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Noise composed of multiplication of two dichotomous noises*

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In this paper, we introduce a noise which is composed of multiplication of two dichotomous noises, and derive the probability density and the statistical properties of this noise. The obtained results can help study the resonant activation phenomenon, the phenomenon of stochastic resonance, the transport of particles, and the nonequilibrium (phase) transition for the systems driven by this noise.

Keywords: Dichotomous noise, probability density, statistical properties
PACC: 0540

1. Introduction

Noise-induced nonequilibrium phenomena have recently attracted a great deal of attention in a variety of contexts. One example of these phenomena is the resonant activation (RA),[1−24] here the mean first passage time (MFPT) of a particle driven by (usually white) noise over a fluctuating potential barrier exhibits a minimum as a function of the parameter of the fluctuating potential barrier. This phenomenon is first identified by Doering and Gadoua[1] and further studied by a number of other investigators.[2−24] Now, except for the theoretical reports about the resonance activation phenomenon,[1−20,24] this phenomenon has been found in experiment.[21−23] A second example of such a phenomenon caused by the appearance of the noise is the phenomenon of stochastic resonance (SR), which has been a subject of great interest in nonlinear physics since the concept of SR was first proposed in answer to questions about the periodically recurrent ice ages.[25−27] The SR has attracted considerable interest due to its many applications in biology, physics, chemistry, and other scientific fields.[28−33] For the phenomenon of SR, the maximum enhancement of a weak periodic signal buried in noise can be achieved with a nonzero optimum amount of noise strength, although it is usually considered to be easier to detect a weak signal when the noise exists. The mechanisms of the SR effects have been elucidated by many theoretical approaches (see Refs.[28–33] and references therein). On the other hand, the SR has been studied experimentally in various kinds of systems including electrical, optical, and neuronal systems (see Ref.[28] and references therein). A third example is the transport of particles induced by noise.[34−38] It has been found that the spatial symmetry breaking and the temporal symmetry breaking are ingredients for the nonzero flux. A fourth example of the noise-induced phenomena is the noise-induced nonequilibrium transition,[38−41] which may be phase transition[38−40] or not.[40,41]

All of the studies of the noise-induced phenomena have been focused on systems subjected to Gaussian noise (Gaussian white noise and Ornstein–Uhlenbeck (O-U) noise) or dichotomous noise (or three-state noise or white shot noise). But practically, there are a lot of complicated noises, which are not Gaussian or dichotomous (or three-state noise or white shot). In this paper, we will introduce a noise, which is composed of the multiplication of two dichotomous noises, and derive the probability density and statistical properties of this noise. Our obtained results can help study the resonant activation phenomenon, the phenomenon of stochastic resonance, the transport of particles, and the nonequilibrium (phase) transition for the systems driven by this noise.

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2. Multiplication of two dichotomous noises and its master equation

In this section, we consider a noise which is the multiplication of two dichotomous noises, i. e.

$$\xi(t) = \xi_1(t)\xi_2(t), \tag{1}$$

where $\xi_1(t)$ is a dichotomous noise that takes two values a and b ($a > b$), the transition rates from a to b and from b to a are both λ; $\xi_2(t)$ is also a dichotomous noise which takes c and d ($c > d$), the transition rates from c to d and from d to c are both μ. In Appendix A, we give an example showing the existence of the noise composed of the multiplication of two dichotomous noises.

The master equation for the noise $\xi(t)$ is

$$\begin{aligned}
\partial_t P(a,c,t \mid x_0,y_0,t_0) &= -(\lambda+\mu)P(a,c,t \mid x_0,y_0,t_0) + \lambda P(b,c,t \mid x_0,y_0,t_0) + \mu P(a,d,t \mid x_0,y_0,t_0),\\
\partial_t P(b,c,t \mid x_0,y_0,t_0) &= -(\lambda+\mu)P(b,c,t \mid x_0,y_0,t_0) + \lambda P(a,c,t \mid x_0,y_0,t_0) + \mu P(b,d,t \mid x_0,y_0,t_0),\\
\partial_t P(a,d,t \mid x_0,y_0,t_0) &= -(\lambda+\mu)P(a,d,t \mid x_0,y_0,t_0) + \lambda P(b,d,t \mid x_0,y_0,t_0) + \mu P(a,c,t \mid x_0,y_0,t_0),\\
\partial_t P(b,d,t \mid x_0,y_0,t_0) &= -(\lambda+\mu)P(b,d,t \mid x_0,y_0,t_0) + \lambda P(a,d,t \mid x_0,y_0,t_0) + \mu P(b,c,t \mid x_0,y_0,t_0),
\end{aligned} \tag{2}$$

