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A field method for integrating the equations of motion of mechanico–electrical coupling dynamical systems*

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A field method for integrating the equations of motion for mechanico–electrical coupling dynamical systems is studied. Two examples in mechanico–electrical engineering are given to illustrate this method.

Keywords: field method, mechanico–electrical coupling system, Lagrange–Maxwell equation
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1. Introduction

It is well known that the Hamilton–Jacobi method is a very effective method in the conservative holonomic mechanics, but using this method to study nonconservative nonholonomic dynamical systems is very difficult and it has very strict constraints.[1,2] In 1984, a field method was provided by Vujanović as an important tool for integrating the equations of motion for nonconservative holonomic dynamical systems. Since then, the field method has become a useful tool for integrating dynamical equations.[3−7] On the other hand, the mechanico–electrical dynamical system which contains coupled mechanical movements and electromagnetic movements, leaves us many topics to study on solving the problems in it. Recently, the first integrals of dynamical systems have been obtained by using the methods of symmetries,[8−18] and many properties of the mechanico–electrical dynamical system such as the Noether theory, form invariance, Lie symmetry, and non-Noether symmetry are presented by using Lie group analysis method.[19−22] In this paper, we will introduce the field method to the system and present two examples to illustrate the applications of the method.

2. Equations of motion of a mechanico–electrical system

In a mechanico–electrical dynamical system a mechanical process couples with an electromagnetic process. The mechanical part consisting of N particles is described by n generalized coordinates $q_s (s = 1, \ldots, n)$, and the electrodynamic part corresponding to m electric circuits is described by m generalized electric quantities e_k $(k = 1, \ldots, m)$. Consider a mechanico–electrical dynamical system composed of m circuits consists of linear conductors and capacitors. For the circuit k, we denote the current by i_k, the electromotive potential by u_k, the capacitor charge by e_k (with $\dot{e}_k = i_k$), the resistance by R_k and the capacitance by C_k.

The Lagrangian for a mechanico–electrical system is

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q}) + W_{\mathrm{m}}(\boldsymbol{q}, \dot{\boldsymbol{e}}) - W_{\mathrm{e}}(\boldsymbol{q}, \boldsymbol{e}), \quad (1)$$

where T and V are, respectively, the kinetic energy and the potential energy. The electric field energy and the magnetic field energy of the mth circuit are,

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respectively, defined by

$$W_{\mathrm{e}}=\frac{1}{2}\frac{e_k^2}{C_k},\quad W_{\mathrm{m}}=\frac{1}{2}L_{kr}i_ki_r,\quad (k,r=1,\ldots,m), \tag{2}$$

where L_{kr} (with $k\neq r$) is the mutual inductance between the kth and the rth circuits and L_{kk} is the self inductance of circuit k, and the summation runs over repeated subscripts.

Motion of the system is governed by the Lagrange–Maxwell equations[23,24]

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}+\frac{\partial F}{\partial\dot{q}_s}=Q_s\quad(s=1,\ldots,n),\\&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{e}_k}-\frac{\partial L}{\partial e_k}+\frac{\partial F}{\partial\dot{e}_k}=u_k\quad(k=1,\ldots,m),\end{aligned} \tag{3}$$

where Q_s $(s=1,\ldots,n)$ are general non-potential forces. This is a system of $n+m$ ordinary differential equations with respect to the n generalized coordinates $q_s(s=1,2,\ldots,n)$ and the m generalized electric quantities $e_k(k=1,2,\ldots,m)$. The dissipative function F of the system is given by

$$F=F_{\mathrm{e}}(\dot{\boldsymbol{e}})+F_{\mathrm{m}}(\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{4}$$

where the electric dissipative function is

$$F_{\mathrm{e}}=\frac{1}{2}R_ki_k^2=\frac{1}{2}R_k\dot{e}_k^2\quad(k=1,\ldots,m), \tag{5}$$

F_{m} is the function for the viscous frictional damping force, and thus $-\partial F/\partial\dot{q}_s$, $-\partial F/\partial\dot{e}_k$ are dissipation forces.

