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Study on phase synchronization of stochastic chaotic system*

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This paper detects and characterizes the diverse roles played by bounded noise in chaotic phase synchronization (CPS) of weakly coupled nonlinear stochastic systems. Analysis of a paradigmatic model of two bidirectional coupled three-level food chains is carried out by various statistical measures such as Shannon entropy and mutual information. The results indicate that inside the synchronous regime, CPS is considerably reduced under the influence of bounded noise; near the onset of phase synchronization, temporal phase locking is diversely changed with the increase of noise, i.e., either weak or strong noise also degrades the degree of CPS, while intermediate noise enhances CPS remarkably, and an optimal noise intensity is detected that maximizes the enhancement.

Keywords: chaotic phase synchronization, coupling, bounded noise
PACC: 0545, 0547

1. Introduction

Starting from the observation of synchronization of two pendulum clocks by Huygens,[1] we find that the notion of chaos synchronization has received substantive attention for its fundamental importance in theory and potential applications to electronic circuits, secure communications and so on.[2−6] In a conventional sense, the term of chaos synchronization means that all involved oscillators evolve identically.[7] In practice, the existence of uncertainty and parameter misfit in the considered oscillators sometimes renders perfect synchronization impossible. Yet, a weak degree of synchronization, such as generalized synchronization and chaotic phase synchronization (CPS)[8−11] can still occur due to the interaction among them. CPS refers to coincidence of the characteristic time scales of coupled oscillators, while their amplitude dynamics remain chaotic and almost uncorrelated. The necessary condition for CPS of two chaotic oscillators can be quantified by phase locking,[9] i.e., $\Delta\phi(t) = |n\phi_1(t) - m\phi_2(t)| < \text{const}$ (m and n are integers) in which $\phi_1(t)$ and $\phi_2(t)$ are two properly defined phase variables, or by frequency entrainment, i.e., $n\Omega_1 = m\Omega_2$ in which $\Omega_{1,2} = \langle\dot{\phi}_{1,2}\rangle$ are average frequencies and brackets mean time averaging.

The phenomenon of CPS is ubiquitous in nature, which has been observed in phase coherent chaotic systems including oscillators driven by periodic force,[9,12] oscillators with parameter mismatch[13,14] even with essentially different structures,[15,16] and array of coupled solid state lasers.[17−19] On the other hand, random disturbance, generally modelled by a certain kind of noise, is unavoidable in experimental and real systems. Thus, it is desirable to explore influences of noise on the phase synchronization process.[20−22] To the best of our knowledge, the noise involved in most of the literatures is described by unbounded Gaussian white noise. Bounded noise[23] is a harmonic function with constant amplitude and random frequency and phase. Its mathematical representation is $\xi(t) = D\cos(\Omega t + \sigma B(t) + \Gamma)$, where D and Ω denote the amplitude and central frequency, respectively, $B(t)$ is a standard Wiener process with strength σ, and Γ is a random variable uniformly distributed in $[0, 2\pi)$. This noise is a bounded stationary stochastic process of non-Gaussian distribution. When σ is small, it is a narrow band random process, while it approaches a white noise if $\sigma \to \infty$. So bounded noise can be a reasonable model for random disturbance in some sense. Therefore, the present work aims at exploring the key roles of bounded noise on CPS of weakly coupled nonlinear oscillators.

The outline of this work is as follows. In Section

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2, a paradigmatic ecology model of two coupled food chains is introduced. Section 3 presents the route to phase synchronization in the model as well as the nontrivial effects induced by bounded noise during this process. Finally, concluding remarks are made at the end of this work.

2. Models

The model under consideration is a bidirectional coupled two-patch food chains. The differential equations governing the biomass of different species are as follows[24]

$$\begin{aligned}
\dot{u}_{1,2} &= a(u_{1,2}-u^*)-\alpha_1 f(u_{1,2},v_{1,2}) \\
&\quad +\varepsilon(u_{2,1}-u_{1,2})+\xi(t), \\
\dot{v}_{1,2} &= -b_{1,2}(v_{1,2}-v^*)+\alpha_1 f(u_{1,2},v_{1,2}) \\
&\quad -\alpha_2 g(v_{1,2},w_{1,2})+\xi(t), \\
\dot{w}_{1,2} &= -c(w_{1,2}-w^*)+\alpha_2 g(v_{1,2},w_{1,2}) \\
&\quad +\varepsilon(w_{2,1}-w_{1,2})+\xi(t), \qquad (1)
\end{aligned}$$