in which we have assumed the stochastic process to be in an initial state $t = t_0$, $\xi_1(t_0) = x_0$ and $\xi_2(t_0) = y_0$, so the initial condition is $P(x,y,t_0 \mid x_0,y_0,t_0) = \delta_{xx_0}\delta_{yy_0}$. $P(a,c,t \mid x_0,y_0,t_0)$, $P(b,c,t \mid x_0,y_0,t_0)$, $P(a,d,t \mid x_0,y_0,t_0)$ and $P(b,d,t \mid x_0,y_0,t_0)$ are the conditional probabilities. $P(a,c,t \mid x_0,y_0,t_0)$ represents that the dichotomous noises are in $\xi_1(t) = a$ and $\xi_2(t) = c$ configuration, and at time t, with the initial state $t = t_0$, $\xi_1(t_0) = x_0$ and $\xi_2(t_0) = y_0$. The same understanding is applicable for $P(b,c,t \mid x_0,y_0,t_0)$, $P(a,d,t \mid x_0,y_0,t_0)$ and $P(b,d,t \mid x_0,y_0,t_0)$.

For convenience, we write $P_1 = P(a,c,t \mid x_0,y_0,t_0)$, $P_2 = P(b,c,t \mid x_0,y_0,t_0)$, $P_3 = P(a,d,t \mid x_0,y_0,t_0)$ and $P_4 = P(b,d,t \mid x_0,y_0,t_0)$. So Eq.(2) can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} P_1\\ P_2\\ P_3\\ P_4 \end{pmatrix} = \begin{pmatrix} -(\lambda+\mu) & \lambda & \mu & 0\\ \lambda & -(\lambda+\mu) & 0 & \mu\\ \mu & 0 & -(\lambda+\mu) & \lambda\\ 0 & \mu & \lambda & -(\lambda+\mu) \end{pmatrix} \times \begin{pmatrix} P_1\\ P_2\\ P_3\\ P_4 \end{pmatrix}. \tag{3}$$

3. Probability densities

It is easy to find that the matrix of the homogeneous part about P_i ($i = 1,2,3,4$) in Eq.(3) has three nonzero eigenvalues and a zero eigenvalue, which are $r_1 = -2\lambda$, $r_2 = -2\mu$, $r_3 = -2(\lambda+\mu)$, and $r_4 = 0$. So, if $\lambda \neq \mu$, the general solution of Eq.(3) is

$$P_i = C_1^{(1)} + C_2^{(1)}\exp[-2\lambda(t-t_0)] + C_3^{(1)}\exp[-2\mu(t-t_0)]C_4^{(1)}\exp[-2(\lambda+\mu)(t-t_0)]; \tag{4}$$

if $\lambda = \mu$, there will be two dependent eigenvalues and the general solution of Eq.(3) is

$$P_i = B_1^{(1)} + \left[B_2^{(1)} + B_3^{(1)}(t-t_0)\exp(-2\lambda(t-t_0))\right] + B_4^{(1)}\exp[-4\lambda(t-t_0)], \tag{5}$$

where $i = 1,2,3,4$; if $\mu \neq \lambda$, the solution of Eq.(3) can be obtained as (see Appendix B)

$$
\begin{aligned}
P_1 &= \frac{1}{4} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)} \\
&\quad + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)}, \\
P_2 &= \frac{1}{4} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)} \\
&\quad - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)}, \\
P_3 &= \frac{1}{4} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)} \\
&\quad - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)}, \\
P_4 &= \frac{1}{4} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)} \\
&\quad + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)}; \qquad (6)
\end{aligned}
$$

if $\mu = \lambda$, the solution of Eq.(3) is (see Appendix B)

$$
\begin{aligned}
P_1 &= \frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} - \delta_{bx_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)}, \\
P_2 &= \frac{1}{4} + \frac{1}{2}(\delta_{bx_0}\delta_{cy_0} - \delta_{ax_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)}, \\
P_3 &= \frac{1}{4} - \frac{1}{2}(\delta_{bx_0}\delta_{cy_0} - \delta_{ax_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)}, \\
P_4 &= \frac{1}{4} - \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} - \delta_{bx_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)}. \qquad (7)
\end{aligned}
$$

