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Perturbation to Lie symmetry and another type of Hojman adiabatic invariants for Birkhoffian systems*

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The perturbation to Lie symmetry and another type of Hojman adiabatic invariants induced from the perturbation to Lie symmetry for Birkhoffian systems are studied. The exact invariants of Lie symmetry for the system without perturbation are given. Based on the concept of adiabatic invariant, the perturbation to Lie symmetry is discussed and another new type of Hojman adiabatic invariants that have the different form from that in [*Acta Phys. Sin.* **55** 3833] for the perturbed system are obtained.

Keywords: Lie symmetry, perturbation, Hojman adiabatic invariant, Birkhoffian system
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1. Introduction

The Birkhoffian dynamics is more general than the Hamiltonian mechanics and usually applied to the fields of modern physics. In 1927, Birkhoff made primary studies on Birkhoffian dynamics.[1] Then the researches on Birkhoffian dynamics, such as the symmetry theory of Birkhoffian systems, have become more and more active.[2−13]

For Birkhoffian systems, the study on perturbation to symmetries and adiabatic invariants of mechanical systems has great significance. As is known, even a small change, which we can call a perturbation, may influence the original symmetries and exact invariants of mechanical systems. Perturbation to symmetries and adiabatic invariants play a very important role. An adiabatic invariant is a certain physical quantity that changes more slowly than a slowly-varying parameter of the system.[14] In these years, some works have been done to discuss this problem for Birkhoffian systems.[15−19]

However, most of the previous studies so far in this field have been on the Noether adiabatic invariants induced from the perturbation to symmetries. There rarely are studies on other types of adiabatic invariants. Recently, Ref.[20] discussed a type of Hojman adiabatic invariants induced from the perturbation to Lie symmetry for Birkhoffian systems.

In this paper, the perturbation to Lie symmetry and another type of Hojman adiabatic invariants induced from the perturbation to Lie symmetry for Birkhoffian systems are studied. The exact invariants of Lie symmetry for the system without perturbation are given. Based on the concept of adiabatic invariant, the perturbation to Lie symmetry is discussed and another new type of Hojman adiabatic invariants that have a different form from that in Ref.[20] of the perturbed system is obtained.

2. Birkhoffian equations

The Birkhoffian equations can be written as

$$\Omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial t} = 0 \quad (\mu, \nu = 1, \cdots, 2n), \quad (1)$$

where $B = B(t, \boldsymbol{a})$ is called a Birkhoffian, $R_{\mu} = R_{\mu}(t, \boldsymbol{a})$ are called Birkhoffian functions, a^{μ} are vari-

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ables, and

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}. \tag{2}$$

The system described by Eq.(1) is called a Birkhoffian system. Suppose the system is non-singular, that is

$$\det(\Omega_{\mu\nu}) \neq 0, \tag{3}$$

then Eq.(1) can be expressed as

$$\dot{a}^\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right), \tag{4}$$

where

$$\Omega^{\mu\nu} = \left[\left(\frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta}\right)^{-1}\right]^{\mu\nu}. \tag{5}$$

Extending Eq.(4), we have

$$\dot{a}^\mu = h_\mu(t, \boldsymbol{a}). \tag{6}$$

3. Lie symmetry and exact invariants

Introduce the infinitesimal transformations

$$\begin{aligned} t^* &= t + \varepsilon\tau^0(t, \boldsymbol{a}), \\ a^{\mu*}(t^*) &= a^\mu(t) + \varepsilon\xi^0_\mu(t, \boldsymbol{a}), \end{aligned} \tag{7}$$

where ε is an infinitesimal parameter and τ^0, ξ^0_μ are infinitesimal generators.