Denoting the space coordinates and general electricity in uniform generalized coordinates $q_s(=1,\cdots,n,n+1,\cdots,n+m)$, where $q_s(s=1,\cdots,n)$ are the components of space coordinates, and $q_s(s=n+1,\cdots,n+m)$ are those of electricity. The Lagrange–Maxwell equation (3) of the mechanico–electrical system can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}+\frac{\partial F}{\partial\dot{q}_s}=Q_s\quad(s=1,\cdots,n,n+1,\cdots,n+m), \tag{6}$$

where $Q_s(s=1,\cdots,n)$ are non-potential generalized forces, $Q_s(s=n+1,\cdots,n+m)$ are general electromotive potentials.

When the mechanico–electrical system satisfies the conditions $Q_s-\partial F/\partial\dot{q}_s=0,(s=1,\cdots,n+m)$, the system (6) reduce to a Lagrange equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}=0\quad(s=1,\cdots,n,n+1,\cdots,n+m). \tag{7}$$

In such a case, the system corresponds to a Lagrange mechanico–electrical dynamical system.

The Lagrange–Maxwell equations (6) for the mechanico–electrical dynamical system can be written in a compact form as

$$\ddot{q}_s=f_s\left(q_k,\dot{q}_k,t\right)\quad(s,k,=1,\cdots,n,n+1,\cdots,n+m). \tag{8}$$

If $x_s=q_s,x_{n+m+s}=\dot{q}_s$, then Eq.(8) can be simplified to equations of order one

$$\dot{x}_s=x_{n+m+s},\dot{x}_{n+m+s}=f_s\left(x_s,x_{n+m+s},t\right)\quad(s=1,\cdots,n,n+1,\cdots,n+m). \tag{9}$$

3. Field method for integrating the equations of motion of a mechanico–electrical dynamical system

Now, we use the field method to integrate Eq.(9). Let a variable such as x_1 be a field function expressed in time t and other variables $x_2,\cdots,x_{2(n+m)}$, i.e.

$$x_1=u\left(t,x_a\right)\quad\left(a=2,\cdots,n+1,\cdots,2(n+m)\right). \tag{10}$$

Differentiating the equations with respect to t and using Eq.(9), we obtain

$$\frac{\partial u}{\partial t}+\sum_{a=2}^{n+m}\frac{\partial u}{\partial x_a}x_{n+m+a}+\sum_{b=1}^{n+m}\frac{\partial u}{\partial x_{n+m+b}}f\left(t,u,x_A\right)-x_{n+m+1}=0. \tag{11}$$

We call the linear partial differential equation (11) the basic partial differential equation.

Assume the complete solution of Eq.(11) is written in the form:

$$x_1=u(t,x_A,C_A)\quad(A=2,\cdots,2(n+m)). \tag{12}$$

Substituting Eq.(12) into Eq.(11) yields an identity. Letting the initial value of x_0 be

$$x_a(0)=x_{a0}\quad(a=1,\cdots,n,n+1,\cdots,n+m) \tag{13}$$

and substituting Eq.(13) into Eq.(12), where the constant C_1 is represented by x_{a0} and other constants $C_A\,(A=2,\cdots,2\,(n+m))$, one can obtain

$$x_1=u(t,x_A,x_{a0},C_A). \tag{14}$$

It is easy to prove that the solution of Eq.(9) with initial values can be determined by formula (14), and the following $(2(n+m)-1)$ algebraic equations hold for any constants C_A

$$\frac{\partial u}{\partial C_A}=0 \quad (A=2,\cdots,2(n+m)). \tag{15}$$

Actually, assuming $\det(\partial^2 u/\partial C_A \partial C_B) \neq 0$ $(A,B=2,\cdots,2(n+m))$ and differentiating Eq.(15) with respect to t, we have

$$\frac{\partial^2 u}{\partial C_A \partial t}+\sum_{a=2}^{n+m}\frac{\partial^2 u}{\partial C_A \partial x_a}\dot{x}_a + \sum_{b=1}^{n+m}\frac{\partial^2 u}{\partial C_A \partial x_{n+m+b}}\dot{x}_{n+m+b}=0. \tag{16}$$