This process $\xi(t) = \xi_1(t)\xi_2(t)$ has the stationary solution [obtained by letting $t_0 \to -\infty$ (or $t \to \infty$) in Eqs.(6) and (7)] as follows:

$$P_{\mathrm{s}}(x, y) = P_1 = P_2 = P_3 = P_4 = \frac{1}{4}. \qquad (8)$$

4. Statistical properties

The first cumulant (i.e. the mean value) of $\xi(t)$ can be straightforwardly computed and given as

$$\langle \xi(t) \mid [x_0, y_0, t_0] = \sum_{(x,y)} xyP(x, y, t \mid x_0, y_0, t_0), \qquad (9)$$

in which $(x, y)\epsilon\{(a, c), (b, c), (a, d), (b, d)\}$.

If $\mu \neq \lambda$, substituting expression (6) into expression (9), we can have

$$
\begin{aligned}
\langle \xi(t) \mid [x_0, y_0, t_0] \rangle &= acP_1 + bcP_2 + adP_3 + bdP_4 = \frac{1}{4}(ac + bc + ad + bd) \\
&\quad + (ac - bc + ad - bd)\left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} \\
&\quad + (ac + bc - ad - bd)\left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)} \\
&\quad + (ac - bc - ad + bd)\left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)}. \qquad (10)
\end{aligned}
$$

Now, there are four cases for the initial condition, which are $(x_0, y_0) = (a, c)$, (b, c), (a, d) and (b, d). For these four cases, we give the expression (10) in Appendix C.

If $\mu = \lambda$, substituting expression (7) into expression (9), we have

$$\begin{aligned}\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = & \frac{1}{4}(ac + bc + ad + bd) + \frac{1}{2}[(ac - bd)(\delta_{ax_0}\delta_{cy_0} - \delta_{bx_0}\delta_{dy_0}) \\ & + (bc - ad)(\delta_{bx_0}\delta_{cy_0} - \delta_{ax_0}\delta_{dy_0})]\mathrm{e}^{-2\lambda(t-t_0)} \\ & + (ac - bc - ad + bd)\left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right]\mathrm{e}^{-4\lambda(t-t_0)}. \end{aligned} \tag{11}$$

Similarly, for expression (11), there are also four cases, which are $(x_0, y_0) = (a, c)$, $(x_0, y_0) = (b, c)$, $(x_0, y_0) = (a, d)$, and $(x_0, y_0) = (b, d)$. For these four cases, we give the expression (11) in Appendix C.

In a stationary state, from the above expressions (10) and (11), we have

$$\langle \xi(t) \rangle_{\mathrm{s}} = \frac{1}{4}(ac + bc + ad + bd).$$

Similarly, the variance (i.e. the second order cumulant) of $\xi(t)$ can also be computed. Here we give only the stationary variance, which is easily computed and given by

$$\begin{aligned} & \langle (\xi(t))^2 \rangle_{\mathrm{s}} - \langle \xi(t) \rangle_{\mathrm{s}}^2 \\ = & \frac{1}{4}(a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2) \\ & - \frac{1}{16}(ac + bc + ad + bd)^2. \end{aligned} \tag{12}$$

The third and higher order cumulants can be obtained by using the above method. In the following, we compute the stationary time correlation function. To do this, we write (assuming $t \geq s$)

$$\begin{aligned} & \langle \xi(t)\xi(s) \rangle_{\mathrm{s}} \\ = & \sum_{(x,y),(x',y')} xyx'y' P(x, y, t \mid x', y', s) P_{\mathrm{s}}(x', y') \\ = & \frac{1}{4}\sum x'y' \langle xy \mid [x', y', s] \rangle \\ = & \frac{1}{4}\sum \xi'(s) \langle \xi(t) \mid [x', y', s] \rangle. \end{aligned} \tag{13}$$