According to Lie symmetry theory, the determining equations of Lie symmetry for the system are

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi^0_\mu - h_\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^0 = X^{(0)}_0(h_\mu), \tag{8}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + h_\mu\frac{\partial}{\partial a^\mu}, \tag{9}$$

$$X^{(0)}_0 = \tau^0\frac{\partial}{\partial t} + \xi^0_\mu\frac{\partial}{\partial a^\mu}. \tag{10}$$

For the Birkhoffian system (1), if the infinitesimal generators τ^0, ξ^0_μ satisfy Eq.(8), then the corresponding symmetry is Lie symmetry of the system.[21]

Proposition 1 For the Birkhoffian system (1), if the infinitesimal generators τ^0, ξ^0_μ satisfy Eq.(8), and there exists a function $\lambda_0 = \lambda_0(t, \boldsymbol{a})$ so that

$$\frac{\partial h_\mu}{\partial a^\mu} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\lambda_0 = 0, \tag{11}$$

then Lie symmetry of the system can lead to a Hojman exact invariant[8]

$$\begin{aligned} I_{\mathrm{H}0} =& \frac{1}{\lambda_0}\frac{\partial}{\partial t}(\lambda_0\tau^0) + \frac{1}{\lambda_0}\frac{\partial}{\partial a^\mu}(\lambda_0\xi^0_\mu) - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^0 \\ =& \text{const}. \end{aligned} \tag{12}$$

4. Perturbation to Lie symmetry and adiabatic invariants

According to the concept of adiabatic invariant,[14] for the Birkhoffian system we have

Definition If $I_z(t, \boldsymbol{a}, \varepsilon)$ is a physical quantity including ε, in which the highest power is z in a Birkhoffian system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a z th order adiabatic invariant of the Birkhoffian system.

Suppose the Birkhoffian system (1) is perturbed by small quantities $\varepsilon W_\mu = \varepsilon W_\mu(t, \boldsymbol{a})$, then the equations of motion for the system become

$$\Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = \varepsilon W_\mu, \tag{13}$$

and can be expressed as

$$\dot{a}^\mu = h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu. \tag{14}$$

Due to the action of εW_μ, the original symmetries and invariants of the system may vary. The infinitesimal generators $\tau(t, \boldsymbol{a})$ and $\xi_\mu(t, \boldsymbol{a})$ under the action of the small quantities are expressed by

$$\begin{aligned} \tau &= \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots, \\ \xi_\mu &= \xi^0_\mu + \varepsilon\xi^1_\mu + \varepsilon^2\xi^2_\mu + \cdots. \end{aligned} \tag{15}$$

After the perturbation the original determining equations of Lie symmetry for the Birkhoffian system become

$$\begin{aligned} &\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\mu - (h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu)\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau \\ =& X^{(0)}(h_\mu) + \varepsilon X^{(0)}(\Omega^{\mu\nu}W_\nu), \end{aligned} \tag{16}$$

where

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_\mu\frac{\partial}{\partial a^\mu}, \tag{17}$$

and

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu)\frac{\partial}{\partial a^\mu}. \tag{18}$$

In order to use Eq.(18) expediently, we write it down simply as

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t} + \varepsilon\Omega^{\mu\nu}W_\nu\frac{\partial}{\partial a^\mu}, \tag{19}$$

where $\bar{\mathrm{d}}/\mathrm{d}t$ has the same form as Eq.(9). Substituting Eqs.(15) and (19) into Eq.(16) and comparing the modulus of ε^m, we have

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu^m+\Omega^{\alpha\beta}W_\beta\frac{\partial\xi_\mu^{m-1}}{\partial a^\alpha}-h_\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\Omega^{\mu\nu}W_\nu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^{m-1}-h_\mu\Omega^{\alpha\beta}W_\beta\frac{\partial\tau^{m-1}}{\partial a^\alpha}-\Omega^{\mu\nu}W_\nu\Omega^{\alpha\beta}W_\beta\frac{\partial\tau^{m-2}}{\partial a^\alpha}\\=&\tau^m\frac{\partial h_\mu}{\partial t}+\xi_\alpha^m\frac{\partial h_\mu}{\partial a^\alpha}+\tau^{m-1}\frac{\partial(\Omega^{\mu\nu}W_\nu)}{\partial t}+\xi_\alpha^{m-1}\frac{\partial(\Omega^{\mu\nu}W_\nu)}{\partial a^\alpha}\\=&X_m^{(0)}(h_\mu)+X_{m-1}^{(0)}(\Omega^{\mu\nu}W_\nu),\end{aligned}\tag{20}$$

where

$$X_m^{(0)}=\tau^m\frac{\partial}{\partial t}+\xi_\mu^m\frac{\partial}{\partial a^\mu}.\tag{21}$$

