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Synchronization of coupled logistic maps on random community networks*

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The collective synchronization of a system of coupled logistic maps on random community networks is investigated. It is found that the synchronizability of the community network is affected by two factors when the size of the network and the number of connections are fixed. One is the number of communities denoted by the parameter m, and the other is the ratio σ of the connection probability p of each pair of nodes within each community to the connection probability q of each pair of nodes among different communities. Theoretical analysis and numerical results indicate that larger m and smaller σ are the key to the enhancement of network synchronizability. We also testify synchronous properties of the system by analysing the largest Lyapunov exponents of the system.

Keywords: complex networks, synchronization, random community networks
PACC: 0250, 0520

1. Introduction

Collective dynamics of coupled maps is of significant interest in many fields of science and technology.[1−7] Synchronization appears to be one of the simplest types of collective dynamics among coupled dynamical systems.[8] Over the past decades, much work on the synchronization of coupled maps has used regular coupling lattices as well as globally coupled maps.[9] Also, random networks with a larger connectivity show synchronized behaviours for larger coupling strengths.[10,11] In Ref.[12], Pecora and Carroll obtained exact results of synchronization on networks with a general structure. In many real systems, however, the structures of actual networks are known to be neither completely regular nor completely random but somewhere in between, which results in the study of complex networks.[13] Recently, much attention has been paid to the synchronization of complex networks. For example, small-world wirings have been proven to enhance the synchronization as compared with regular topologies.[14,15] But smaller average distance between a pair of vertices does not always improve the synchronizability. For scale-free networks, the synchronizability is reduced with the increase of the heterogeneity in the connectivity distribution even though the average network distance decreases at the same time.[16] Motivated by this counterintuitive effect, researchers have pursued many ways to enhance network synchronizability.[17−21]

It has been demonstrated that many realistic networks have a so-called community structure.[22−25] That is, there are groups of nodes with dense connections within each group and sparse connections with the rest of the network. In this paper, we will investigate the community effect on the synchronization of coupled maps, i.e. the introducing of nonlinear dynamics to each node in the network. Consider a community network with N nodes (for more details, see Ref.[26]). Suppose that the N nodes are divided into m groups randomly. Each pair of nodes is connected with a probability p within the same group and a probability q among different groups, and $p \gg q$. Throughout this paper, we denote the degree of community by a ratio $\sigma = p/q$, which reflects the heterogeneous topological structure of real social networks. We assume that the total number of connections is N_c. When the values of N and N_c are fixed, the structure of the community network will be determined by the parameters m and σ. Here, the question arises, how is the network synchronizability influenced by m and σ? The aim of the present work is to provide a positive answer.

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This paper is organized as follows. In Section 2, theoretical analysis and numerical results are given. In Section 3 we discuss the largest Lyapunov exponents of the system, which act in accord with the results displayed in Section 2. Finally we end this paper with a short conclusion in Section 4.

2. Theoretical analysis and numerical results

Consider a system consisting of N identical coupled maps on the network, which can be written as

$$x_{n+1}^i = f(x_n^i) + \frac{\varepsilon}{k_i}\sum_{j=1}^{N} G_{ij} f(x_n^j), \tag{1}$$

where $i, j \in [1, N]$, x_n^i is the dynamical variable of node i at the nth time step, k_i is the degree of the node i, ε is the overall coupling strength parameter, and G is the Laplacian matrix

$$G_{ij} = \begin{cases} -k_i, \text{ if } i = j, \\ 1, \quad \text{if node } i \text{ connects node } j, \\ 0, \quad \text{otherwise.} \end{cases} \tag{2}$$

The dynamics of each individual node (i.e. with $\varepsilon = 0$) is described by

$$x_{n+1}^i = f(x_n^i). \tag{3}$$

In the present work, we use the well-known logistic map

$$f(x) = 1 - ax^2, \tag{4}$$

where the free parameter a is restricted to the range $-0.25 \leq a \leq 2$ and contains all possible dynamical regimes from a fixed point (e.g. $a = 0$) to fully developed chaotic orbits (e.g. $a = 2$). Some attention must be paid to the parameter ranges, for the basin of attraction of the fully synchronized states may be bounded by regions of phase space where amplitudes diverge. In particular, for maps of the range one has $0 \leq \varepsilon \leq 1$ in order to guarantee convergence of any initial configuration.[27] Since the matrix of coupling coefficients $\{G_{ij}\}$ satisfies $\sum_{j=1}^{N} G_{ij} = 0$, the system permits an exact synchronized solution: $x_n^1 = x_n^2 = \cdots = x_n^N = s_n$ as $n \to \infty$, where $s_{n+1} = f(s_n)$. If the system is synchronizable, starting from a random initial condition, it will approach the synchronized states. To estimate the collective behaviour, we introduce a synchronization error