There are two cases for expression (13), which are $\mu \neq \lambda$ and $\mu = \lambda$. If $\mu \neq \lambda$, from expression (13) and expressions (C1)–(C4), we can have

$$\begin{aligned} & \langle \xi(t)\xi(s) \rangle_{\mathrm{s}} \\ = & \langle \xi(t) \rangle_{\mathrm{s}}^2 + \frac{1}{16}(ac + ad)(ac - bc + ad - bd)\mathrm{e}^{-2\lambda(t-s)} + \frac{1}{16}(ac + bc)(ac + bc - ad - bd)\mathrm{e}^{-2\mu(t-s)} \\ & + \frac{1}{16}(ac + bd)(ac - bc - ad + bd)\mathrm{e}^{-2(\lambda+\mu)(t-s)}, \end{aligned} \tag{14}$$

in which $t \geq s$; if $\mu = \lambda$, from expression (13) and expressions (C5)–(C8), we can obtain

$$\begin{aligned} \langle \xi(t)\xi(s) \rangle_{\mathrm{s}} = & \langle \xi(t) \rangle_{\mathrm{s}}^2 + \frac{1}{8}(a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 - 4acbd)\mathrm{e}^{-2\lambda(t-s)} \\ & + \frac{1}{16}(ac + bd)(ac - bc - ad + bd)\mathrm{e}^{-4\lambda(t-s)}, \end{aligned} \tag{15}$$

where $t \geq s$.

5. Conclusion and discussion

In this paper, we have introduced a noise composed of multiplication of two dichotomous noises, and investigated the probability density and the statistical properties of this noise. We have found that the probability density and the statistical properties of this noise (composed of multiplication of two dichotomous noises) all depend on the initial condition of the noise; while the stationary ones do not depend on the initial condition of the noise. The results obtained in the paper can help study the resonant activation phenomenon, the phenomenon of stochastic resonance, the transport of particles, and the nonequilibrium (phase) transition for the systems driven by this noise.

In addition, in this paper only the noise that is composed of the multiplication of two dichotomous noise is studied. For this noise, the values ac, ad, bc and bd, i.e. four values are taken. The transition rates from ac to ad and from ad to ac are both μ; The transition rates from ac to bc and from bc to ac are both λ; the transition rates from bc to bd and from bd to bc are both μ; and the transition rates from ad to bd and from bd to ad are both λ. The noise can be vividly presented in Fig.1. Thus, the noise that is composed of the multiplication of two dichotomous noise is a special case for the four-state noise. In Fig.2, we present a general four-state noise. It can be clearly seen from Figs.1 and 2 that the noise shown in Fig.1 is a special case for the noise shown in Fig.2. In the next paper of mine, the probability density and statistical properties for a four-state Markovian noise will be investigated, for the detailed investigation into its probability density and statistical properties has not been made though a lot of phenomena induced by four-state noise have been found.

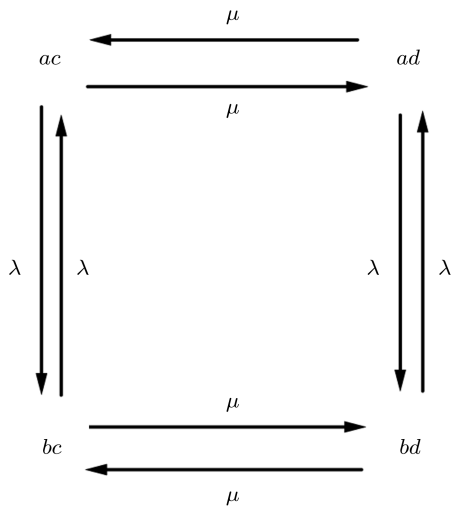


Fig.1. A vivid representation for the noise $\xi(t) = \xi_1(t)\xi_2(t)$ considered in the paper. The noise takes values ac, ad, bc and bd. The transition rates from ac to ad and from ad to ac are both μ; The transition rates from ac to bc and from bc to ac are both λ; The transition rates from bc to bd and from bd to bc are both μ; And the transition rates from ad to bd and from bd to ad are both λ.

