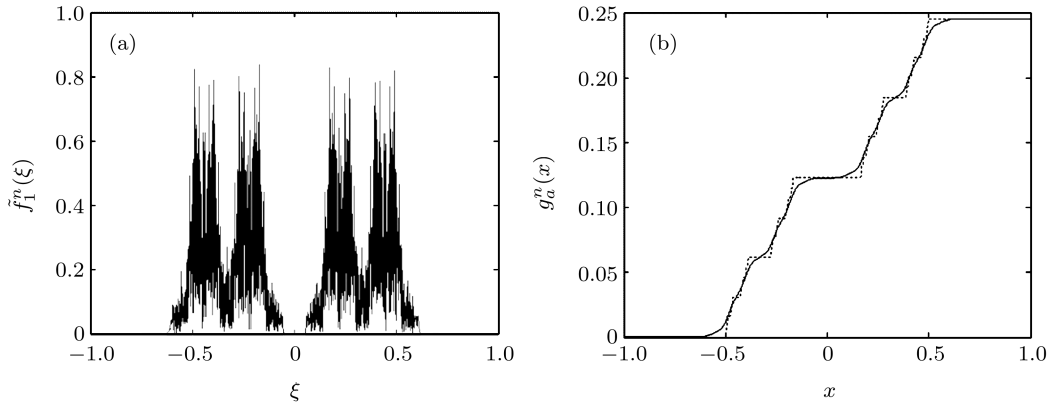


Fig.1. Amplitude of FRFT of Cantor set and its integral of form Eq.(7), where the relevant Cantor set is at fifth stage with sampling interval 2^{-12}, and the order of FRFT is 1/20. In (b), devil staircase response to the Cantor set is also shown by dashed line (a constant is multiplied to match the integral function shown by solid line).

As clarified in Refs.[15,16], FRFT is equivalent to the propagation of optical field in quadratic graded-index medium, and its order is proportional to propagation distance. So here the space evolution of fractal sets is studied in essence. This paper only considers FRFT of small orders, so that the self-similarity of Cantor sets is degraded but partially preserved. The reason of our preference of FRFT to other optical transforms (such as Fresnel transform) is that its order can fully determine the FRFT of a function, which makes our model being concise. One may criticize the triviality of our model for using the most simple fractal sets: Cantor sets. In fact, this study is inspired by optical diffraction of fractal screen.[5,17,18] FRFT of more complex fractals seems to be more meaningful, but the severe problems of mathematical definition and physical realization are encountered. We hope that both theoretical and experimental researchers are interested in our model.

This paper uses the famous WTMM in the following multifractal analysis, in which the wavelets are successive derivatives of Gaussian function

$$\psi^{(N)}(x) = \mathrm{d}^N \exp(-x^2/2)/\mathrm{d}x^N, \quad N = 1, 2, \ldots. \tag{8}$$

The details of this method can be found in Refs.[11–14], which is omitted here. Let's denote the scaling function as $\tau(q)$ and the singularity spectrum as (h, $D(h)$), which are connected through Legendre transform

$$D(h) = \min_q [qh - \tau(q)], \quad h(q) = \partial\tau(q)/\partial q. \tag{9}$$

Provided that the vanishing moments of wavelet is large enough, the correct singularity spectrum can be obtained.[13,14] In addition, the method of filtering the divergent maxima lines is also adopted.[19] Parameter q can be any real number for pure mathematical fractals. However, non-continuity of singularity spectrum often occurs for fractals in numerical and experimental studies, which is called 'phase transition' in many references.[20−24] In the authors' opinion, this phenomenon reflects some kind of limitation of multifractal theory in its application. In fact, the validity of multifractal formalism is proved rigorously only for self-similar functions.[25] Following the basic idea of Ref.[1], we find that multifractal phase transition implies that the range of q should be restricted properly. Here two properties of multifractal analysis are applied to determine the range of q:

$$\mathrm{d}D(h(q))/\mathrm{d}q = q, \tag{10a}$$

$$\mathrm{d}^2\tau(q)/\mathrm{d}q^2 \leq 0. \tag{10b}$$

The first one can be found in the paper of Halsey *et al*,[26] and the second one is proved by Jaffard.[25] Equations (10a) and (10b) are further consequences of the extremal condition of Legendre transform Eq.(9).