$$e = \frac{1}{N}\sum_{i=1}^{N} |x_n^i - \langle x_n \rangle|, \tag{5}$$

where $\langle \cdot \rangle$ indicates the average over all nodes. In numerical simulations, the synchronization is defined as $e < 10^{-10}$. Here and below, we set the parameters $N = 1000$ and $N_c = 10000$. In Fig.1, we plot the synchronization error e versus the coupling strength ε at $a = 1.6$. For $a = 1.6$, the individual dynamics is chaotic, with the largest Lyapunov exponent approximated to 0.37. Starting from $\varepsilon = 0$, we increase its value at small intervals. One can find that there is a critical value $\varepsilon_c \approx 0.62$ above which e becomes zero (within the numerical accuracy), indicating that the system is synchronized.

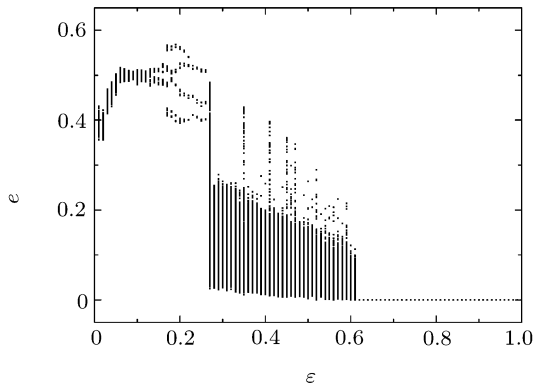


Fig.1. The synchronization error e as a function of coupling strength ε, for $m = 60$, $\sigma = 20$ and $a = 1.6$.

To study how the synchronizability is affected by the parameter m, we first examine the relation between e and m. Figure 2(a) displays the behaviour of e as a function of m for $\sigma = 20$, $\varepsilon = 0.7$, and $a = 1.6$. In each grid corresponding to a value of m, 1000 final iterations of e are plotted after discarding about 500 time steps as initial transients. For $m \geq 44$, the system described by expression (1) is in a synchronized state. For more insight into the effect of m on synchronization of networks, we plot ε_c versus m in Fig.2(b) while the value of σ is fixed at 20. The critical coupling strength ε_c decreases as m becomes larger. Figure 2 implies that larger m will enhance the network synchronizability when σ is fixed. In the following, we shall derive a theoretical formula to understand the dependence of ε_c on m.

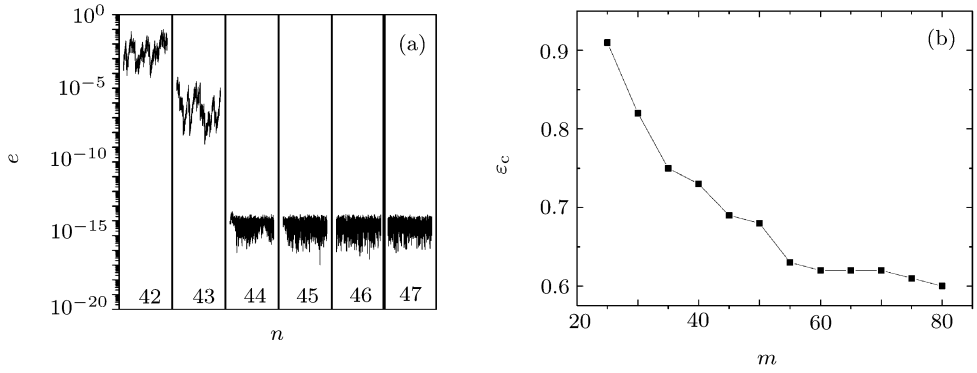


Fig.2. The synchronization error e versus n (denoted inside the grid) for $\sigma = 20$, $\varepsilon = 0.7$ and $a = 1.6$, where 1000 iterations of e are plotted between two grid lines (a), and the critical coupling strength ε_c as a function of m for $\sigma = 20$ and $a = 1.6$ (b).