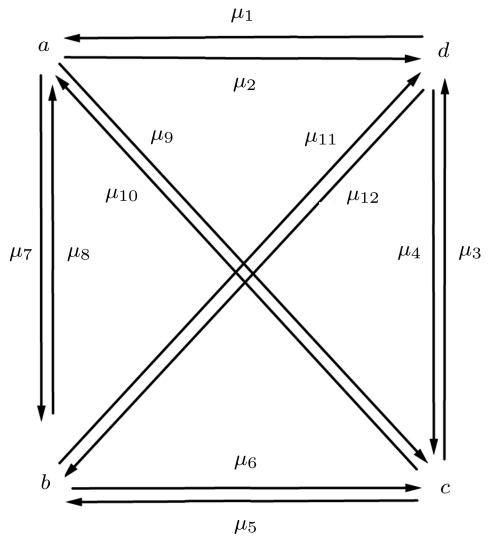


Fig.2. A representation for a general four-state noise, in which the values a, b, c and d are taken and the transition rates from one value to another are also presented.

Appendix A

In this Appendix, we give an example showing the existence of the noise composed of the multiplication of two dichotomous noises.

See an electric circuit in Fig.A1, where R is a resistance (constant), and $r(t)$ a dichotomous resistor that fluctuates between two values r_a and r_b, with mean waiting times t_a and t_b, respectively. The dichotomous resistor can be obtained by inserting into the circuit a point contact whose conductance is controlled by an asymmetric two-level system tunnelling incoherently between the two states with rates $\gamma = 1/t_a$ and $\gamma' = 1/t_b$.[42,43] Accordingly, the fluctuations in the point contact resistor can be modelled as a stationary Markovian dichotomous noise.

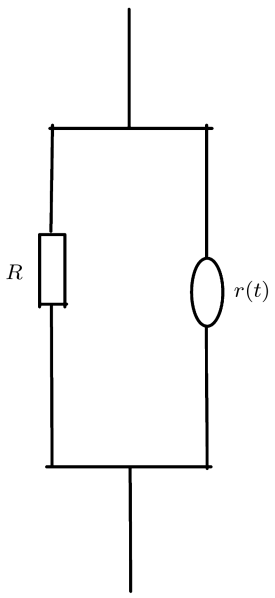


Fig.A1. An electric circuit, where R is a resistance (constant) and $r(t)$ a dichotomous resistor that fluctuates between two values r_a and r_b, with mean waiting times t_a and t_b, respectively.

For this circuit, the total resistance is

$$R_0(t) = \frac{Rr(t)}{R+r(t)} = \frac{R}{R+r(t)} r(t) = \xi_1(t)\xi_2(t),$$

in which $\xi_1(t) = R/(R+r(t))$ and $\xi_2(t) = r(t)$. $\xi_1(t)$ fluctuates between $a = R/(R+r_a)$ and $b = R/(R+r_b)$ (the transition rate from a to b is γ and transition rate from b to a is γ'), and $\xi_2(t)$ fluctuates between two values $c = r_a$ and $d = r_b$ (the transition rate from c to d is γ and that from d to c is γ').

Then, we have a noise that is composed of the multiplication of two dichotomous noises.

Appendix B

Substituting expressions (4) and (5) into Eq.(3) and using the comparing-coefficient method, we obtain $C_1^{(1)} = C_1^{(2)} = C_1^{(3)} = C_1^{(4)}$, $C_2^{(1)} = -C_2^{(2)} = C_2^{(3)} = -C_2^{(4)}$, $C_3^{(1)} = C_3^{(2)} = -C_3^{(3)} = -C_3^{(4)}$ and $C_4^{(1)} = -C_4^{(2)} = -C_4^{(3)} = C_4^{(4)}$, and $B_1^{(1)} = B_1^{(2)} = B_1^{(3)} = B_1^{(4)}$, $B_4^{(1)} = -B_4^{(2)} = -B_4^{(3)} = C_4^{(4)}$, $B_3^{(1)} = B_3^{(2)} = B_3^{(3)} = B_3^{(4)} = 0$, $B_2^{(4)} = -B_2^{(1)}$ and $B_2^{(3)} = -B_2^{(2)}$. So, the solution of Eq.(3) is, if $\lambda \neq \mu$,