Taking partial derivation of Eq.(11) with respect to C_A obtains

$$\frac{\partial^2 u}{\partial C_A \partial t}+\sum_{a=2}^{n+m}\frac{\partial^2 u}{\partial C_A \partial x_a}x_{n+m+a} + \sum_{b=1}^{n+m}\frac{\partial^2 u}{\partial C_A \partial x_{n+m+b}}f_b + \sum_{b=1}^{n+m}\frac{\partial u}{\partial x_{n+m+b}}\frac{\partial f_b}{\partial u}\frac{\partial u}{\partial C_A}=0. \tag{17}$$

Comparing Eq.(16) with Eq.(17) and making use of Eq.(15), we have

$$\dot{x}_a = x_{n+m+a}, \dot{x}_{n+m+b}=f_b\left(x_v, x_{n+m+v}, t\right), \quad (a=2,\cdots,n,n+1,\cdots,n+m; b=1,\cdots, n,n+1,\cdots,n+m). \tag{18}$$

These are the last $(2(n+m)-1)$ equations of mechanico–electrical dynamical system in Eq.(11). Differentiating Eq.(14) with respect to t, and using the basic differential equations (11), we verify that the first equation of the mechano–electrical system is satisfied, too.

We would show that using the field method to integrate the equations of motion has a common nature (they have no refrain like Hamilton–Jacobi equations) and nimbleness (could choose any field variable to substitute x^1); via finding the whole solutions of the basic differential equation (11), without the need of any integrating further, we can use Eqs.(14) and (15) to find the solutions of the mechanico–electrical system. The main difficulty of the field method is to find the whole solutions of the basic differential equations.

4. Examples

Example 1

An electric meter made up of a coil rotating in a uniform magnetic field (Fig.1). The coil is fixed to a spiral spring with a rigid coefficient k. The pointer of the meter has a deflection when one end of the coil winding is linked to a power supply E; Fig.2 shows the equivalent circuits of the electric meter. Let i_0 denote the current in the spiral coil and φ the deflection angle of the rotating coil. When the magnetic axis is perpendicular to the coil axis, φ is zero. For this electric meter, we denote the rotational inertia by J, the viscous friction damping coefficient by h, the self inductive coefficient by L_1, and the mutual induction between the coil and the spiral by $M(\varphi)$.

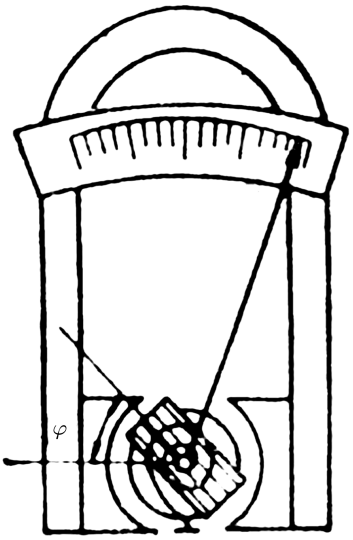


Fig.1. Electric meter.

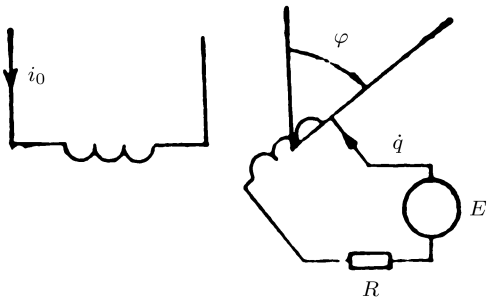


Fig.2. Equivalent circuits for the meter.