where the subscripts imply the food web 1 and 2, respectively. Each food web sustains a vertical three-species trophic system consisted of interacting predators, consumers and resources. The resources $u_{1,2}$ are consumed by the consumers $v_{1,2}$, which in return are preyed on by the top predators $w_{1,2}$. The classical Holling type II term $f(u,v)=uv/(1+ku)$ is used to describe the interactions between the consumers and resources. The competitions among the predators and preys are also incorporated into the equations through the standard Lotka–Volterra term $g(v,w)=vw$. The two colonies are assumed to be different and specified by parameter mismatch of $b_{1,2}$. Diffusion migration between the two nonidentical communities can be achieved via mutual coupling $u_{1,2}$ and $w_{1,2}$ with the magnitude set by ε. Suppose uncertain fluctuations such as measurement, demographic or environmental stochasticity are modelled by the above introduced bounded noise $\xi(t)$. The parameters $a, b_{1,2}$, and c represent the respective net growth rates of each trophic species in the absence of intrinsic interactions, diffusive migration and extrinsic noise perturbation. Without species interacting and noise stirring, each independent food chain is assumed to possess an equilibrium state (u^*, v^*, w^*).

The three-level food web model is universal in ecological systems, and it might, for example, outline the principal ecological transfers involved in the Canadian lynx-hare-vegetation food chain and in the oceanic zooplankton-phytoplankton -nutrients food chain.[24,25] When the parameters in the unforced model (1) ($\varepsilon=0, D=0$) are selected as $a=1.0$, $b_1=0.975$, $b_2=0.945$, $c=10.0$, $\alpha_1=0.2$, $\alpha_2=1.0$, $k=0.05$, $u^*=0.0$, $v^*=0.0$, $w^*=0.006$, each oscillator in the model can generate phase coherent chaotic oscillation. That is to say, the phase coherent oscillator has a relatively simple topology of oscillations, as can be seen from the spiral chaotic attractors in Fig.1(a) as well as from the well-pronounced peaks in the power spectrum in Fig.1(b). Then an instantaneous phase variable for oscillator can be conveniently introduced[9]

$$\phi(t)=\mathrm{arctg}(v(t)-v_0/u(t)-u_0), \qquad (2)$$

where (u_0, v_0) is the rotation centre in the projection plane (u, v).

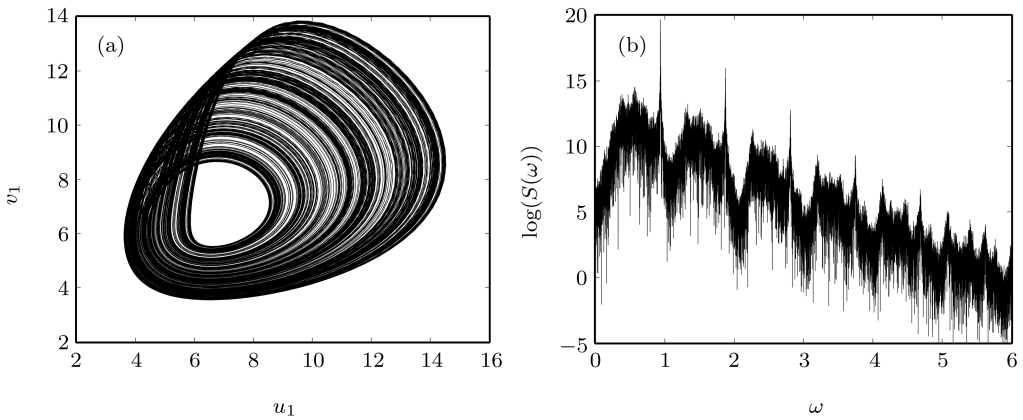


Fig.1. (a) The phase portrait and (b) the power spectrum of u_1 in model (1).

3. Results

In this section, at first we show how the two coupled noise-free oscillators in model (1) arrive at CPS with the increase of coupling. Then we explore the influences of bounded noise on the phase synchronization process by various statistical measures.