$$\begin{aligned}
P_1 &= C_1^{(1)} + C_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} \\
&\quad + C_3^{(1)}\mathrm{e}^{-2\mu(t-t_0)} + C_4^{(1)}\mathrm{e}^{-2(\lambda+\mu)(t-t_0)}, \\
P_2 &= C_1^{(1)} - C_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} \\
&\quad + C_3^{(1)}\mathrm{e}^{-2\mu(t-t_0)} - C_4^{(1)}\mathrm{e}^{-2(\lambda+\mu)(t-t_0)}, \\
P_3 &= C_1^{(1)} + C_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} \\
&\quad - C_3^{(1)}\mathrm{e}^{-2\mu(t-t_0)} - C_4^{(1)}\mathrm{e}^{-2(\lambda+\mu)(t-t_0)}, \\
P_4 &= C_1^{(1)} - C_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} \\
&\quad - C_3^{(1)}\mathrm{e}^{-2\mu(t-t_0)} + C_4^{(1)}\mathrm{e}^{-2(\lambda+\mu)(t-t_0)};
\end{aligned}$$

if $\lambda = \mu$,

$$\begin{aligned}
P_1 &= B_1^{(1)} + B_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} + B_4^{(1)}\mathrm{e}^{-4\lambda(t-t_0)}, \\
P_2 &= B_1^{(1)} + B_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} - B_4^{(1)}\mathrm{e}^{-4\lambda(t-t_0)}, \\
P_3 &= B_1^{(1)} - B_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} - B_4^{(1)}\mathrm{e}^{-4\lambda(t-t_0)}, \\
P_4 &= B_1^{(1)} - B_2^{(1)}\mathrm{e}^{-2\lambda(t-t_0)} + B_4^{(1)}\mathrm{e}^{-4\lambda(t-t_0)}.
\end{aligned}$$

After considering the initial condition, we have, if $\mu \neq \lambda$,

$$\begin{aligned}
&P(a, c, t_0 \mid x_0, y_0, t_0) \\
&= C_1^{(1)} + C_2^{(1)} + C_3^{(1)} + C_4^{(1)} = \delta_{ax_0}\delta_{cy_0}, \\
&P(b, c, t_0 \mid x_0, y_0, t_0) \\
&= C_1^{(1)} - C_2^{(1)} + C_3^{(1)} - C_4^{(1)} = \delta_{bx_0}\delta_{cy_0}, \\
&P(a, d, t_0 \mid x_0, y_0, t_0) \\
&= C_1^{(1)} + C_2^{(1)} - C_3^{(1)} - C_4^{(1)} = \delta_{ax_0}\delta_{dy_0}, \\
&P(b, d, t_0 \mid x_0, y_0, t_0) \\
&= C_1^{(1)} - C_2^{(1)} - C_3^{(1)} + C_4^{(1)} = \delta_{bx_0}\delta_{dy_0},
\end{aligned}$$

from which we can obtain

$$\begin{aligned}
C_1^{(1)} &= \frac{1}{4}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0} + \delta_{bx_0}\delta_{dy_0}) \\
&= \frac{1}{4}, \\
C_2^{(1)} &= -\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0}), \\
C_3^{(1)} &= -\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0}), \\
C_4^{(1)} &= -\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0}),
\end{aligned}$$

where we have used $P_1 + P_2 + P_3 + P_4 = 1$; if $\mu = \lambda$, from the initial condition, we have

$$\begin{aligned}
B_1^{(1)} + B_2^{(1)} + B_4^{(1)} &= \delta_{ax_0}\delta_{cy_0}, \\
B_1^{(1)} + B_2^{(1)} - B_4^{(1)} &= \delta_{bx_0}\delta_{cy_0}, \\
B_1^{(1)} - B_2^{(1)} - B_4^{(1)} &= \delta_{ax_0}\delta_{dy_0}, \\
B_1^{(1)} - B_2^{(1)} + B_4^{(1)} &= \delta_{bx_0}\delta_{dy_0},
\end{aligned}$$

from which we can obtain (considering $P_1 + P_2 + P_3 + P_4 = 1$)

$$\begin{aligned}
B_1^{(1)} &= \frac{1}{4}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0} + \delta_{bx_0}\delta_{dy_0}) \\
&= \frac{1}{4}, \\
B_2^{(1)} &= \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} - \delta_{bx_0}\delta_{dy_0}), \\
B_2^{(2)} &= \frac{1}{2}(\delta_{bx_0}\delta_{cy_0} - \delta_{ax_0}\delta_{dy_0}), \\
B_4^{(1)} &= -\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0}).
\end{aligned}$$