3.2. Numerical results

We take the function shown in Fig.1(b) (solid line) as an example to illustrate our method. Some results obtained from wavelet $\psi^{(1)}$ are shown in Fig.2, where the first derivative of $D(h(q))$ with respect to q is shown in Fig.2(a) (the dashed line is q vs q) and the second derivative of scaling function $\tau''(q)$ is shown in Fig.2(b). It can be easily found that there is not a proper range of q which can make Eqs.(10a) and (10b) be correct, except for the ones at left and right sides of the figure. However, Hausdorff dimension ($q = 0$), information dimension ($q = 1$) and correlation dimensions ($q = 2, 3, \ldots$) can not be studied if we accept these qs, which departs from the basic idea of multifractal theory.[26,14]

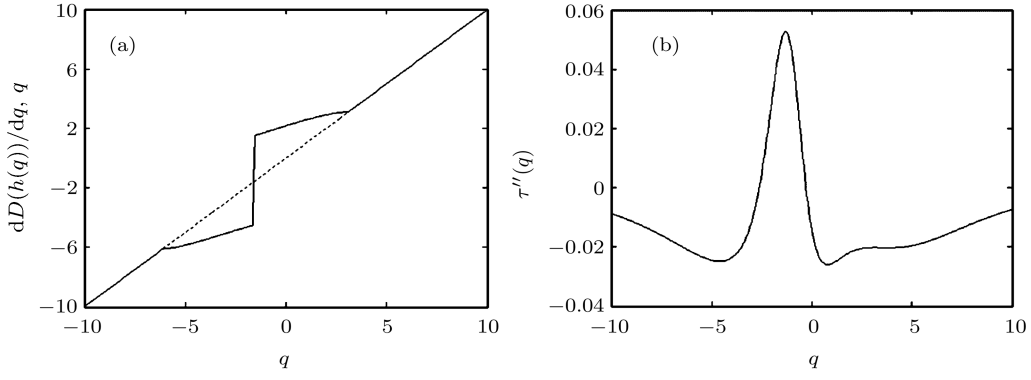


Fig.2. Some results of WTMM analysis of the function in Fig.1(b) (solid line) with wavelet $\psi^{(1)}$. (a) q vs $\mathrm{d}D(h(q))/\mathrm{d}q$ (solid line) and q vs q (dashed line); (b) q vs $\tau''(q)$.

The origins of this phenomenon may come from two aspects. The first one is due to FRFT which breaks the pure self-similarity of Cantor set, and the second one is due to the analyzing wavelet. The partial breakdown of self-similarity is what this paper concerns, and we choose different wavelets to reveal it. As pointed out in Refs.[13,14], wavelets with larger number of vanishing moments make the entire singularity more accessible when studying perturbed functions. Figure 3 shows the results obtained from $\psi^{(3)}$. It can be found approximately that Eq.(10a) holds for $q \in [-2.25, 3.45]$ and that Eq.(10b) holds for $q \in [-4.30, 5.45]$. Observe the four short lines in the left and right sides of Fig.3(a). If we use the procedure in Ref.[1] to determine the range of q in this example, we have to undergo at least four steps. The method of this paper about verifying directly Eqs.(10a) and (10b) can simplify this process and put the results on a more rigorous base.