3.1. Transition to phase synchronization

The two oscillators in model (1) are nonidentical, hence, without coupling there exists a difference in their respective natural frequencies. When coupled, the coupling can gradually overcome the frequency difference between interacting oscillators and sufficiently large coupling can entrain them a common one. The dependence of their average frequencies $\Omega_{1,2}$ on the coupling ε is depicted in Fig.2(a), from which one can see that the two coupled food chains obtain CPS when the coupling exceeds a critical value $\varepsilon_{\rm ps1}(\approx 0.0116)$.

In addition, the Lyapunov exponents (LEs) can give more insight into the phase synchronization transition of chaotic oscillators. Model (1) has six LEs and the evolution of the four largest LEs during the coupling is plotted in Fig.2(b). In this figure one can observe that one of the zero LEs becomes negative at a certain critical coupling, while the two largest LEs remain positive in the whole range of coupling, which implies that the amplitudes of coupled oscillators are chaotic and independent. Further observation implies that phase locking and frequency entrainment occur approximately at the same value of coupling for which one of the zero LEs becomes negative.

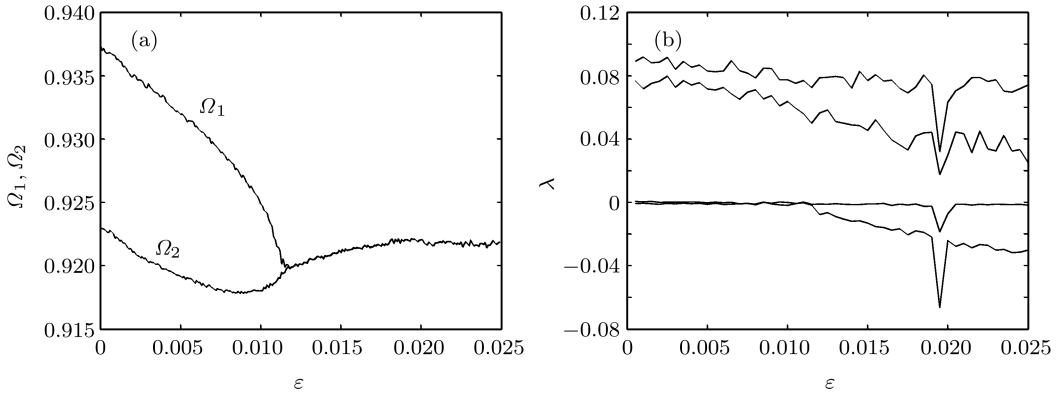


Fig.2. (a) The dependence of the two average frequencies and (b) the four largest LEs on the coupling in model (1).

3.2. Noise-induced dual roles outside the synchronous regime

When the coupling strength is below the onset of synchronization, the influences of bounded noise on the phase synchronization process are first analysed. Without loss of generality, the parameters for bounded noise are chosen as $\Omega = 1.0$ and $\sigma = 1.0$.

Adding some noise to the two coupled food webs, we find that the noise-induced effects are clearly demonstrated in the evolution of instantaneous phase variable error $\phi_1 - \phi_2$, as can be seen in Fig.3(a) for $\varepsilon = 0.0115$. At $D = 0$, phase locking appears at large intervals of time for the coupling has an ability to produce some plateaus where the phase difference keeps nearly constant, though some phase slips lie between these epochs of phase locking. The addition of a weak noise, for instance $D = 0.0003$, makes phase jumps arise frequently. Interestingly, an intermediate noise, for example $D = 0.0006$ and $D = 0.0008$, prolongates the phase locking periods prominently. While strong noise, for example $D = 0.003$, breaks down the long phase locking epochs and brings more phase jumps than ever.

To further understand the noise-induced effects on CPS, we turn to calculate quantitatively the average duration $\tau_{\rm mean}$ of the phase synchronization epochs. The dependence of average duration on the noise intensity D is shown in Fig.4(a) for several coupling such as $\varepsilon = 0.01, 0.011, 0.0115$. From that one can see that when the coupling is very near $\varepsilon_{\rm ps1}$, $\tau_{\rm mean}$ firstly decreases at the beginning of noise, then increases as adding much noise and reaches a maximum value, after which it decreases instead with a further increase in noise.