And in Eq.(15) the terms with the minimum order in τ and ξ_μ are τ^0 and ξ_μ^0, so here when $m=0$, τ^{m-1}, τ^{m-2}, ξ_μ^{m-1} and ξ_μ^{m-2} would not exist. We note that when $m=0$, $\tau^{m-1}=\tau^{m-2}=\xi_\mu^{m-1}=\xi_\mu^{m-2}=0$, then Eq.(20) would turn into Eq.(8). Also, when $m=1$, $\tau^{m-2}=\xi_\mu^{m-2}=0$ holds also.

For the Birkhoffian system, when it is perturbed by small quantities εW_μ, if the infinitesimal generators $\tau^m(t,\boldsymbol{a})$ and $\xi_\mu^m(t,\boldsymbol{a})$ satisfy Eq.(20), the change of the symmetry is called perturbation to Lie symmetry of the system.

Proposition 2 For the Birkhoffian system perturbed by small quantities εW_μ, if the infinitesimal generators $\tau^m(t,\boldsymbol{a})$ and $\xi_\mu^m(t,\boldsymbol{a})$ satisfy Eq.(20) and there exists such a function $\lambda=\lambda(t,\boldsymbol{a})$ that

$$\frac{\partial h_\mu}{\partial a^\mu}+\varepsilon\frac{\partial(\Omega^{\mu\nu}W_\nu)}{\partial a^\mu}+\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\ln\lambda=0,\tag{22}$$

than the perturbation to Lie symmetry of the system will lead to a z th order Hojman adiabatic invariant

$$\begin{aligned}I_{\mathrm{Hz}}=&\varepsilon^m\left[\frac{1}{\lambda}\frac{\partial(\lambda\tau^m)}{\partial t}+\frac{1}{\lambda}\frac{\partial(\lambda\xi_\mu^m)}{\partial a^\mu}-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m\right.\\&\left.-\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-1}}{\partial a^\mu}\right]=\text{const.}\\&(m=0,1,2,\cdots,m).\end{aligned}\tag{23}$$

Proof Taking the derivative of I_{Hz} with respect to t, and using Eq.(19), we have

$$\begin{aligned}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{Hz}}=&\varepsilon^m\left\{\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\tau^m\frac{\partial\ln\lambda}{\partial t}\right)+\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tau^m}{\partial t}+\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\xi_\mu^m\frac{\partial\ln\lambda}{\partial a^\mu}\right)+\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_\mu^m}{\partial a^\mu}-\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-1}}{\partial a^\mu}\right)\right\}\\=&\varepsilon^m\left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}[X_m^{(0)}(\ln\lambda)]+\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}[X_m^{(0)}(\ln\lambda)]+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tau^m}{\partial t}+\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial^2\tau^m}{\partial a^\alpha\partial t}\right.\\&+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_\mu^m}{\partial a^\mu}+\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial^2\xi_\mu^m}{\partial a^\alpha\partial a^\mu}-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-1}}{\partial a^\mu}\right)\\&\left.-\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}\left(\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-1}}{\partial a^\mu}\right)\right\}.\end{aligned}\tag{24}$$

By using Eq.(16), we obtain

$$X^{(0)}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\ln\lambda\right)+\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\ln\lambda=\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}X^{(0)}(\ln\lambda).\tag{25}$$