We rewrite expression (1) as

$$x_{n+1}^i = f(x_n^i) + \varepsilon \sum_{j=1}^{N} \Gamma_{ij} f(x_n^j), \tag{6}$$

where $\Gamma_{ij} = G_{ij}/k_i$ and the eigenvalue of Γ is real and nonpositive,[28] we write it as $-\lambda_i$. And then the eigenvalue equation is

$$\Gamma u_i + \lambda_i u_i = 0. \tag{7}$$

Thus, we can order the eigenvalues as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. The synchronization state of this system is stable if[28]

$$\frac{1 - \mathrm{e}^{-\beta_0}}{\lambda_2} < \varepsilon < \frac{1 + \mathrm{e}^{-\beta_0}}{\lambda_N}, \tag{8}$$

where β_0 is the largest Lyapunov exponent of a single map. Since in most relevant cases, the upper limit $(1 + \mathrm{e}^{-\beta_0})/\lambda_N$ is greater than 1, so we discuss just the lower limit $(1 - \mathrm{e}^{-\beta_0})/\lambda_2$. According to Ref.[29], we can obtain λ_2 as follows. By definition, $\Gamma \cdot \boldsymbol{u}_2 = -\lambda_2 \boldsymbol{u}_2$ and $\boldsymbol{u}_2 \cdot \boldsymbol{u}_2 = 1$, we have $\lambda_2 = -\boldsymbol{u}_2^T \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{u}_2 = -\sum_{i,j=1}^{N} u_{2i} \Gamma_{ij} u_{2j}$, where u_{2i} is the ith component of $\boldsymbol{u}_2$. If i and j belong to the same community, then $\Gamma_{ij} = 1/k_i$ with a probability p; if $i = j$, then $\Gamma_{ij} = -1$; if i and j belong to the dissimilar communities, then $\Gamma_{ij} = 1/k_i$ with a probability q. At the same time, for a community network, the components of the eigenvector $\boldsymbol{u}_2$ have the same values within the same community, while they are different among different communities. So we have $\lambda_2 = \sum_{i=1}^{N} u_{2i}[N(q/k_i)u_I - (q/k_i)\sum_{J=1}^{m} n_J u_J]$, where u_I is the component associated with the community that contains node i, $1 \leq I \leq m$. Its average degree is $k = [N^2 q + (p - q)\sum_{I=1}^{m} n_I^2]/N$. Since $\sum_{I=1}^{m} n_I u_I^2 \approx \sum_{i=1}^{N} u_{2i}^2 = 1$, $\sum_{I=1}^{m} n_I u_I = \sum_{i=1}^{N} u_{2i}$ and because Γ is symmetric, replacing k_i with k, we can obtain

$$\lambda_2 \approx \frac{N^2}{N^2 + (\sigma - 1)\sum_{I=1}^{m} n_I^2}. \tag{9}$$

In Ref.[29], authors discussed only how the network synchronizability was influenced by the connection probability p_s of each pair of nodes within each community and p_l of each pair of nodes among different communities for a fixed m. Here, we also concern about the relation between the network synchronizability and m when the value of $\sigma = p_s/p_l$ is fixed. We first consider a special case in which there is $n = N/m$ nodes in each community. Expression (9) can be simplified into

$$\lambda_2 \approx \frac{m}{m + (\sigma - 1)}, \tag{10}$$

which is consistent with the result given in Ref.[29]. Hence, for a fixed σ, a qualitative relation between λ_2 and m is $\lambda_2 \sim m$. We obtain an inverse relation between the critical value ε_c and m

$$\varepsilon_c \sim \frac{1}{m}, \tag{11}$$

from inequality (8). Those with different clustered sizes have the same qualitative properties because they are both random community networks. This theoretical result is strongly corroborated with numerical results as shown in Fig.2(b).

Next, we explore how the parameter σ affects the synchronizability of the network. From expression (9), one can obtain $\lambda_2 \sim \dfrac{1}{\sigma}$ in the same way when the values of other variables are fixed. Combining Eq.(8), we obtain a qualitative relation

$$\varepsilon_c \sim \sigma. \tag{12}$$

For checking the above analysis, we also provide numerical calculations. In Fig.3(a), we plot e for different values of σ in the case of $m = 60$, $\varepsilon = 0.7$, and $a = 1.6$. In each grid corresponding to a value of σ, 1000 final iterations of e are plotted after discarding about 500 time steps as initial transients. One can find that synchronization arises when $1 < \sigma \leq 27$. Figure 3(b) displays the dependence of the critical coupling strength ε_c on the parameter σ for $m = 60$ and $a = 1.6$. As σ becomes larger, ε_c increases. Figure 3 indicates that smaller σ will enhance the network synchronizability while m is fixed.

