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Existence and asymptotic estimates of periodic solutions of a sea–air oscillator model of decadal variations in subtropical cells and equatorial Pacific SST*

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This paper studies a time delay equation for sea–air oscillator model. The existence and asymptotic estimates of periodic solutions of corresponding problem are obtained by employing the technique of upper and lower solution, and by using the continuation theorem of coincidence degree theory.

Keywords: nonlinear, time delay, El Niño–Southern Oscillator, periodic solution
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1. Introduction

Interactions between the ocean and atmosphere contribute to climate fluctuations over a broad spectrum of time scales. Studies of those interactions have far focused on El Niño Southern Oscillation (ENSO) phenomena that have a period of 3 to 4 years and whose principal signature is in the tropical Pacific.[1,2] Equatorial Pacific sea surface temperature (SST) is of central importance to global climate. In addition to ENSO, eastern equatorial Pacific SST anomalies (SSTAs) also display significant decadal or interdecadal variations that modulate the amplitude and frequency of ENSO.[3] The cause of decadal SSTAs in the equatorial Pacific is not yet clear. Nonaka *et al*[4] consider that the nonlinearity of the tropical ocean–atmosphere can give rise to chaotic modulation of ENSO on decadal and longer time scales without involving the extratropics. The phenomenon of the ENSO is a very attractive object of study in the international academic circles.[5−8] Feng *et al*[5] considered an instable model of air–sea oscillator, Wang[7] studied an especial ENSO mechanism. Mo *et al*[9−17] Lin W T and Mo,[18] and Lin Y H *et al*[19−22] also considered some problems in atmospheric physics, ocean weather, dynamic system and so on.

Recently, Li have studied the periodic solution to some problems for the ENSO.[23] In this paper a time delay equation for sea–air oscillator model is studied. The existence and asymptotic estimates of periodic solutions of corresponding problem are obtained by using the technique of upper and lower solution, and using continuation theorem of coincidence degree theory and the theory of perturbation.[24,25]

2. Model of equatorial Pacific SSTA

The authors in Ref.[4] illustrated some experiments to demonstrate that wind variability in the off-equatorial tropics is an effective mechanism for generating equatorial decadal SSTAs. The heat-anomaly budget for the tropical upper ocean is an useful diagnostic. It is defined by the volume integral of the model's temperature-anomaly equation in a box that extends across the basin from 11°S to 15°N and from –550 m to the ocean surface. The resulting heat-anomaly equation can be written as[4]

$$\frac{\mathrm{d}H'}{\mathrm{d}t} = v' - Q', \tag{1}$$

where H' is the anomalous heat content in the box, t is a time variable, v' is the heat transport into the box

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through its sides and bottom which is deep enough to encompass almost all of the mean sub-tropical cells, $Q' = \gamma \int \int_A T_0' \mathrm{d}x\mathrm{d}y$ is the integral of surface heat flux out of the box over its horizontal A, and γ is a heat flux coefficient and T_0' is the solution's SSTA field. The authors in Ref.[4] obtained a close relationship between Q' and H', with $H' = aQ'$, $a = 841\,\mathrm{m}$. And the experiments demonstrate[4] that the Q' is very similar to decadal T in the eastern equatorial region 140°W–90°W, 1°S–1°N, and to a good approximation $Q' = bT, b = 2.83 \times 10^{14}\mathrm{W}^\circ \cdot \mathrm{C}^{-1}$. These two relations imply that the Eq.(1) can be rewritten for decadal time scales as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{1}{a}T + \frac{1}{ab}v'. \tag{2}$$

From the demonstration of the experiments, and Eqs.(1), (2), we can consider that the heat transport v' signifies $T^2T(t-\delta)$, where δ is a time delay of T. Thus we obtain the following time delay model of equatorial Pacific:

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -\frac{1}{a}T(t) + \frac{\gamma}{ab}T^2(t)T(t-\delta). \tag{3}$$

The authors in Ref.[15] obtained the approximate solution of Eq.(3) by employing the method of variational iteration. In this paper, we continue to study Eq.(3), the existence and asymptotic estimates of periodic solutions of Eq.(3) are obtained by using the technique of upper and lower solution, and using continuation theorem of coincidence degree theory. Throughout this paper, we assume that a, b are two positive constants.