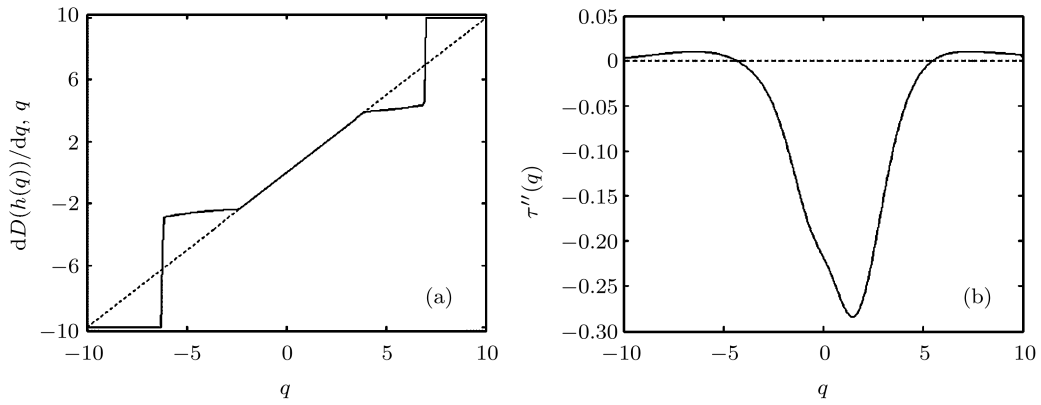


Fig.3. Some results of WTMM analysis of the function in Fig.1(b) (solid line) with wavelet $\psi^{(3)}$. (a) q vs $\mathrm{d}Dh(q))/\mathrm{d}q$ (solid line) and q vs q (dashed line); (b) q vs $\tau''(q)$ (solid line) and q vs 0 (dashed line).

The singularity spectrum of the function in Fig.1(b) (solid line) is shown in Fig.4 by circles, which is obtained with wavelet $\psi^{(3)}$ and $q \in [-2.25, 3.45]$. An important feature of this spectrum is its negative part for h of lower value, which is a sign of very rare events.[27,28] Also, the singularity spectrum of the function similar to the one in Fig.1(b) (solid line) but with the order of FRFT being 1/40 is shown in Fig.4 by squares, which is obtained with wavelet $\psi^{(4)}$ and $q \in [-0.25, 2.1]$. It is worth noting that there is not a definite rule for the choice of analyzing wavelet. The computation of this paper chooses the wavelet which makes the range of q where Eqs.(10a) and (10b) hold as large as possible, and the statement in the first paragraph of Subsection 3.2 is also concerned. Some clues for the evolution of singularity spectrum can be inferred from Fig.4. Remembering that the spectrum of Cantor set is a single point (ln2/ln3, ln2/ln3), we find that for FRFT order of small value, there exists the non-monotonic behaviour of the Holder exponent corresponding to the largest dimension, which reflects the leading singular behaviour. Also, the evolution of singularity spectrum with FRFT order is quite abrupt from 0 to very small value (such as 1/40).

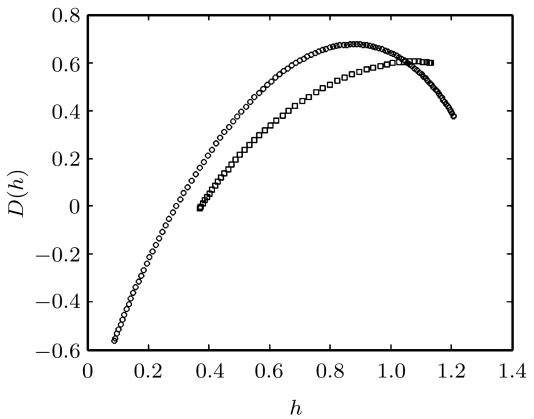


Fig.4. Singularity spectra of amplitudes of FRFT of Cantor set at the fifth stage with the order being 1/40 (squares) and 1/20 (circles).

4. Conclusion and discussion

We have analysed the amplitude of FRFT of Cantor set from the viewpoint of multifractal. This problem can be equivalent to the study of perturbed devil staircase, where the perturbation is generated by FRFT of small order. The validity of multifractal for-

malism is guaranteed by restricting parameter q to a proper range, so that the phenomenon of multifractal phase transition can be explained reasonably. Also, the method of determining this range in Ref.[1] is developed to be more operational and rigorous. In fact, Lashermes *et al*[29] have proposed an operational definition of multifractal by constraining parameter q to a proper range, but the method of how to determine it is not given.

Let us stress that our results are some-what dependent on sampling interval. Although Section 2 of this paper gives an upper bound for sampling interval by sampling theorem, we conjecture that the existence of a great number of discontinuous points of Cantor set (in the sense of function) makes the numerical computation of FRFT not be uniformly convergent, which can be considered as an extension of the Gibbs phenomenon in the classical Fourier domain.[30]

In addition, the perturbation generated by FRFT of small order is interesting in itself, which offers a new perspective into the study of perturbed multifractal function.

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