Vol 17 No 6, June 2008
1674-1056/2008/17(06)/1962-05

Chinese Physics B

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New conserved quantities of Noether–Mei symmetry of mechanical system in phase space

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(Received 30 September 2007; revised manuscript received 22 October 2007)

This paper studies two new types of conserved quantities deduced by Noether–Mei symmetry of mechanical system in phase space. The definition and criterion of Noether–Mei symmetry for the system are given. A coordination function is introduced, and the conditions under which the Noether–Mei symmetry leads to the two types of conserved quantities and the forms of the two types of conserved quantities are obtained. An illustrative example is given. The coordination function can be selected according to the demand for finding the gauge function, and the choice of the coordination function has multiformity, so more conserved quantities deduced from Noether–Mei symmetry of mechanical system can be obtained.

Keywords: mechanical system, phase space, Noether–Mei symmetry, new conserved quantity
PACC: 0320

1. Introduction

There are intimate relations between symmetries and conserved quantities. The study of symmetries and conserved quantities of a mechanical system was of more mathematical and physical significance. It possesses a crucial position in research of modern mathematics, mechanics and physics. The modern symmetry of a mechanical system includes mainly Noether, Lie and Mei symmetries.[1−6] The known conserved quantities mainly consist of Noether, Hojmam and Mei symmetries. In recent years, more attention has been paid to the study of three symmetries and conserved quantities of mechanical systems which makes such field a hotspot. Many achievements have been obtained in this field.[7−12] Recently, Mei Fengxiang *et al* combined these three main symmetries mentioned above, studied the united and unified symmetries of related mechanical system, and gave the conserved quantities induced by the united and unified symmetries.[12−17] These studies upgrade the theoretical level and depth of the research on symmetries and conserved quantities.

In this paper, we investigated the united and uniform symmetries–Noether–Mei symmetries and its' induced conserved quantities of mechanical system in phase space. The conditions under which the Noether–Mei symmetry directly lead to the two types of conserved quantities and the forms of the two types of conserved quantities are also given. An example is illustrated to show the application of our results. The results in this paper have the universal sense. The Noether conserved quantity and Mei conserved quantity of mechanical system in phase space are the special cases of the two new conserved quantities respectively given in this paper.

2. The definition of Noether–Mei symmetry of the system and its criterion equations

The differential equations of motion for mechanical system of phase space can be written in the form

$$\begin{aligned} \dot{q}_s &= \frac{\partial H}{\partial p_s}, \\ \dot{p}_s &= -\frac{\partial H}{\partial q_s} + Q_s, \quad (s = 1, \cdots, s), \end{aligned} \tag{1}$$

where $H = H(t, \boldsymbol{q}, \boldsymbol{p})$ is the Hamiltonian of the system, $Q_s = Q_s(t, \boldsymbol{q}, \boldsymbol{p})$ are the non-potential generalized forces.

Equations (1) can be expressed as follows

$$\dot{q}_s = g_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad \dot{p}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}). \tag{2}$$

Definition The symmetry that is both a Noether

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symmetry and a Mei symmetry is called Noether–Mei symmetry.

Introduce the infinitesimal transformations

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^*(t^*) &= p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \tag{3}$$

where ε is the infinitesimal parameter, and ξ_0, ξ_s, η_s are the generators of infinitesimal transformations.

For the mechanical system Eq.(1) in phase space, under the infinitesimal transformations Eq.(3), the Noether identity can be written in the form

$$\begin{aligned} &H\dot{\xi}_0 + \frac{\partial H}{\partial t}\xi_0 + \frac{\partial H}{\partial q_s}\xi_s - p_s\dot{\xi}_s \\ &- Q_s(\xi_s - \dot{q}_s\xi_0) - \dot{G}_{\mathrm{N}} = 0, \end{aligned} \tag{4}$$

where $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q}, \boldsymbol{p})$ is a gauge function.

The criterion equations (or determining equations) of the Mei symmetry of the system Eq.(1) are

$$\begin{aligned} &\frac{\partial}{\partial p_s}\{X^{(0)}(H)\} = 0, \\ &\frac{\partial}{\partial q_s}\{X^{(0)}(H)\} - X^{(0)}(Q_s) = 0, \end{aligned} \tag{5}$$

where

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}. \tag{6}$$

According to the above definition, using Eqs.(4) and (5), we have the following criterion.

