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The He–McKellar–Wilkens effect for spin-1 particles on non-commutative space*

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By using star product method, the He–McKellar–Wilkens (HMW) effect for spin-one neutral particle on non-commutative (NC) space is studied. After solving the Kemmer-like equations on NC space, we obtain the topological HMW phase on NC space where the additional terms related to the space–space non-commutativity are given explicitly.

Keywords: non-commutative quantum mechanics, non-commutative space, HMW effect
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1. Introduction

The study of physics effects on non-commutative (NC) space has attracted much attention in recent years. Because the effects of the space non-commutativity may become significant at the very high energy scale. Besides the quantum field theory, there are many papers devoted to the study of various aspects of quantum mechanics on NC space.[1−9] For example, the Aharonov–Bohm (AB) phase on NC space and NC phase space has been studied in Refs.[1–3]. The Aharonov–Casher (AC) phase for the spin-1/2 and spin-1 particles on NC space and NC phase space has been studied in Refs.[4–7]. The He–McKellar–Wilkens (HMW) effect on commutative space was first discussed in 1993 by He and Meckellar[10] and a year later, independently by Wilkens.[11] The HMW effect corresponds to a topological phase related to a neutral spin-1/2 particle with non-zero electric dipole moment moving in the magnetic field, and in 1998, Dowling, Williams and Franson pointed out that the HMW effect can be partially tested by using metastable hydrogen atoms.[12] HMW phase for a spin-1/2 particle on NC space and NC phase space has been studied in Ref.[8]. But the HMW effect for neutral particle of spin-1 on NC space is still missing, so the purpose of this paper is to find HMW phase on NC space.

We briefly review some basic properties of non-commutative space. On NC space the coordinate $\hat{x}_\nu$ and momentum $\hat{p}_\nu$ operators satisfy the following commutation relations (taking $\hbar = c = 1$)

$$[\hat{x}_\nu, \hat{x}_\lambda] = \mathrm{i}\Theta_{\nu\lambda}, \quad [\hat{p}_\nu, \hat{p}_\lambda] = 0, \quad [\hat{x}_\nu, \hat{p}_\lambda] = \mathrm{i}\delta_{\nu\lambda}, \quad (1)$$

where $\Theta_{\nu\lambda}$ is an element of an antisymmetric matrix, and is related to the energy scale. It represents the non-commutativity of the NC space.

Non-commutative quantum mechanics can be constructed from commutative quantum mechanics by replacing the usual product between Hamiltonian and wave function with a star product. The star product between the two functions is defined as

$$\begin{aligned}&(f * g)(x)\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta_{\nu\lambda}\partial_{x_\nu}\partial_{x_\lambda}} f(x_\nu)g(x_\lambda)\\ &= f(x)g(x) + \frac{\mathrm{i}}{2}\Theta_{\nu\lambda}\partial_\nu f\partial_\lambda g\big|_{x_\nu = x_\lambda} + \mathcal{O}(\Theta^2)\,, \quad (2)\end{aligned}$$

where $f(x)$ and $g(x)$ are two arbitrary functions.

2. The Kemmer equation and HMW effect

Like Aharonov–Bohm and Aharonov–Casher topological effects, the HMW effect can also be studied in $2+1$ dimensions. The ordinary configuration

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of HWM effect is a neutral particle with nonzero electric dipole moment $\mu_{\rm e}$ that moves in a pure magnetic field produced by an infinitely long filament which is uniformly charged with magnetic charge (monopoles), and the filament is perpendicular to the $x-y$ plane. We use the conventions $g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$. The Lagrange of a spin-1 neutral particle with electric dipole $\mu_{\rm e}$ moving in the electromagnetic field is given by

$$L=\bar{\phi}\left(\mathrm{i}\beta^{\nu}\partial_{\nu}-\frac{1}{2}\mu_{\rm e}S_{\lambda\rho}\tilde{F}^{\lambda\rho}-m\right)\phi, \tag{3}$$

where $\tilde{F}_{\mu\nu}=\dfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is the $(3+1)$-dimensional dual of the electromagnetic field tensor; the 10×10 matrices β^{ν} are generalization of the 4×4 Dirac gamma matrices, and the definition of these matrices can be found in Refs.[13–16]. They satisfy the following relation

$$\beta_{\nu}\beta_{\lambda}\beta_{\rho}+\beta_{\rho}\beta_{\lambda}\beta_{\nu}=\beta_{\nu}g_{\lambda\rho}+\beta_{\rho}g_{\nu\lambda}. \tag{4}$$