Choosing the generalized coordinates of the system as q, φ, its Lagrangian is

$$L=\frac{1}{2}\left[J\dot{\varphi}^2+L_1\dot{q}^2+2Mi_0q-k\varphi^2\right]. \tag{19}$$

The void work of the non-potential force and the dissipation force is

$$\delta A = (E - R\dot{q})\delta q - h\dot{\varphi}\delta\varphi. \tag{20}$$

By the of Lagrange–Maxwell equations, we have

$$\begin{aligned} &L_1\ddot{q} + R\dot{q} + i_0\frac{\mathrm{d}M}{\mathrm{d}\varphi} = E, \\ &J\ddot{\varphi} + h\dot{\varphi} + k\varphi - i_0\frac{\mathrm{d}M}{\mathrm{d}\varphi}\dot{q} = 0. \end{aligned} \tag{21}$$

When the deflexion angle is small, i.e. $M(\varphi) \approx \frac{1}{2}M_0\varphi^2$, we can write Eq.(21) in the form

$$\begin{aligned} &\ddot{q} + \frac{R}{L_1}\dot{q} + \frac{M_0 i_0}{L_1}\varphi = \frac{E}{L_1}, \\ &\ddot{\varphi} + \frac{h}{J}\dot{\varphi} + \frac{k}{J}\varphi - \frac{M_0 i_0}{J}\dot{q} = 0. \end{aligned} \tag{22}$$

If $a_1 = R/L_1$, $a_2 = M_0 i_0/L_1$, $a_3 = E/L_1, b_1 = h/J$, $b_2 = k/L_1$, $b_3 = M_0 i_0/J$, then Eq.(22) can be written as

$$\begin{aligned} &\ddot{q} = -a_1\dot{q} - a_2\varphi + a_3, \\ &\ddot{\varphi} = -b_1\dot{\varphi} - b_2\varphi + b_3\dot{q}. \end{aligned} \tag{23}$$

Now, we use the field method to obtain the solution of formula (23).

If $x_1 = q$, $x_2 = \varphi$, $x_3 = \dot{q}$, $x_4 = \dot{\varphi}$, then Eq.(23) is given by

$$\begin{aligned} &\dot{x}_1 = x_3, \dot{x}_2 = x_4, \\ &\dot{x}_3 = -a_1 x_3 - a_2 x_4 + a_3, \\ &\dot{x}_4 = -b_1 x_4 - b_2 x_2 + b_3 x_3. \end{aligned} \tag{24}$$

Firstly, for $x_4 = u(t, x_1, x_2, x_3)$, we have the basic partial differential equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}x_3 + \frac{\partial u}{\partial x_2}u \\ &+\frac{\partial u}{\partial x_3}(-a_1 x_3 - a_2 u + a_3) \\ &+b_1 u + b_2 x_2 - b_3 x_3 = 0. \end{aligned} \tag{25}$$

Secondly, suppose Eq.(25) has a solution of the form

$$x_4 = u = f_0(t) + f_1(t)x_1 + f_2 x_2 + f_3 x_3. \tag{26}$$

Substituting Eq.(26) into Eq.(25), and comparing the free item and the items with x_1, x_2, x_3, we have

$$\begin{aligned} &\dot{f}_1 + (f_2 - a_2 f_3 + b_1)f_1 = 0, \\ &\dot{f}_2 + (f_2 - a_2 f_3 + b_1)f_2 + b_2 = 0, \\ &\dot{f}_3 - a_1 f_3 + f_1 + (f_2 - a_2 f_3 + b_1)f_3 - b_3 = 0, \\ &\dot{f}_0 + (f_2 - a_2 f_3 + b_1)f_0 + a_3 f_3 = 0. \end{aligned} \tag{27}$$