3. The periodic solution of a nonlinear problem

In order to study Eq.(3), first we discuss the following nonlinear problem:

$$T'(t) = f(t, T(t), T(t-\delta)), \tag{4}$$

where $f \in C(\boldsymbol{R}^3, \boldsymbol{R})$, $f(t+\omega, x, y) \equiv f(t, x, y)$ for arbitrary $(x, y) \in \boldsymbol{R}^2$, $\omega > 0$ is a given constant.

Firstly, we give some definitions as follows.

Definition 1 If $\alpha(t) \in C^1(\boldsymbol{R}, \boldsymbol{R})$ with $\alpha(t+\omega) \equiv \alpha(t)$, and

$$\alpha'(t) < f(t, \alpha(t), \alpha(t-\delta)), \forall t \in [0, \omega],$$

then $\alpha(t)$ is called a strict ω-periodic lower solution of Eq.(4).

Definition 2 If $\beta(t) \in C^1(\boldsymbol{R}, \boldsymbol{R})$ with $\beta(t+\omega) \equiv \beta(t)$, and

$$\beta'(t) > f(t, \beta(t), \beta(t-\delta)), \forall t \in [0, \omega],$$

then $\beta(t)$ is called a strict ω-periodic upper solution of Eq.(4).

Further, we let $C_\omega = \{T | T \in C(\boldsymbol{R}, \boldsymbol{R}), T(t+\omega) \equiv T(t)\}$ with the norm $|\varphi|_0 = \max\limits_{t \in [0,\omega]} |\varphi(t)|, \forall \varphi \in C_\omega$. Clearly, C_ω is a Banach space. We also define operator L as

$$L : D(L) \subset C_\omega \longrightarrow C_\omega, \quad LT = T',$$

where $D(L) = \{T | T \in C^1(\boldsymbol{R}, \boldsymbol{R}), T(t+\omega) \equiv T(t)\}$. It is easy to see that $\mathrm{Ker}L = \boldsymbol{R}$ and $\mathrm{Im}L = \{T | T \in C_\omega, \int_0^\omega T(s)\mathrm{d}s = 0\}$. Thus, L is a Fredholm operator with index zero.[26] Let

$$P : C_\omega \longrightarrow \mathrm{Ker}L, \quad Q : C_\omega \longrightarrow \frac{C_\omega}{\mathrm{Im}L}$$

defined respectively by

$$PT = T(0), \quad QT = \frac{1}{\omega}\int_0^\omega T(s)\mathrm{d}s$$

and let

$$L_P = L|_{D(L) \bigcap \mathrm{Ker}P} : D(L) \cap \mathrm{Ker}P \longrightarrow \mathrm{Im}L.$$

Then $\mathrm{Im}P = \mathrm{Ker}L, \mathrm{Ker}Q = \mathrm{Im}L$, and L_P has a unique continuous pseudo-inverse K_P on $\mathrm{Im}L$ defined by

$$[K_P y](t) = \int_0^t y(s)\mathrm{d}s. \tag{5}$$

Let $N : C_\omega \longrightarrow C_\omega, [NT](t) = f(t, T(t), T(t-\delta))$. It is easy to check that Eq.(4) is equivalent to following operator equation

$$T = MT,$$

where $MT = PT + QNT + K_P(I-Q)NT$. From Eq.(5), one can find that M is compact on any bounded subset of C_ω.

Theorem 1 Suppose following conditions hold.

(A_1) $f(t, x, y)$ is non-decreasing with respect to y.