Other algebraic properties of the Kemmer β-matrices were given in Ref.[14]. $S_{\lambda\rho}$ is the Dirac $\sigma_{\lambda\rho}$ like spin operator, which is defined as

$$S_{\lambda\rho}=\frac{1}{2}(\beta_{\lambda}\beta_{\rho}-\beta_{\rho}\beta_{\lambda}). \tag{5}$$

Thus in an electromagnetic field, the Kemmer equation of motion of a spin-1 neutral particle with electric dipole moment $\mu_{\rm e}$ is

$$\left(\mathrm{i}\beta^{\nu}\partial_{\nu}-\frac{1}{2}\mu_{\rm e}S_{\lambda\rho}\tilde{F}^{\lambda\rho}-m\right)\phi=0. \tag{6}$$

We write the solution of the above equation in the following form

$$\phi=\mathrm{e}^{-\mathrm{i}\xi_3\int^r\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{r}}\ \phi_0, \tag{7}$$

where $\boldsymbol{a}$ is a vector which will be determined soon; ϕ_0 is a solution of the free Kemmer equation; and the spin-1 pseudo-vector operator ξ_{ν} is defined as

$$\xi_{\nu}=\frac{1}{2}\mathrm{i}\varepsilon_{\nu\lambda\rho\sigma}\beta^{\lambda}\beta^{\rho}\beta^{\sigma}, \tag{8}$$

where $\varepsilon_{\nu\lambda\rho\sigma}$ is the Levi–Civita symbol in four dimensions. To find the explicit form of the vector $\boldsymbol{a}$ in Eq.(7), first we write the free Kemmer equation for ϕ_0 in terms of ϕ

$$(\mathrm{i}\beta^{\nu}\partial_{\nu}-m)\ \mathrm{e}^{\mathrm{i}\xi_3\int^r\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{r}}\ \phi=0. \tag{9}$$

Then the equivalence of Eqs.(6) and (9) can be obtained by imposing the following two conditions:

$$\mathrm{e}^{-\mathrm{i}\xi_3\int^r\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{r}}\ \beta^{\nu}\ \mathrm{e}^{\mathrm{i}\xi_3\int^r\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{r}}=\beta^{\nu}, \tag{10}$$

and

$$\beta^{\nu}\xi_3 a_{\nu}\phi=\frac{1}{2}\mu_{\rm e}S_{\lambda\rho}\tilde{F}^{\lambda\rho}\phi=\mu_{\rm e}S_{0l}\tilde{F}^{0l}\phi. \tag{11}$$

By comparing Eq.(10) with the Baker–Hausdorf formula

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\lambda\xi_3}\ \beta^{\nu}\ \mathrm{e}^{\mathrm{i}\lambda\xi_3}\\ &=\beta^{\nu}+\wp(-\mathrm{i}\lambda)[\xi_3,\beta^{\nu}]\\ &\quad+\frac{1}{2!}\wp(-\mathrm{i}\lambda)^2[\xi_3,[\xi_3,\beta^{\nu}]]\cdots,\end{aligned} \tag{12}$$

one obtains $[\xi_3,\beta^{\nu}]=0$, where $\wp$ stands for path ordering of the integral in the phase. If $\nu\neq 3$ this commutation relation is automatically satisfied; however, for $\nu=3$, by using Eqs.(4) and (8), one finds that the commutator does not vanish. Therefore, in order to fulfil the first condition the particle should be restricted to move in $x-y$ plane, that is, $B_z=0$. In particular, $\partial_3\phi=0$ and $a_3=0$. From the second condition in Eq.(11), by using Eqs.(4), (5) and (8), one obtains