Taking the derivative of Eq.(19) with respect to a^μ, we have

$$\begin{aligned}&\frac{\partial}{\partial a^\mu}\left[\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\mu-(h_\mu+\varepsilon\Omega^{\mu\nu}W_\nu)\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-X^{(0)}(h_\mu+\varepsilon\Omega^{\mu\nu}W_\nu)\right]\\=&\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_\mu}{\partial a^\mu}+\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tau}{\partial t}-\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-\left[\frac{\partial}{\partial a^\mu}(h_\mu+\varepsilon\Omega^{\mu\nu}W_\nu)\right]\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-X^{(0)}\left[\frac{\partial}{\partial a^\mu}(h_\mu+\varepsilon\Omega^{\mu\nu}W_\nu)\right]\\=&0.\end{aligned}\tag{26}$$

Substituting Eqs.(15), (19) into Eqs.(25), (26), and comparing the modulus of ε^m, we obtain

$$X_m^{(0)}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\lambda\right)+X_{m-1}^{(0)}\left(\Omega^{\alpha\beta}W_\beta\frac{\partial\ln\lambda}{\partial a^\alpha}\right)+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m\right)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\lambda+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^{m-1}\right)\Omega^{\mu\nu}W_\nu\frac{\partial\ln\lambda}{\partial a^\mu}$$

$$+\Omega^{\alpha\beta}W_\beta\frac{\partial\tau^{m-1}}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\lambda+\Omega^{\alpha\beta}W_\beta\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-2}}{\partial a^\alpha}\frac{\partial\ln\lambda}{\partial a^\mu}$$
$$=\frac{\bar{\mathrm{d}}}{\mathrm{d}t}[X_m^{(0)}(\ln\lambda)]+\Omega^{\mu\nu}W_\nu\frac{\partial}{\partial a^\mu}[X_{m-1}^{(0)}(\ln\lambda)], \tag{27}$$

and

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_\mu^m}{\partial a^\mu}+\Omega^{\alpha\beta}W_\beta\frac{\partial^2\xi_\mu^{m-1}}{\partial a^\alpha\partial a^\mu}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tau^m}{\partial t}+\Omega^{\alpha\beta}W_\beta\frac{\partial^2\tau^{m-1}}{\partial a^\alpha\partial t}-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m$$
$$-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-1}}{\partial a^\mu}\right)-\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^{m-1}-\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}\left(\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-2}}{\partial a^\mu}\right)$$
$$-\frac{\partial h_\mu}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\frac{\partial h_\mu}{\partial a^\mu}\Omega^{\alpha\beta}W_\beta\frac{\partial\tau^{m-1}}{\partial a^\alpha}-\frac{\partial(\Omega^{\mu\nu}W_\nu)}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^{m-1}-\Omega^{\alpha\beta}W_\beta\frac{\partial(\Omega^{\mu\nu}W_\nu)}{\partial a^\mu}\frac{\partial\tau^{m-2}}{\partial a^\alpha}$$
$$-X_m^{(0)}\left(\frac{\partial h_\mu}{\partial a^\mu}\right)-X_{m-1}^{(0)}\left[\frac{\partial(\Omega^{\mu\nu}W_\nu)}{\partial a^\mu}\right]=0. \tag{28}$$