If $f_2 - a_2 f_3 + b_1 = \omega$ is constant, integrating Eq.(27) leads to

$$\begin{aligned} f_1 &= C_1\mathrm{e}^{-\omega t}, \quad f_2 = C_2\mathrm{e}^{-\omega t} - \frac{b_2}{\omega}, \\ f_3 &= C_3\mathrm{e}^{-(\omega - a_1)t} + \frac{C_1}{a_1}\mathrm{e}^{-\omega t} - \frac{b_3}{\omega - a_1}, \\ f_0 &= C_0\mathrm{e}^{-\omega t} - \frac{a_3 C_3}{a_1}\mathrm{e}^{-(\omega - a_1)t} \\ &\quad -\frac{a_3 C_1}{a_1}t\mathrm{e}^{-\omega t} - \frac{a_3 b_3}{w(w - a_1)}, \end{aligned} \tag{28}$$

with ω being determined by

$$\begin{aligned} &\omega^3 - (a_1 + b_1)\omega^2 \\ &+(b_2 + a_2 b_3 - a_1 b_1)\omega - a_1 b_2 = 0. \end{aligned} \tag{29}$$

So we have

$$\begin{aligned} x_4 &= u \\ &= C_0\mathrm{e}^{-\omega t} - \frac{a_3 C_3}{a_1}\mathrm{e}^{-(\omega - a_1)t} - \frac{a_3 C_1}{a_1}t\mathrm{e}^{-\omega t} \\ &\quad -\frac{a_3 b_3}{w(w - a_1)} + C_1\mathrm{e}^{-\omega t}x_1 + \left(c_2\mathrm{e}^{-\omega t} - \frac{b_2}{\omega}\right)x_2 \\ &\quad + \left(C_3\mathrm{e}^{-(\omega - a_1)t} + \frac{C_1}{a_1}\mathrm{e}^{-\omega t} + \frac{b_3}{w - a_1}\right)x_3, \end{aligned} \tag{30}$$

for the initial conditions $x_\alpha(0) = x_{\alpha 0} \quad (\alpha = 1, 2, 3, 4)$, then we obtain

$$\begin{aligned} C_0 &= x_{40} + \frac{a_3 b_3}{w(w - a_1)} - C_1 x_{10} - \left(c_2 - \frac{b_2}{\omega}\right)x_{20} \\ &\quad - \left(c_3 + \frac{C_1}{a_1} + \frac{b_3}{w - a_1}\right)x_{30}. \end{aligned} \tag{31}$$

Substituting Eq.(31) into Eq.(30), we have

$$\begin{aligned} x_4 &= u \\ &= \left[x_{40} + \frac{a_3 b_3}{w(w - a_1)} - C_1 x_{10} - \left(\frac{a_2}{a_1}c_1 - \frac{b_2}{\omega}\right)x_{20}\right. \\ &\quad \left. - \left(\frac{C_1}{a_1} + \frac{b_3}{w - a_1}\right)x_{30}\right]\mathrm{e}^{-\omega t} - \frac{a_3 C_3}{a_1}\mathrm{e}^{-(\omega - a_1)t} \\ &\quad -\frac{a_3 C_1}{a_1}t\mathrm{e}^{-\omega t} - \frac{a_3 b_3}{w(w - a_1)} + C_1\mathrm{e}^{-\omega t}x_1 \\ &\quad + \left(c_2\mathrm{e}^{-\omega t} - \frac{b_2}{\omega}\right)x_2 \\ &\quad + \left(C_3\mathrm{e}^{-(\omega - a_1)t} + \frac{C_1}{a_1}\mathrm{e}^{-\omega t} + \frac{b_3}{w - a_1}\right)x_3. \end{aligned} \tag{32}$$

Thirdly, since $\partial u/\partial C_\alpha = 0, (\alpha = 1, 2, 3)$, one has

$$\begin{aligned} x_1 &= x_{10} + \frac{x_{30}}{a_1} + \frac{a_3}{a_1}t - \frac{1}{a_1}x_{30}\mathrm{e}^{-\omega t} \\ &\quad -\frac{a_3}{a_1^2}\mathrm{e}^{\omega t} - \frac{a_3}{a_1^2}, \\ x_2 &= x_{20}, \\ x_3 &= x_{30}\mathrm{e}^{-\omega t} + \frac{a_3}{a_1}\mathrm{e}^{\omega t} + \frac{a_3}{a_1}. \end{aligned} \tag{33}$$