$$a_l=2\mu_{\rm e}\varepsilon_{lk}B_k,\quad l,k=1,2. \tag{13}$$

Thus the HMW phase for a neutral spin-1 particle moving in a $2+1$ space time under the influence of a pure magnetic field produced by an infinitely long filament which is uniformly charged with magnetic monopoles, is given by

$$\begin{aligned}\phi_{\rm HMW}&=\xi_3\oint\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{r}=2\mu_{\rm e}\xi_3\varepsilon^{lk}\oint B_l\mathrm{d}x_k\\ &=2\mu_{\rm e}\xi_3\oint(-B_1\mathrm{d}x_2+B_2\mathrm{d}x_1).\end{aligned} \tag{14}$$

The above equation can also be written as

$$\begin{aligned}\phi_{\rm HMW}&=\xi_3\oint\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{r}=\xi_3\int_S(\nabla\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S}\\ &=2\mu_{\rm e}\xi_3\int_S(\nabla\cdot\boldsymbol{B})\mathrm{d}S=2\mu_{\rm e}\xi_3\lambda_{\rm m},\end{aligned} \tag{15}$$

where $\lambda_{\rm m}$ is the magnetic charge density of the filament. This spin-1 HMW phase is also purely a quantum mechanical effect and has no classical interpretation. One may note that the HMW phase for spin-1 particles is exactly the same as that for spin-1/2 particles, except that the spin and spinor have changed. The factor of two shows that the phase is twice that accumulated by a spin-1/2 particle with the same electric dipole moment coupling constant, in the same magnetic field.

3. The HMW effect on NC space

In this section we will study the HMW phase by using star product method on non-commutative space. From the discussion of the introduction part, the equation of motion for a neutral spin-1 particle with a nonzero electric dipole moment moving in an electromagnetic field, on NC space, is given by

$$\left(\mathrm{i}\beta^{\nu}\partial_{\nu}-\frac{1}{2}\mu_{\mathrm{e}}S_{\lambda\rho}\tilde{F}^{\lambda\rho}-m\right)*\phi=0. \tag{16}$$

Considering Eq.(7), we assume that on NC space the solution of the above equation can be set to

$$\phi=\mathrm{e}^{-\mathrm{i}\xi_3 f}\phi_0, \tag{17}$$

where f is the time-independent scalar phase which can be determined from the following calculation. By using Eq.(12) and $[\beta_\mu,\xi_3]=0$, Eq.(16) can be rewritten as

$$\begin{aligned}&\mathrm{i}\beta_{\mu}\partial^{\mu}(\xi_3 f)\left(\mathrm{e}^{-\mathrm{i}\xi_3 f}\phi_0\right)\\&+\frac{\mu_{\mathrm{e}}}{2}S_{\alpha\beta}\tilde{F}^{\alpha\beta}*\left(\mathrm{e}^{-\mathrm{i}\xi_3 f}\phi_0\right)=0.\end{aligned} \tag{18}$$

Define k_j as

$$\partial_j\phi_0=(\mathrm{i}k_j)\phi_0. \tag{19}$$

Considering that a neutral particle moves in a pure magnetic field and expanding the second term in Eq.(18) up to the first order of θ, one obtains

$$\begin{aligned}&\Big[\mathrm{i}\beta^{\mu}\partial_{\mu}(\xi_3 f)-\mu_{\mathrm{e}}S_{0l}\tilde{F}^{0l}-\frac{\mathrm{i}}{2}\mu_{\mathrm{e}}\theta_{ij}\partial_i(S_{0l}\tilde{F}^{0l})\partial_j(\xi_3 f)\\&-\frac{\mathrm{i}}{2}\mu_{\mathrm{e}}\theta_{ij}\partial_i(S_{0l}\tilde{F}^{0l})\partial_j(\mathrm{i}k^j)\Big]\exp[-\mathrm{i}\xi_3 f]\phi_0=0.\end{aligned} \tag{20}$$

We expand f up to the first order in θ

$$f=f^{(0)}+\theta f^{(1)}+\cdots, \tag{21}$$

and notice that in a two-dimensional NC space, $\theta^{ij}=\theta\varepsilon^{ij}$, then we obtain the following equations.