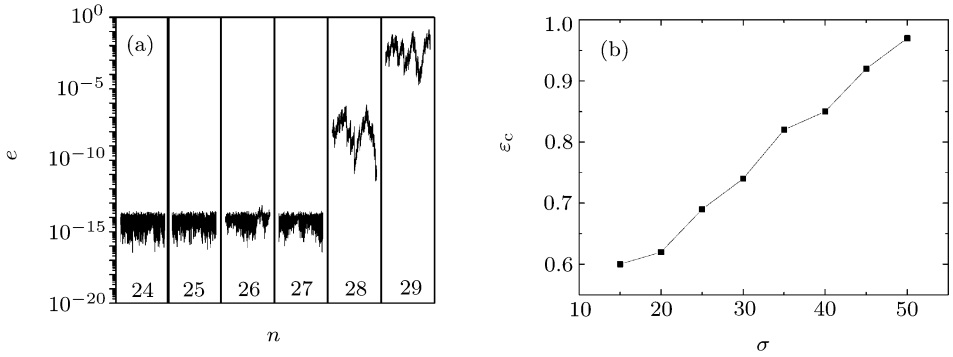


Fig.3. The synchronization error e versus n (denoted inside the grid) for $m = 60$, $\varepsilon = 0.7$, and $a = 1.6$, where 1000 iterations of e are plotted between two grid lines (a), and the critical coupling strength ε_c as a function of σ for $m = 60$ and $a = 1.6$ (b).

3. Analysing the largest Lyapunov exponents of the system

To complete our study on synchronization of the random community network, we also study how the complete synchronization is affected by m and σ through analysing the largest Lyapunov exponent $\beta_{\max}$ of the system. The sign of $\beta_{\max}$ will reveal the stability of the synchronized state. Only for negative $\beta_{\max}$, can the synchronous state be stable. According to Refs.[12,29–31], around the synchronized states s_n, the variational equations of Eq.(6) can be written as

$$\delta x^i_{n+1} = Df(s_n)\cdot\delta x^i_n + \varepsilon\sum_{j=1}^{N}\Gamma_{ij}Df(s_n)\cdot\delta x^j_n$$
$$i = 1, 2, \cdots, N, \tag{13}$$

where $\delta x^i \equiv x^i - s$ and Df is the Jacobian matrix of the corresponding vector function. This transformation can be diagonalized into

$$\delta y^i_{n+1} = (1 + \varepsilon\lambda_i)Df(s_n)\delta y^i_n, \tag{14}$$

where λ_i $(i = 1, 2, \cdots, N)$ represent the eigenvalues of Γ. Let $\beta_{\max}$ denotes the largest Lyapunov exponent of the system (expression (6)), we obtain $\beta_{\max} = \lim_{n\to\infty}\frac{1}{n}\ln\frac{|\delta y^i_n|}{|\delta y^i_0|} = \ln|1 + \varepsilon\lambda_i| + \lim_{n\to\infty}\frac{1}{n}\ln\prod_{j=0}^{n-1}|f'(s_j)|$, where the second term on the right-hand side is the largest Lyapunov exponent β_0 of a single map. Then we have

$$\beta_{\max} = \beta_0 + \ln|1 + \varepsilon\lambda_i|. \tag{15}$$

We compute the largest Lyapunov exponent $\beta_{\max}$ of the system as a function of m (σ) in Fig.4(a) (Fig.4(b)). Figure 4(a) displays the relation between $\beta_{\max}$ and m for $\sigma = 20$, $\varepsilon = 0.7$, and $a = 1.6$. The largest Lyapunov exponent $\beta_{\max}$ is negative (the synchronous state is stable) for $m \geq 44$, which consists with numerical results as shown in Fig.2(a). In Fig.4(b), We plot the dependence of $\beta_{\max}$ on σ for $m = 60$, $\varepsilon = 0.7$, and $a = 1.6$. The largest Lyapunov exponent $\beta_{\max}$ is negative for $\sigma \leq 27$, which is consistent with numerical results as shown in Fig.3(a).