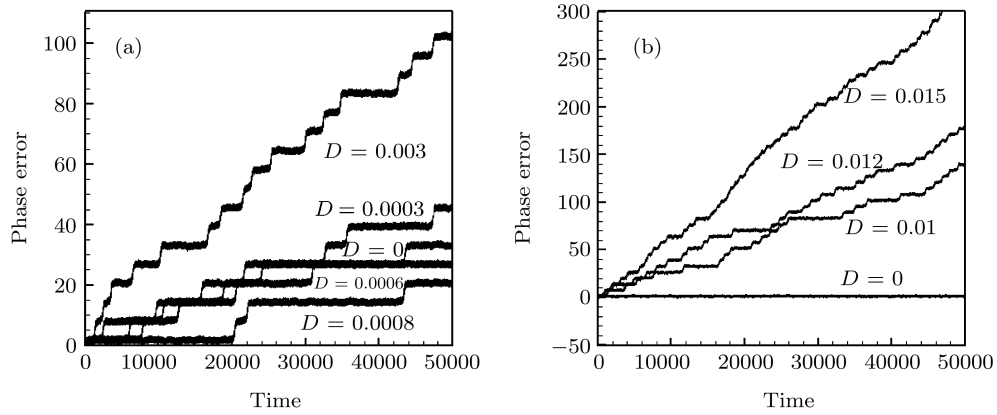


Fig.3. The evolution of phase variable error for various noise intensities in model (1) at (a) $\varepsilon = 0.0115$ and (b) $\varepsilon = 0.013$.

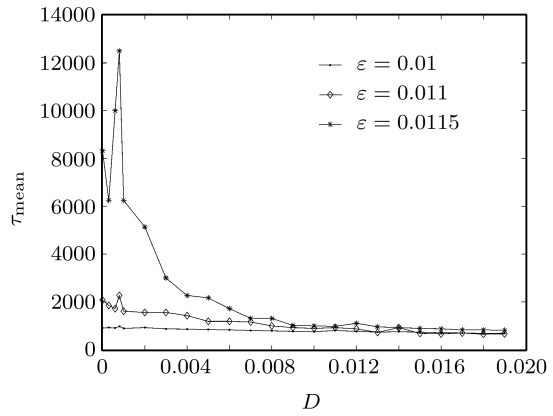


Fig.4. The dependence of the mean duration on the noise intensity for different coupling intensities in model (1).

To better scale the degree of CPS under the influence of noise, another two indices based on statistical measure are employed. The first index is qualified by $\rho = S/S_{\max}$, where $S = -\sum_{k=1}^{N} p_k \ln p_k$ is the Shannon entropy[26] of the distribution of the cyclic relative phase difference ($\tilde{\phi}(t) = [\phi_1(t) - \phi_2(t)] \bmod 2\pi$), p_k is the probability that the phase difference lies in the kth bin, and $S_{\max} = \ln N (N = 100$ is the number of bins) is the maximal entropy of the uniform distribution. The degree of CPS is higher for smaller ρ. The second index is the mutual information $M_0 = \sum\limits_{i,j} p(i,j) \ln[p(i,j)/p_1(i)p_2(j)]$ between the cyclic phases ($\Psi_{1,2}(t) = \phi_{1,2}(t) \bmod 2\pi$),[22] in which $p(i,j)$ is the joint probability density that $\Psi_1(t)$ lies in the ith bin and $\Psi_2(t)$ lies in the jth bin, $p_1(i)$ and $p_2(j)$ are the probabilities that the phases $\Psi_1(t)$ and $\Psi_2(t)$ lie in the ith and jth bins, respectively. M_0 is normalized by $M = M_0/S_{\max}$. The degree of CPS is higher with larger M.

The results of ρ along with M as a function of the noise intensity at different coupling are displayed in Figs.5(a) and 5(b), respectively. When the coupling is well below the threshold, the values of ρ and M are basically unaffected by the appearance of noise. The