If $C_1 = C_2 = C_3 = 0$, and substituting Eq.(33) into Eq.(32), we have

$$\begin{aligned} x_4 = & \left[x_{40} + \frac{a_3 b_3}{w(w-a_1)} + \frac{b_2}{\omega}x_{20} - \frac{b_3}{w-a_1}x_{30}\right] \mathrm{e}^{-\omega t} \\ & - \frac{a_3 b_3}{w(w-a_1)} - \frac{b_2}{\omega}x_{20} \\ & + \frac{b_3}{w-a_1}\left(x_{30}\mathrm{e}^{-\omega t} + \frac{a_3}{a_1}\mathrm{e}^{\omega t} + \frac{a_3}{a_1}\right). \end{aligned} \tag{34}$$

Finally, writing Eqs.(33) and (34) in generalized coordinates, and considering that the a_1, a_2, a_3, b_1, b_2, b_3 are constants, we obtain the solution of Eq.(22), in which there are three independent constants.

Example 2

If the term of $L_1 \mathrm{d}^2 q/\mathrm{d}t^2$ in Eq.(21) can be omitted, then it can be written in a differential equation of second order

$$\ddot{\varphi} + \left(\frac{h}{J} + \frac{i_0^2 M_0^2}{JR}\right)\dot{\varphi} + \frac{k}{J}\varphi = \frac{i_0 M_0 E}{JR}. \tag{35}$$

If $a_1 = \dfrac{h}{J} + \dfrac{i_0^2 M_0^2}{JR}$, $a_2 = \dfrac{k}{J}$, $a_3 = \dfrac{i_0 M_0 E}{JR}$, then Eq.(35) can be written as

$$\ddot{\varphi} = -a_1\dot{\varphi} - a_2\varphi + a_3. \tag{36}$$

When $x_1 = \varphi$, $x_2 = \dot{\varphi}$, Eq.(36) can be expressed as

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a_1 x_2 - a_2 x_1 + a_3. \tag{37}$$

Letting $x_2 = u(t, x_1)$, we have the basic differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}u + a_1 u + a_2 x_1 - a_3 = 0. \tag{38}$$

If Eq.(38) has a solution of the form

$$x_2 = u = f_0(t) + f_1(t)x_1, \tag{39}$$

then substituting Eq.(39) into Eq.(38), and comparing the free terms and the terms including x_1, we obtain

$$\begin{aligned} & \dot{f}_0 + f_1 f_0 + a_1 f_0 - a_3 \\ = \ & \dot{f}_0 + (f_1 + a_1) f_0 - a_3 = 0, \end{aligned} \tag{40}$$

$$\begin{aligned} & \dot{f}_1 + f_1^2 + a_1 f_1 + a_2 \\ = \ & \dot{f}_1 + (f_1 + a_1) f_1 + a_2 = 0. \end{aligned} \tag{41}$$

Solving Eq.(39) leads to

$$f_1 = \frac{a_1 c_1 \mathrm{e}^{-a_1 t}}{1 - c_1 \mathrm{e}^{-a_1 t}} - \frac{a_1}{2} + \frac{\sqrt{a_1^2 - 4a_2}}{2} \tag{42}$$

or

$$f_1 = \frac{a_1 c_1 \mathrm{e}^{-a_1 t}}{1 - c_1 \mathrm{e}^{-a_1 t}} - \frac{a_1}{2} - \frac{\sqrt{a_1^2 - 4a_2}}{2}. \tag{43}$$

Substituting Eq.(42) into Eq.(40), we have

$$\begin{aligned} & \dot{f}_0 + \left(\frac{a_1 c_1 \mathrm{e}^{-a_1 t}}{1 - c_1 \mathrm{e}^{-a_1 t}} - \frac{a_1}{2} + \frac{\sqrt{a_1^2 - 4a_2}}{2}\right) f_0 \\ = \ & a_3. \end{aligned} \tag{44}$$