Our considerations so far have been limited to the case in which the parameter $a = 1.6$. For the logistic map described by Eq.(4), it becomes chaotic when $a \approx 1.40116$, going through a period doubling bifurcation sequence. Figure 5(a) shows the dependence of $\beta_{\max}$ on m for $\sigma = 20$, $\varepsilon = 1.0$, and different values of a, which demonstrates that for a given m, as a increases, $\beta_{\max}$ increases. That is to say, the synchronizability is decreasing. In Fig.5(b), we depict the behaviour of $\beta_{\max}$ with respect to σ for $m = 50$, $\varepsilon = 1.0$, and different values of a. It can be seen that for a given σ, as a increases, $\beta_{\max}$ increases. As can be seen in both Figs.5(a) and 5(b), an exception occurs for $a = 1.8$, which is below the case for $a = 1.7$,

due to the fact that for $a = 1.8$ the largest Lyapunov exponent of the logistic map is smaller than that for $a = 1.7$.

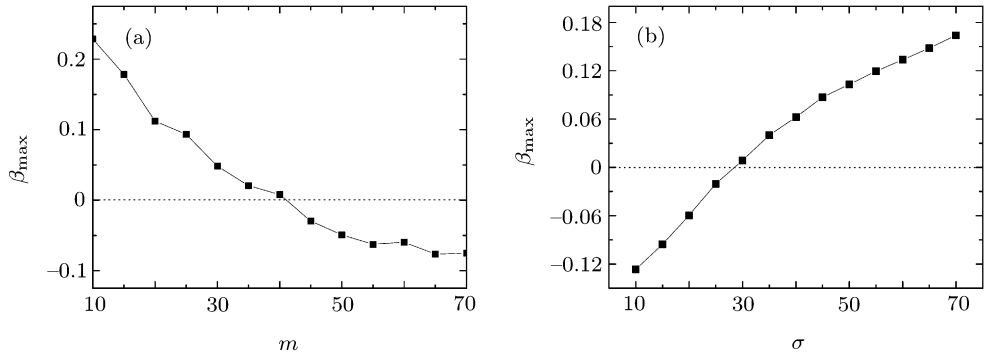


Fig.4. The maximum Lyapunov exponent $\beta_{\max}$ of the system versus m for $\sigma = 20$, $\varepsilon = 0.7$, and $a = 1.6$ (a), and the largest Lyapunov exponents $\beta_{\max}$ versus σ for $m = 60$, $\varepsilon = 0.7$, and $a = 1.6$ (b).

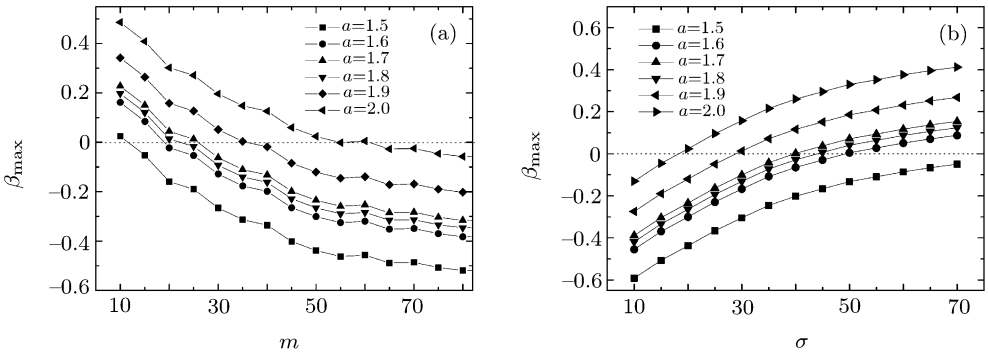


Fig.5. The maximum Lyapunov exponents $\beta_{\max}$ of the system versus m for $\sigma = 20$ and $\varepsilon = 1.0$ (a), and the largest Lyapunov exponents $\beta_{\max}$ versus σ for $m = 50$ and $\varepsilon = 1.0$ (b). From bottom to top, a=1.5, 1.6, 1.8, 1.7, 1.9, and 2.0, respectively.

4. Conclusion

We have analytically and numerically studied the collective synchronization in the system of coupled logistic maps on random community network. This kind of network has been recognized to be important for real network systems. It is found that the synchronization of community networks is dependent on the values of m and σ while the size of the network and the total number of connections between nodes are fixed. Our results implies that a better synchronizability for the random community networks is induced by increasing the number of communities or by decreasing the degree of community. The synchronous stability of the networks has also been verified by analysing the largest Lyapunov exponents of the system.

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