(A_2) There exists a constant $R_0 > 0$ such that

$$f(t, R_0, R_0) < 0 \quad \text{and} \quad f(t, -R_0, -R_0) > 0, \quad \forall t \in \boldsymbol{R}.$$

Then $\deg\{I - M, \Omega, \theta\} \neq 0$, where $\Omega = \{T : T \in C_\omega, |T|_0 < R_0\}$.

Proof Consider operator equation as follows

$$LT = \lambda NT \ , \ \forall \lambda \in (0, 1]. \tag{6}$$

In what follows, we will prove that

$$LT \neq \lambda NT, \quad \forall \lambda \in (0, 1], \quad \forall T \in \partial\Omega. \tag{7}$$

Otherwise, if $T \in \partial\Omega$ satisfies

$$LT = \lambda NT \ , \ \text{for some } \lambda \in (0,1],$$

then

$$T'(t) = \lambda f(t, T(t), T(t-\delta)). \tag{8}$$

In view of $T \in \partial\Omega$, one can find that $|T|_0 = R_0$. Then there exists a $t^* \in [0,\omega]$ such that $|T(t^*)| = \max\limits_{t\in[0,\omega]} |T(t)| = |T|_0 = R_0$. Without loss of generality, we may assume that

$$T(t^*) = R_0, \tag{9}$$

then t^* is a maximum point of $T(t)$ on $\boldsymbol{R}$. So

$$T'(t^*) = 0. \tag{10}$$

Substituting Eqs.(9) and (10) into Eq.(8), we have

$$f(t^*, R_0, T(t^*-\delta)) = 0. \tag{11}$$

As $T(t^*-\delta) \le R_0$, it follows from condition (A_1) that

$$f(t^*, R_0, R_0) \ge f(t^*, R_0, T(t^*-\delta)) = 0,$$

which contradicts to condition (A_2). Therefore

$$|T|_0 < R_0.$$

Thus, Eq.(7) is satisfied.

One can easily verify that the equation $LT = \lambda NT$ is equivalent to the equation of the type

$$T = M(T,\lambda) := PT + QNT + \lambda K_P(I-Q)NT, \tag{12}$$

and $M(T,\lambda)$ is compact on $\overline{\Omega} \times [0,1]$. Further, we can check that

$$T \ne M(T,\lambda), \quad \forall T \in \partial\Omega, \quad \lambda \in [0,1].$$

In fact, from the above proof, it is easy to see that

$$T \ne M(T,\lambda), \quad \forall T \in \partial\Omega, \quad \lambda \in (0,1]. \tag{13}$$

Now we need to prove that $T \ne M(T,0), \forall T \in \partial\Omega$. If the statement is not true, then

$$M(T,0) = T \ , \ \text{for some } T \in \partial\Omega,$$

i.e.

$$T = QNT + PT,$$

which yields that

$$T \in \mathrm{Ker}L \ \text{ and } \ QNT = 0. \tag{14}$$

That is $QN(R_0) = 0$ or $QN(-R_0) = 0$. But by the definition of Q and N, we know that $QN(R_0) = \dfrac{1}{\omega}\displaystyle\int_0^\omega f(t, R_0, R_0)\mathrm{d}t < 0$ and $QN(-R_0) = \dfrac{1}{\omega}\displaystyle\int_0^\omega f(t, -R_0, -R_0)\mathrm{d}t > 0$, which contradicts to Eq.(14). Thus

$$T \ne M(T,\lambda), \ \text{for } \forall (T,\lambda) \in \partial\Omega \times [0,1],$$

and

$$QNT \ne 0, \quad \forall T \in \partial\Omega \cap \mathrm{Ker}L.$$

Therefore,

$$\begin{aligned}&\deg\{I - M, \Omega, \theta\}\\ =&\mathrm{d_B}\{-QN|_{\mathrm{Ker}L\cap\Omega}, \mathrm{Ker}L \cap \Omega, 0\}.\end{aligned}$$

As $R_0 QN(R_0) \ne 0$, it follows that

$$\deg\{I - M, \Omega, \theta\} \ne 0.$$

The proof is completed.