Equation (44) has the following solution

$$\begin{aligned} f_0 = & \frac{a_3}{(1 - c_1 \mathrm{e}^{-a_1 t})\left(-\dfrac{a_1}{2} + \dfrac{\sqrt{a_1^2 - 4a_2}}{2}\right)} \\ & - \frac{c_1 a_3 \mathrm{e}^{-a_1 t}}{(1 - c_1 \mathrm{e}^{-a_1 t})\left(-\dfrac{3a_1}{2} + \dfrac{\sqrt{a_1^2 - 4a_2}}{2}\right)} \\ & + c_0 \frac{\mathrm{e}^{\frac{a_1}{2}t - \frac{\sqrt{a_1^2 - 4a_2}}{2}t}}{(1 - c_1 \mathrm{e}^{-a_1 t})}. \end{aligned} \tag{45}$$

Substituting Eq.(43) into Eq.(40) yields

$$\begin{aligned} & \dot{f}_0 + \left(\frac{a_1 c_1 \mathrm{e}^{-a_1 t}}{1 - c_1 \mathrm{e}^{-a_1 t}} - \frac{a_1}{2} - \frac{\sqrt{a_1^2 - 4a_2}}{2}\right) f_0 \\ = \ & a_3. \end{aligned} \tag{46}$$

Integrating Eq.(46), we have

$$\begin{aligned} f_0 = & \frac{a_3}{(1 - c_1 \mathrm{e}^{-a_1 t})\left(-\dfrac{a_1}{2} - \dfrac{\sqrt{a_1^2 - 4a_2}}{2}\right)} \\ & - \frac{a_3 \mathrm{e}^{-a_1 t}}{(1 - c_1 \mathrm{e}^{-a_1 t})\left(-\dfrac{3a_1}{2} - \dfrac{\sqrt{a_1^2 - 4a_2}}{2}\right)} \\ & + \frac{c_0}{(1 - c_1 \mathrm{e}^{-a_1 t})} \mathrm{e}^{\frac{a_1}{2}t + \frac{\sqrt{a_1^2 - 4a_2}}{2}t}. \end{aligned} \tag{47}$$

Substituting Eqs.(42) and (45) into Eq.(39) leads to

$$\begin{aligned} x_2 = u = & \frac{a_3}{(1 - c_1 \mathrm{e}^{-a_1 t})\left(-\dfrac{a_1}{2} + \dfrac{\sqrt{a_1^2 - 4a_2}}{2}\right)} \\ & - \frac{a_3 \mathrm{e}^{-a_1 t}}{(1 - c_1 \mathrm{e}^{-a_1 t})\left(-\dfrac{3a_1}{2} + \dfrac{\sqrt{a_1^2 - 4a_2}}{2}\right)} \\ & + \frac{c_0}{(1 - c_1 \mathrm{e}^{-a_1 t})} \mathrm{e}^{\frac{a_1}{2}t - \frac{\sqrt{a_1^2 - 4a_2}}{2}t} \\ & + \left(\frac{a_1 c_1 \mathrm{e}^{-a_1 t}}{1 - c_1 \mathrm{e}^{-a_1 t}} - \frac{a_1}{2} + \frac{\sqrt{a_1^2 - 4a_2}}{2}\right) x_1, \end{aligned} \tag{48}$$

or

$$\begin{aligned} x_2 &= u \\ &= \frac{a_3}{(1-c_1\mathrm{e}^{-a_1t})\left(-\frac{a_1}{2}-\frac{\sqrt{a_1^2-4a_2}}{2}\right)} \\ &\quad -\frac{a_3\mathrm{e}^{-a_1t}}{(1-c_1\mathrm{e}^{-a_1t})\left(-\frac{3a_1}{2}-\frac{\sqrt{a_1^2-4a_2}}{2}\right)} \\ &\quad +\frac{c_0}{(1-c_1\mathrm{e}^{-a_1t})}\mathrm{e}^{\frac{a_1}{2}t+\frac{\sqrt{a_1^2-4a_2}}{2}t} \\ &\quad +\left(\frac{a_1c_1\mathrm{e}^{-a_1t}}{1-c_1\mathrm{e}^{-a_1t}}-\frac{a_1}{2}-\frac{\sqrt{a_1^2-4a_2}}{2}\right)x_1. \end{aligned} \tag{49}$$