Criterion If there exists such a gauge function $G_{\mathrm{N}}(t, \boldsymbol{q}, \boldsymbol{p})$ that the infinitesimal generators ξ_0, ξ_s and η_s satisfy the condition

$$\begin{aligned} &\left\{H\dot{\xi}_0 + \frac{\partial H}{\partial t}\xi_0 + \frac{\partial H}{\partial q_s}\xi_s - p_s\dot{\xi}_s \right. \\ &\left. - Q_s(\xi_s - \dot{q}_s\xi_0) - \dot{G}_{\mathrm{N}}\right\}^2 \\ &+ \left\{\frac{\partial}{\partial p_s}[X^{(0)}(H)]\right\}^2 \\ &+ \left\{\frac{\partial}{\partial q_s}[X^{(0)}(H)] - X^{(0)}(Q_s)\right\}^2 = 0, \end{aligned} \tag{7}$$

then the symmetry is a Noether–Mei symmetry of mechanical system Eq.(1) in phase space.

3. Two new types of conserved quantities of Noether–Mei symmetry of the system

The conditions and the forms of the two new conserved quantities which are induced by Noether–Mei symmetry of a mechanical system in phase space are given in the following theorems.

Theorem 1 For a mechanical system Eq.(1) in phase space, if the infinitesimal generators ξ_0, ξ_s, η_s of Noether–Mei symmetry and a gauge function $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{q}, \boldsymbol{p})$ satisfy

$$\begin{aligned} &X^{(0)}(p_s)\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}[X^{(0)}(p_s)]\xi_s \\ &-\frac{\partial}{\partial t}[X^{(0)}(H)]f - \xi_s\frac{\partial}{\partial q_s}[X^{(0)}(H)] \\ &-X^{(0)}(H)\dot{f} + X^{(0)}(Q_s)(\xi_s - \dot{q}_s f) + \dot{G}_{\mathrm{M}} = 0, \end{aligned} \tag{8}$$

then the system has the following new conserved quantities

$$I_{\mathrm{M}} = X^{(0)}(p_s)\xi_s - X^{(0)}(H)f + G_{\mathrm{M}} = \mathrm{const}. \tag{9}$$

Where $f = f(t, \boldsymbol{q}, \boldsymbol{p})$ is an arbitrary function to make G_{M} exist. We call f a coordination function.

Proof Taking the derivation of I_{M} to t, we have

$$\begin{aligned} \frac{\mathrm{d}I_{\mathrm{M}}}{\mathrm{d}t} =& X^{(0)}(p_s)\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}[X^{(0)}(p_s)]\xi_s \\ &- X^{(0)}(H)\dot{f} - \left[\frac{\partial}{\partial t}X^{(0)}(H) + \dot{q}_s\frac{\partial}{\partial q_s}X^{(0)}(H)\right. \\ &\left. + \dot{p}_s\frac{\partial}{\partial p_s}X^{(0)}(H)\right]f + \dot{G}_{\mathrm{M}}. \end{aligned} \tag{10}$$

According to the definition, the Noether–Mei symmetry of the system must be the Mei symmetry. Substituting the criterion equation of Mei symmetry Eq.(5) and condition Eq.(8) into Eq.(10), we obtain

$$\frac{\mathrm{d}I_{\mathrm{M}}}{\mathrm{d}t} = 0. \tag{11}$$

Special case When $f = \xi_0$, Eqs.(8) and (9) respectively become

$$\begin{aligned} &X^{(0)}(p_s)\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}[X^{(0)}(p_s)]\xi_s \\ &- X^{(0)}[X^{(0)}(H)] - X^{(0)}(H)\dot{\xi}_0 \\ &+ X^{(0)}(Q_s)(\xi_s - \dot{q}_s\xi_0) + \dot{G}_{\mathrm{M}} = 0, \end{aligned} \tag{12}$$

$$I_{\mathrm{M}} = X^{(0)}(p_s)\xi_s - X^{(0)}(H)\xi_0 + G_{\mathrm{M}} = \mathrm{const}. \tag{13}$$

Equation (12) is the condition to which symmetry can lead the Mei conserved quantity. Equation (13) is just the Mei conserved quantity induced by symmetry of mechanical system in phase space. We call the new conserved quantity Eq.(9) a generalized Mei conserved quantity of mechanical system in phase space.