Theorem 2 Suppose following conditions hold.

(B_1) $f(t,x,y)$ is non-decreasing with respect to y.

(B_2) $\alpha(t), \beta(t)$ are the strict lower and strict upper ω-periodic solution of Eq.(4) with $\alpha(t) < \beta(t)$ respectively, $\forall t \in [0,\omega]$.

Then Eq.(4) exists at least one ω-periodic solution $T(t)$ such that

$$\alpha(t) < T(t) < \beta(t) \ , \ \forall t \in [0,\omega].$$

Proof Take R_0 large enough such that

$$f(t, \beta(t), \quad \beta(t-\delta)) - R_0 + \beta(t) < 0, \quad \forall t \in [0,\omega],$$

$$f(t, \alpha(t), \quad \alpha(t-\delta)) + R_0 + \alpha(t) > 0, \quad \forall t \in [0,\omega],$$

and

$$R_0 > \max\{-\min_{t\in[0,\omega]} \alpha(t), \max_{t\in[0,\omega]} \beta(t)\}.$$

Meanwhile, define auxiliary function $F_1(t,x,y)$ as follows

$$\begin{cases} f(t,x,\beta(t-\delta)), |x| \le R_0, \beta(t-\delta) \le y \le R_0, \\ \qquad f(t,x,y), |x| \le R_0, \alpha(t-\delta) < y < \beta(t-\delta), \\ f(t,x,\alpha(t-\delta)), |x| \le R_0, -R_0 \le y \le \alpha(t-\delta), \end{cases}$$

and define auxiliary function $F_2(t,x,y)$ as follows

$$\begin{cases} F_1(t,\beta(t),y) - x + \beta(t), \beta(t) \le x \le R_0, \\ \qquad F_1(t,x,y), \alpha(t) < x < \beta(t), \\ F_1(t,\alpha(t),y) - x + \alpha(t), -R_0 \le x \le \alpha(t). \end{cases}$$

Clearly, $F_2 \in C(\boldsymbol{R} \times [-R_0,R_0] \times [-R_0,R_0], \boldsymbol{R})$ and

$$F_2(t, R_0, R_0) = f(t, \beta(t), \beta(t-\delta)) - R_0 + \beta(t) < 0, \tag{15}$$

$$F_2(t,-R_0,-R_0)=f(t,\alpha(t),\alpha(t-\delta))+R_0+\alpha(t)>0. \tag{16}$$

From Eqs.(15) and (16), we know that the function $F_2(t,x,y)$ satisfies all the conditions of Theorem 1. Thus

$$\deg\{I-M_{F_2},\Omega,\theta\}\neq 0, \tag{17}$$

where $\Omega=\{T:T\in C_\omega,|T|_0<R_0\}$. Further, we let $\Delta=\{T:T\in C_\omega,\alpha(t)<T(t)<\beta(t)\}$. Obviously, $\Delta\subset\Omega$. In what follows, we shall prove that

$$LT\neq N_{F_2}T\ ,\ \forall T\in\Omega\setminus\overline{\Delta}.$$

If the statement is not true, there must be a $t_0\in[0,\omega]$ such that

$$\max_{t\in[0,\omega]}\{T(t)-\beta(t)\}=T(t_0)-\beta(t_0)\geq 0, \tag{18}$$

or

$$\min_{t\in[0,\omega]}\{T(t)-\alpha(t)\}=T(t_0)-\alpha(t_0)\leq 0. \tag{19}$$

If Eq.(18) holds, then

$$T'(t_0)=\beta'(t_0). \tag{20}$$

But, from Eq.(18) and the definition of $\beta(t)$, we have

$$\begin{aligned}&T'(t_0)-\beta'(t_0)\\=&F_2(t_0,T(t_0),T(t_0-\delta))-\beta'(t_0)\\=&f(t_0,\beta(t_0),\beta(t_0-\delta))-T(t_0)+\beta(t_0)-\beta'(t_0)\\<&0,\end{aligned}$$