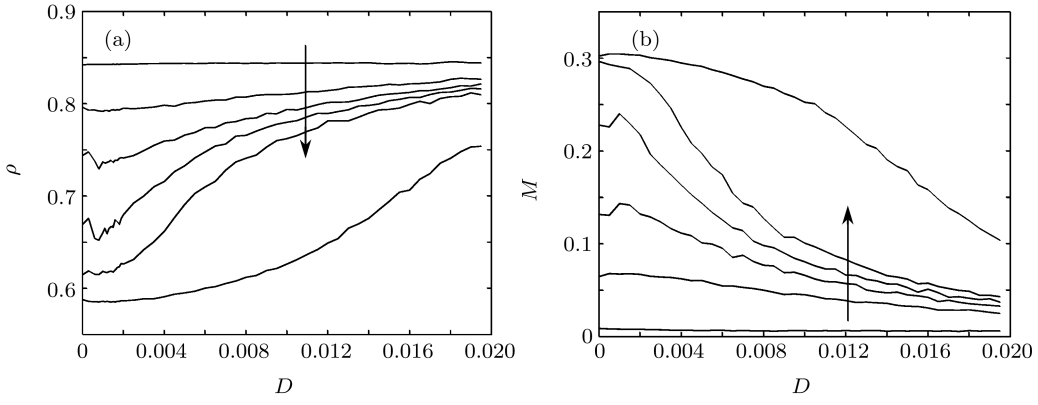


Fig.5. The entropy and mutual information versus the noise intensity with the coupling intensities of 0.005, 0.01, 0.011, 0.0115, 0.012, 0.016 in the direction of the arrows.

ascending tendency of ρ together with descending tendency of M implies that the degree of CPS is weakened when the coupling is much larger than the threshold. Nevertheless, in the case of the coupling close to the threshold, ρ firstly enlarges for the presence of noise, then reduces with the increase of noise and reaches a minimum at an optimal noise intensity, after which it increases again with further increase of noise. That is to say, the variation of ρ indicates that when the coupling is close to the threshold either weak or strong noise can degrade the degree of CPS, while a proper amount of intermediate noise can enhance CPS remarkably, and an optimal noise intensity that maximizes the enhancement is detected. This interesting phenomenon is further verified by the evolution of mutual information in the course of noise, namely, it firstly decreases, then increases until arrives at a maximum, after that it decreases again.

In a word, the phenomenon that an enhancement of CPS resulted from a proper amount of bounded noise is contrast with the influence of Gaussian white noise on phase synchronization of coupled periodic oscillators that noise has an destructive effect on phase synchronization.[27] The bounded noise-induced dual roles may be interpreted from the actions of noise on unstable periodic orbits (UPOs) embedded in chaotic attractors for phase synchronization of coupled chaotic oscillators can be viewed as phase locking of a number of pairs of UPOs.[2,21]

3.3. Noise-induced degraded role inside the synchronous regime

Outside the phase synchronization region, the bounded noise has demonstrated a constructive role in prolongating the periods of synchronization epochs as well as a destructive role in inducing more phase slips. While in the synchronous region, perfect phase locking in model (1) is always interrupted by noise-induced phase slips of locked UPOs. The noise-induced degraded influence on CPS can be clearly illustrated by the evolution of instantaneous phase variable error, which is displayed in Fig.3(b) in the case of the coupling $\varepsilon = 0.013$. From that one can observe that in the absence of noise, the coupling is so strong that the two oscillators are entrained and their phase variable error keeps bounded in the course of time. While the loss of synchronization occurs once the bounded noise, for example $D = 0.01, 0.012$ and 0.015, is presented. Now, phase jumps in units of approximate 2π set in and temporary phase locking can only be maintained in the short time intervals between successive phase jumps. Moreover, phase jumps become extremely frequent with an increase in noise, i.e., stronger noise causes more phase slips.

4. Conclusions

This work has extended to explore the diverse influences of bounded noise on weakly coupled nonlinear oscillators in terms of CPS. Employing a typical ecological model of two coupled three-level food webs, we find that the noise has exhibited a destructive role of inducing phase slips in synchronous region. On the contrary, when the coupling is close to the onset of phase synchronization, the noise has displayed different influences on CPS with the increase of noise intensity, i.e., either weak or strong noise also degrades phase synchronization by inducing more phase slips, while a proper amount of intermediate noise enhances CPS remarkably by prolongating the durations of phase locking, and an optimal noise intensity that maximizes the enhancement is detected.

The findings discovered in this work on the one hand complement the previous results about noise influences on phase synchronization of weakly coupled periodic as well as chaotic oscillators.[2,20−22,27] On the other hand, the model considered here is a classical system in ecology and bounded noise is a suitable model for describing random fluctuations. Thus, the results are significant for further understanding the influence of fluctuations on phase synchronization of biological and ecological systems, and may have practical applications in the future.

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