Theorem 2 For mechanical system in phase space Eq.(1), if the generators ξ_s of Noether–Mei symmetry and a gauge function $\bar{G}_{\rm N}=\bar{G}_{\rm N}(t,\boldsymbol{q},\boldsymbol{p})$ satisfy

$$H\dot{f}_s+\frac{\partial H}{\partial t}f+\frac{\partial H}{\partial q_s}\xi_s-p_s\dot{\xi}-Q_s(\xi_s-\dot{q}_sf)-\dot{\bar{G}}_{\rm N}=0, \tag{14}$$

then the system will lead to the following new conserved quantities:

$$I_{\rm N}=p_s\xi_s-Hf+\bar{G}_{\rm N}=\text{const}, \tag{15}$$

where $f=f(t,\boldsymbol{q},\boldsymbol{p})$ is an arbitrary function to make $\bar{G}_{\rm N}$ exist. We call f a coordination function.

Proof Taking the derivation of $I_{\rm N}$ to t, we have

$$\frac{{\rm d}I_{\rm N}}{{\rm d}t}=\dot{p}_s\xi_s+p_s\dot{\xi}_s-\frac{\partial H}{\partial t}f-\frac{\partial H}{\partial q_s}\dot{q}_sf-\frac{\partial H}{\partial p_s}\dot{p}_sf-H\dot{f}+\dot{\bar{G}}_{\rm N}. \tag{16}$$

Substituting Eq.(14) into Eq.(16) and using Eq.(1), we have

$$\frac{{\rm d}I_{\rm N}}{{\rm d}t}=0. \tag{17}$$

Special case When $f=\xi_0$, we know that $\dot{\bar{G}}_{\rm N}=\dot{G}_{\rm N}$ forms Eqs.(4) and (14). Under the condition, Eq.(14) becomes Noether identity Eq.(4) and the new conserved quantity Eq.(15) becomes

$$I_{\rm N}=p_s\xi_s-H\xi_0+G_{\rm N}=\text{const}. \tag{18}$$

We call the new conserved quantity Eq.(15) a generalized Noether conserved quantity of mechanical system in phase space.

4. Illustrative example

The Hamiltonian of a mechanical system is

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+q_1+q_2, \tag{19}$$

the non-potential generalized forces are

$$Q_1=p_3,\quad Q_2=-p_3,\quad Q_3=0. \tag{20}$$

Try to study its Noether–Mei symmetry and two types of new conserved quantities.

Equations (1) and (2) give

$$\begin{aligned}&g_1=\dot{q}_1=p_1,\\&g_2=\dot{q}_2=p_2,\\&g_3=\dot{q}_3=p_3,\\&h_1=\dot{p}_1=-1+p_3,\\&h_2=\dot{p}_2=-1-p_3,\\&h_3=\dot{p}_3=0.\end{aligned} \tag{21}$$

After the calculation, we have

$$\begin{aligned}&X^{(0)}(H)=\eta_1p_1+\eta_2p_2+\eta_3p_3+\xi_1+\xi_2,\\&X^{(0)}(Q_1)=\eta_3,\quad X^{(0)}(Q_2)=-\eta_3,\\&X^{(0)}(Q_3)=0.\end{aligned} \tag{22}$$

Choosing the infinitesimal generators as

$$\begin{aligned}&\xi_0=1,\quad \xi_1=p_1+t,\\&\xi_2=p_2,\quad \xi_3=t,\\&\eta_1=-1,\quad \eta_2=-1,\quad \eta_3=0,\end{aligned} \tag{23}$$

then we have

$$\begin{aligned}&X^{(0)}(H)=t,\\&X^{(0)}(Q_1)=X^{(0)}(Q_2)=X^{(0)}(Q_3)=0.\end{aligned} \tag{24}$$