Finally, if $\mu \neq \lambda$,

$$\begin{aligned}
P_1 &= \frac{1}{4} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right]\mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right]\mathrm{e}^{-2\mu(t-t_0)} \\
&\quad + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right]\mathrm{e}^{-2(\lambda+\mu)(t-t_0)},
\end{aligned}$$

$$P_2 = \frac{1}{4} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)}$$
$$- \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)},$$
$$P_3 = \frac{1}{4} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)}$$
$$- \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)},$$
$$P_4 = \frac{1}{4} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{ax_0}\delta_{dy_0})\right] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{cy_0})\right] \mathrm{e}^{-2\mu(t-t_0)}$$
$$+ \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-2(\lambda+\mu)(t-t_0)};$$

if $\mu = \lambda$,

$$P_1 = \frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} - \delta_{bx_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)},$$
$$P_2 = \frac{1}{4} + \frac{1}{2}(\delta_{bx_0}\delta_{cy_0} - \delta_{ax_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)},$$
$$P_3 = \frac{1}{4} - \frac{1}{2}(\delta_{bx_0}\delta_{cy_0} - \delta_{ax_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} - \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)},$$
$$P_4 = \frac{1}{4} - \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} - \delta_{bx_0}\delta_{dy_0})\Big] \mathrm{e}^{-2\lambda(t-t_0)} + \left[-\frac{1}{4} + \frac{1}{2}(\delta_{ax_0}\delta_{cy_0} + \delta_{bx_0}\delta_{dy_0})\right] \mathrm{e}^{-4\lambda(t-t_0)}.$$

Appendix C

For expression (10), there are four cases. In this Appendix, we give them as follows:

Case 1 $(x_0, y_0) = (a, c)$:

$$\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = \frac{1}{4}(ac + bc + ad + bd) + \frac{1}{4}(ac - bc + ad - bd)\mathrm{e}^{-2\lambda(t-t_0)}$$
$$+ \frac{1}{4}(ac + bc - ad - bd)\mathrm{e}^{-2\mu(t-t_0)} + \frac{1}{4}(ac - bc - ad + bd)\mathrm{e}^{-2(\lambda+\mu)(t-t_0)}; \tag{C1}$$

Case 2 $(x_0, y_0) = (b, c)$:

$$\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = \frac{1}{4}(ac + bc + ad + bd) + \frac{1}{4}(ac + bc - ad - bd)\mathrm{e}^{-2\mu(t-t_0)}; \tag{C2}$$

Case 3 $(x_0, y_0) = (a, d)$:

$$\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = \frac{1}{4}(ac + bc + ad + bd) + \frac{1}{4}(ac - bc + ad - bd)\mathrm{e}^{-2\lambda(t-t_0)}; \tag{C3}$$

Case 4 $(x_0, y_0) = (b, d)$:

$$\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = \frac{1}{4}(ac + bc + ad + bd) + \frac{1}{4}(ac - bc - ad + bd)\mathrm{e}^{-2(\lambda+\mu)(t-t_0)}. \tag{C4}$$

For expression (11), there are also four cases. They are given as follows:

Case 1 $(x_0, y_0) = (a, c)$:

$$\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = \frac{1}{4}(ac + bc + ad + bd) + \frac{1}{2}(ac - bd)\mathrm{e}^{-2\lambda(t-t_0)}$$
$$+ \frac{1}{4}(ac - bc - ad + bd)\mathrm{e}^{-4\lambda(t-t_0)}; \tag{C5}$$

Case 2 $(x_0, y_0) = (b, c)$:

$$\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = \frac{1}{4}(ac + bc + ad + bd) + \frac{1}{2}(bc - ad)\mathrm{e}^{-2\lambda(t-t_0)}; \tag{C6}$$

Case 3 $(x_0, y_0) = (a, d)$:

$$\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = \frac{1}{4}(ac + bc + ad + bd) - \frac{1}{2}(bc - ad)\mathrm{e}^{-2\lambda(t-t_0)}; \tag{C7}$$

Case 4 $(x_0, y_0) = (b, d)$:

$$\begin{aligned}\langle \xi(t) \mid [x_0, y_0, t_0] \rangle = {} & \frac{1}{4}(ac + bc + ad + bd) - \frac{1}{2}(ac - bd)\mathrm{e}^{-2\lambda(t-t_0)} \\ & + \frac{1}{4}(ac - bc - ad + bd)\mathrm{e}^{-4\lambda(t-t_0)}.\end{aligned} \tag{C8}$$

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