Inserting Eqs.(27) and (28) into Eq.(24), and using Eq.(22) we obtain

$$\begin{aligned}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{Hz}}=&\varepsilon^m\Bigg\{-\left[\varepsilon X_m^{(0)}\left(\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}\right)-X_{m-1}^{(0)}\left(\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}\right)\right]-\left[\varepsilon X_m^{(0)}\left(\Omega^{\alpha\beta}W_\beta\frac{\partial\ln\lambda}{\partial a^\alpha}\right)-X_{m-1}^{(0)}\left(\Omega^{\alpha\beta}W_\beta\frac{\partial\ln\lambda}{\partial a^\alpha}\right)\right]\\&-\left[\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial\ln\lambda}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\Omega^{\alpha\beta}W_\beta\frac{\partial\ln\lambda}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^{m-1}\right]-\left[\varepsilon\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^{m-1}\right]\\&+\left[\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}X_m^{(0)}(\ln\lambda)-\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}X_{m-1}^{(0)}(\ln\lambda)\right]+\left[\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial^2\tau^m}{\partial a^\alpha\partial t}-\Omega^{\alpha\beta}W_\beta\frac{\partial^2\tau^{m-1}}{\partial a^\alpha\partial t}\right]\\&+\left[\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial^2\xi_\mu^m}{\partial a^\alpha\partial a^\mu}-\Omega^{\alpha\beta}W_\beta\frac{\partial^2\xi_\mu^{m-1}}{\partial a^\alpha\partial a^\mu}\right]-\left[\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^m-\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^{m-1}\right]\\&-\left[\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial\tau^{m-1}}{\partial a^\alpha}\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}-\Omega^{\alpha\beta}W_\beta\frac{\partial\tau^{m-2}}{\partial a^\alpha}\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}\right]-\left[\varepsilon\Omega^{\alpha\beta}W_\beta\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-1}}{\partial a^\alpha}\frac{\partial\ln\lambda}{\partial a^\mu}\right.\\&\left.-\Omega^{\alpha\beta}W_\beta\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{m-2}}{\partial a^\alpha}\frac{\partial\ln\lambda}{\partial a^\mu}\right]-\left[\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\alpha}\frac{\partial\tau^{m-1}}{\partial a^\mu}-\Omega^{\alpha\beta}W_\beta\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\alpha}\frac{\partial\tau^{m-2}}{\partial a^\mu}\right]\\&-\left[\varepsilon\Omega^{\alpha\beta}W_\beta\Omega^{\mu\nu}W_\nu\frac{\partial^2\tau^{m-1}}{\partial a^\alpha\partial a^\mu}-\Omega^{\alpha\beta}W_\beta\Omega^{\mu\nu}W_\nu\frac{\partial^2\tau^{m-2}}{\partial a^\alpha\partial a^\mu}\right]\Bigg\}\\=&\varepsilon^{z+1}\Bigg\{-X_z^{(0)}\left(\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}\right)-X_z^{(0)}\left(\Omega^{\alpha\beta}W_\beta\frac{\partial\ln\lambda}{\partial a^\alpha}\right)-\Omega^{\alpha\beta}W_\beta\frac{\partial\ln\lambda}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^z-\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^z\\&+\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}X_z^{(0)}(\ln\lambda)+\Omega^{\alpha\beta}W_\beta\frac{\partial^2\tau^z}{\partial a^\alpha\partial t}+\Omega^{\alpha\beta}W_\beta\frac{\partial^2\xi_\mu^z}{\partial a^\alpha\partial a^\mu}-\Omega^{\alpha\beta}W_\beta\frac{\partial}{\partial a^\alpha}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau^z\\&-\Omega^{\alpha\beta}W_\beta\frac{\partial\tau^{z-1}}{\partial a^\alpha}\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\mu}-\Omega^{\alpha\beta}W_\beta\Omega^{\mu\nu}W_\nu\frac{\partial\tau^{z-1}}{\partial a^\alpha}\frac{\partial\ln\lambda}{\partial a^\mu}-\Omega^{\alpha\beta}W_\beta\frac{\partial\Omega^{\mu\nu}W_\nu}{\partial a^\alpha}\frac{\partial\tau^{z-1}}{\partial a^\mu}\\&-\Omega^{\alpha\beta}W_\beta\Omega^{\mu\nu}W_\nu\frac{\partial^2\tau^{z-1}}{\partial a^\alpha\partial a^\mu}\Bigg\}.\end{aligned} \tag{29}$$

This shows that $\tilde{\mathrm{d}}I_{\mathrm{Hz}}/\mathrm{d}t$ is in direct proportion to ε^{z+1}.