For the infinitesimal generators Eq.(23), when we take

$$G_{\rm N}=q_2-\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(1-p_3)t^2-p_3t, \tag{25}$$

the Criterion equation of the Noether–Mei symmetry Eq.(7) is satisfied, so the infinitesimal generators Eq.(23) are Noether–Mei symmetrical. For generators Eq.(23), we can obtain from Eqs.(8) and (14)

$$\dot{G}_{\rm M}=\dot{p}_1+\dot{p}_2+1+f+t\dot{f}, \tag{26}$$

$$\dot{\bar{G}}_{\rm N}=H\dot{f}+t+p_2-p_3-p_1\dot{p}_1-p_2\dot{p}_2-p_3(p_1+t-\dot{q}_1f)+p_3(p_2-\dot{q}_2f). \tag{27}$$

1) If select $f=q_3-p_3t$, then

$$G_{\rm M}=p_1+p_2+\frac{1}{2}(q_3-p_3t+t)^2-\frac{1}{2}t^2+t, \tag{28}$$

$$\bar{G}_{\rm N}=q_2+\frac{1}{2}t^2-p_3t-\frac{1}{2}(p_1^2+p_2^2)-\frac{1}{2}p_3t^2-p_3(q_1-q_2)(1-q_3+p_3t). \tag{29}$$

From Eqs.(9) and (15), we can get

$$I_{\rm M} = \frac{1}{2}(q_3 - p_3 t)^2 = \text{const}, \tag{30}$$

$$I_{\rm N} = \left[\frac{1}{2}(p_1^2 + p_2^2) - p_3(q_1 - q_2)\right](1 - q_3 + p_3 t) + q_2 + p_1 t + \frac{1}{2}t^2 - \frac{1}{2}p_3 t^2 - \left(q_1 + q_2 + \frac{1}{2}p_3^2\right)(q_3 - p_3 t) = \text{const}. \tag{31}$$

2) If select $f = p_1 + t - p_3 t$, then

$$G_{\rm M} = p_1 + p_2 + q_1 + t + \frac{1}{2}t^2(1 - p_3), \tag{32}$$

$$\bar{G}_{\rm N} = q_2 + \frac{1}{2}t^2 - p_3 t - \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{2}p_3 t^2 - p_3(q_1 - q_2)(1 - p_1 - t + p_3 t). \tag{33}$$

From Eqs.(9) and (15), we can get

$$I_{\rm M} = q_1 + \frac{1}{2}p_3 t^2 - p_1 t - \frac{1}{2}t^2 = \text{const}, \tag{34}$$

$$I_{\rm N} = \left[\frac{1}{2}(p_1^2 + p_2^2) - p_3(q_1 - q_2)\right](1 - p_1 - t - p_3 t) + q_2 + \frac{1}{2}t^2 + p_1 t - \frac{1}{2}p_3 t^2 - \left(q_1 + q_2 + \frac{1}{2}p_3^2\right)(p_1 + t - p_3 t) = \text{const}. \tag{35}$$

3) If select $f = \xi_0 = 1$, then

$$G_{\rm M} = 0, \tag{36}$$

$$\bar{G}_{\rm N} = G_{\rm N} = q_2 - \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(1 - p_3)t^2 - p_3 t. \tag{37}$$

From Eqs.(9) and (15), we can get

$$I_{\rm M} = -p_1 - p_2 - 2t = \text{const}, \tag{38}$$

$$I_{\rm N} = p_1 t - \frac{1}{2}p_3^2 - q_1 + \frac{1}{2}t^2 - \frac{1}{2}p_3 t^2 = \text{const}. \tag{39}$$

5. Conclusion

Two new types of conserved quantities induced by Noether–Mei symmetry of mechanical system given in this paper—the generalized Mei conserved quantity and generalized Noether conserved quantity have more general significance. The Mei conserved quantity and the Noether conserved quantity of the mechanical system are special cases of the generalized Mei conserved quantity and the generalized Noether conserved quantity respectively when $f = \xi_0$. The coordination function f in the result of this paper can be selected properly according to the demand for finding out the gauge function, and the choice of the coordination function has multiformity, so more conserved quantities deduced from Noether–Mei symmetry of mechanical system can be obtained.

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