For the initial conditions $x_\alpha(0)=x_{\alpha0}\quad(\alpha=1,2)$, from Eq.(48) we can derive

$$\begin{aligned} c_0 &= (1-c_1)x_{20}-\frac{a_3}{\left(-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)} \\ &\quad +\frac{c_1a_3}{\left(-\frac{3a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)} \\ &\quad -(1-c_1)\left(\frac{a_1c_1}{1-c_1}-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)x_{10}. \end{aligned} \tag{50}$$

Substituting Eq.(50) into Eq.(48), we have

$$\begin{aligned} x_2 = u = {} & \frac{a_3}{(1-c_1\mathrm{e}^{-a_1t})\left(-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)}-\frac{a_3\mathrm{e}^{-a_1t}}{(1-c_1\mathrm{e}^{-a_1t})\left(-\frac{3a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)} \\ & +\left[(1-c_1)x_{20}-\frac{a_3}{\left(-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)}+\frac{c_1a_3}{\left(-\frac{3a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)}\right. \\ & \left.-(1-c_1)\left(\frac{a_1c_1}{1-c_1}-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)x_{10}\right]\frac{\mathrm{e}^{\frac{a_1}{2}t-\frac{\sqrt{a_1^2-4a_2}}{2}t}}{(1-c_1\mathrm{e}^{-a_1t})} \\ & +\left(\frac{a_1c_1\mathrm{e}^{-a_1t}}{1-c_1\mathrm{e}^{-a_1t}}-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)x_1. \end{aligned} \tag{51}$$

Then according to $\partial u/\partial C_\alpha=0,(\alpha=1,2,3)$, one has

$$\begin{aligned} \frac{\partial u}{\partial c_1} = {} & \frac{a_3\mathrm{e}^{-a_1t}}{(1-c_1\mathrm{e}^{-a_1t})^2\left(-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)}-\frac{a_3\mathrm{e}^{-2a_1t}}{(1-c_1\mathrm{e}^{-a_1t})^2\left(-\frac{3a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)} \\ & \times\left[-x_{20}+\frac{a_3}{\left(-\frac{3a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)}-\frac{3a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)x_{10}\right]\frac{\mathrm{e}^{\frac{a_1}{2}t-\frac{\sqrt{a_1^2-4a_2}}{2}t}}{(1-c_1\mathrm{e}^{-a_1t})} \\ & +\left[(1-c_1)x_{20}-\frac{a_3}{\left(-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)}+\frac{c_1a_3}{\left(-\frac{3a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)}\right. \\ & \left.-(1-c_1)\left(\frac{a_1c_1}{1-c_1}-\frac{a_1}{2}+\frac{\sqrt{a_1^2-4a_2}}{2}\right)x_{10}\right]\frac{\mathrm{e}^{-a_1t}}{(1-c_1\mathrm{e}^{-a_1t})^2}\mathrm{e}^{\frac{a_1}{2}t-\frac{\sqrt{a_1^2-4a_2}}{2}t} \\ & +\frac{a_1\mathrm{e}^{-a_1t}}{(1-c_1\mathrm{e}^{-a_1t})^2}x_1 \\ = {} & 0. \end{aligned} \tag{52}$$

Solving c_1 from Eq.(52), and then substituting it into Eq.(51), we have the integration of x_2.

Using the familiar steps, then we, from Eq.(49), derive the other solutions for the mechanico–electrical dynamical system.

5. Conclusion

The main contribution of this paper is that it extends the field integral method to the mechanico–electrical coupling dynamical system. Firstly, these studies suggest a new method to solve mechanico–electrical coupling dynamical system. Secondly, they make a basis to apply the field method to the study on problems in other fields.

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