5. An example

The Birkhoffian system is expressed as

$$R_1=a^3,\quad R_2=a^4,\quad R_3=R_4=0,$$
$$B=\frac{1}{2}\left[a^3-\frac{1}{b}\mathrm{arctg}bt\right]^2+\frac{1}{2}\left[a^4-\frac{1}{2b}\ln(1+b^2t^2)\right]^2, \tag{30}$$

and the system is perturbed by the small quantities

$$\varepsilon W_1=-\varepsilon bt,\quad \varepsilon W_2=\frac{-\varepsilon bt}{1+b^2t^2},\quad \varepsilon W_3=\varepsilon W_4=0. \tag{31}$$

Let us study the first order Hojman adiabatic invariants induced from the perturbation to Lie symmetry of the system.

The equations of motion for the system without perturbation are

$$\begin{aligned}
&\dot{a}^1 = a^3 - \frac{1}{b}\mathrm{arctg}bt,\\
&\dot{a}^2 = a^4 - \frac{1}{2b}\ln(1+b^2t^2),\\
&\dot{a}^3 = 0,\\
&\dot{a}^4 = 0.
\end{aligned} \tag{32}$$

Firstly, we seek the exact invariants. Choose infinitesimal generators as

$$\begin{aligned}
&\tau^0 = 1, \quad \xi_1^0 = t - \frac{1}{b}\mathrm{arctg}bt,\\
&\xi_2^0 = t - \frac{1}{2b}\ln(1+b^2t^2),\\
&\xi_3^0 = 1, \quad \xi_4^0 = 1.
\end{aligned} \tag{33}$$

It is easy to verify that they satisfy Eq.(8), so the infinitesimal generators (33) are Lie symmetrical for the system (30). From Eq.(11), we obtain

$$\lambda_0 = a^3.$$

According to Proposition 1, Lie symmetry of the system (30) can lead to a Hojman exact invariant

$$I_{\mathrm{H0}} = \frac{1}{a^3} = \mathrm{const.} \tag{34}$$

Then we seek the first order adiabatic invariants. Choose infinitesimal generators as

$$\begin{aligned}
&\tau^1 = 1, \quad \xi_1^1 = \frac{1}{2}bt^2 - \frac{1}{b}\mathrm{arctg}bt,\\
&\xi_2^1 = 0, \quad \xi_3^1 = bt, \quad \xi_4^1 = \frac{bt}{1+b^2t^2}.
\end{aligned} \tag{35}$$

We can verify that they satisfy Eqs.(20). From Eq.(22), we have

$$\lambda = a^3 - \varepsilon\frac{1}{2}bt^2. \tag{36}$$

According to Proposition 2, we will obtain the first Hojman order adiabatic invariant induced from the perturbation to Lie symmetry of the system.

$$I_{\mathrm{H1}} = \frac{1-\varepsilon bt}{a^3 - \frac{1}{2}\varepsilon bt^2} + \varepsilon\left[\frac{bt - \varepsilon bt}{a^3 - \frac{1}{2}\varepsilon bt^2}\right]. \tag{37}$$

6. Discussion

In this work, we present a new type of Hojman adiabatic invariants which has a different form from that in Ref.[20] for the perturbed Birkhoffian system. The infinitesimal ε included in $\tilde{\mathrm{d}}/\mathrm{d}t$ was not considered in Ref.[20] when the author compared the modulus of ε^m in the formulae. However, in this paper ε included in $\tilde{\mathrm{d}}/\mathrm{d}t$ (Eq.(19)) is considered (for example the deduction from Eqs.(16)–(20)). Based on our deduction, different methods in dealing with the physical quantity including ε will lead to dissimilar types of adiabatic invariants. For a physical quantity including ε of the perturbed mechanical systems, when comparing the modulus of ε^m, both the two methods are reasonable if it is kept identical in the deduction process. Due to the deduction related to ε included in $\tilde{\mathrm{d}}/\mathrm{d}t$, the type of adiabatic invariants obtained in this paper has one term more than that in Ref.[20]. When $z = 0$, the adiabatic invariants obtained in this paper will also turn into the exact invariants naturally.

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