which contradicts to Eq.(20). This contradiction implies that $T(t)<\beta(t),\forall t\in[0,\omega]$. Similarly, we have $T(t)>\alpha(t),\forall t\in[0,\omega]$. So

$$LT\neq N_{F_2}T\ ,\ \forall T\in\Omega\setminus\overline{\Delta}.$$

Thus,

$$\begin{aligned}&\deg\{I-M,\Delta,\theta\}\\=&\deg\{I-M_{F_2},\Delta,\theta\}\\=&\deg\{I-M_{F_2},\Omega,\theta\}-\deg\{I-M_{F_2},\Omega\setminus\overline{\Delta},\theta\}\\\neq&0.\end{aligned}$$

Thus Eq.(4) exists at least one ω-periodic solution $T(t)\in\Delta$.[26] That is

$$\alpha(t)<T(t)<\beta(t)\ ,\ \forall t\in[0,\omega].$$

This completes the proof.

4. The periodic solutions for model (3)

By applying Theorem 2, we study the existence and asymptotic estimates of periodic solutions of Eq.(3).

(1) We first consider the reduced problem of Eq.(3), i.e.

$$T'(t)=-\frac{1}{a}T(t). \tag{21}$$

By theorem 2, we let $f(t,x,y)=-\frac{1}{a}x,\alpha(t)\equiv -R_1<0,\beta(t)\equiv R_1>0$, where R_1 is a constant. Obviously, $f(t,x,y)$ is non-decreasing with respect to y, and $\alpha(t),\beta(t)$ are the strict lower and strict upper ω-periodic solutions of Eq.(21) with $\alpha(t)<\beta(t)$, respectively $\forall t\in[0,\omega]$.

By applying the Theorem 2, we know that the reduced problem of Eq.(3) exists at least one ω-periodic solution $T_0(t)$ satisfying

$$-R_1<T_0(t)<R_1\ ,\ \forall t\in[0,\omega].$$

(2) We now consider the Eq.(3)

$$T'(t)=-\frac{1}{a}T(t)+\frac{\gamma}{ab}T^2(t)T(t-\delta).$$

From conclusion (1), there is a constant R_2 satisfies $|T_0(t)|\leq R_2$. By Theorem 2, we let $f(t,x,y)=-\frac{1}{a}x+\frac{\gamma}{ab}x^2y,\alpha(t)=T_0(t)-l\gamma,\beta(t)=T_0(t)+l\gamma$, where l is a constant satisfies $l>\frac{R_2^3}{b}$. In the following, we assume sufficiently small $\gamma>0$. Then $f(t,x,y,\gamma)$ is non-decreasing with respect to y, and

$$\begin{aligned}&\alpha'(t)-f(t,\alpha(t),\alpha(t-\delta))\\=&\left(\frac{R_2^3}{ab}-\frac{l}{a}\right)\gamma+o(\gamma)<0\ ,\ 0<\gamma\ll 1,\end{aligned}$$

i.e.

$$\alpha'(t)<f(t,\alpha(t),\ \alpha(t-\delta)),\ \ \forall t\in[0,\omega].$$

Similarly,

$$\beta'(t)>f(t,\beta(t),\ \beta(t-\delta)),\ \ \forall t\in[0,\omega].$$

By applying Theorem 2, for sufficiently small $0<\gamma\ll 1$, Eq.(3) has at least one ω-periodic solution $T(t,\gamma)$ such that

$$|T(t,\gamma)-T_0(t)|\leq O(\gamma).$$

5. Conclusions

Atmospheric physics studies a very complicated natural phenomenon. Hence we need to reduce basic models for the sea–air oscillator and study the periodic behaviours of the temperature anomaly in the equatorial eastern Pacific. The interesting thing is that the asymptotic estimates of periodic solutions can be used to obtain the approximation of period. Thus, we can forecast when the ENSO appears.

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