$$\left[\beta^{\mu}\partial_{\mu}\Big(\mathrm{i}\xi_3 f^{(0)}-\mu_{\mathrm{e}}(S_{0l}\tilde{F}^{0l})\Big)\right]\phi_0=0, \tag{22}$$

$$\begin{aligned}&\Big[\beta^{\mu}\partial_{\mu}\Big(\xi_3 f^{(1)}-\frac{\mathrm{i}}{2}\mu_{\mathrm{e}}\varepsilon_{ij}\partial_i\left(S_{0l}\tilde{F}^{0l}\right)\partial_j\left(\xi_3 f^{(0)}\right)\\&-\frac{\mathrm{i}}{2}\mu_{\mathrm{e}}\varepsilon_{ij}\partial_i\left(S_{0l}\tilde{F}^{0l}\right)\partial_j(\mathrm{i}k^j)\Big]\phi_0=0.\end{aligned} \tag{23}$$

After a straightforward calculation, from Eq.(22), we obtain

$$\bigtriangledown f^{(0)}=2\mu_{\mathrm{e}}(\boldsymbol{B}\times\hat{k}). \tag{24}$$

Then, the zero-order approximation of the phase is given by

$$\begin{aligned}\phi_0&=\xi_3\oint\boldsymbol{\nabla}f^{(0)}\cdot\mathrm{d}\boldsymbol{r}\\&=2\mu_{\mathrm{e}}\xi_3\oint(\boldsymbol{B}\times\hat{k})\cdot\mathrm{d}\boldsymbol{r}\\&=2\mu_{\mathrm{e}}\xi_3\oint(-B_1\mathrm{d}x_2+B_2\mathrm{d}x_1),\end{aligned} \tag{25}$$

which is just the HMW phase in the commuting space (refer to Eq.(14)). Now consider the first-order correction. Substituting Eq.(25) in Eq.(23), and after a long but straightforward calculation (the MATHEMATICA software is used), the following correction to ϕ_0 is obtained for a neutral particle with nonzero electric dipole moment μ_{e} with spin-1 ($s_3=1,0,-1$),

$$\begin{aligned}\triangle\phi_\theta&=\theta\xi_3\oint\boldsymbol{\nabla}f^{(1)}\cdot\mathrm{d}\boldsymbol{r}\\&=\theta\mu_{\mathrm{e}}\xi_3\varepsilon^{ij}\oint k_j(\partial_i B_2\mathrm{d}x_1-\partial_i B_1\mathrm{d}x_2)\\&\quad+2\theta s_3\mu_{\mathrm{e}}^2\xi_3\varepsilon^{ij}\oint\left(\hat{k}\times\boldsymbol{B}\right)_j\\&\quad\times[(\partial_i B_2)\mathrm{d}x_1-(\partial_i B_1)\mathrm{d}x_2].\end{aligned} \tag{26}$$

The solution (17) then reads

$$\phi=\phi_0\mathrm{e}^{\mathrm{i}(\phi_0+\triangle\phi_\theta)}, \tag{27}$$

and the total phase shift for the HMW effect is given by

$$\phi=\phi_0+\triangle\phi_\theta. \tag{28}$$

The $\triangle\phi$ is the correction of the space-space non-commutativity for HMW effect. If $\mu_{\mathrm{e}}\to\mu_{\mathrm{m}},\boldsymbol{B}\to-\boldsymbol{E}$, then the HMW effect will coincide with the non-commutative Aharonov–Casher (AC) effect.

4. Conclusion

In this paper we have studied the topological HMW phase on NC space. First the Kemmer equation of spin-1 neutral particle in an electromagnetic field on NC space can be obtained by replacing the usual product in the Kemmer equation on commuting space with a star product. Then after calculation of the star production, we obtain the solution of the NC Kemmer equation. Finally the NC topological phase is obtained explicitly up to the first order of NC parameter θ. The θ-dependent term ($\triangle\phi_\theta$) is divided into two parts (see Eq.(26)), the first part is a velocity-dependent correction, which does not have the topological properties of the commutative HMW effect and could modify the phase shift. The second part is a correction to the vortex, which does not contribute to the line spectrum. The method employed in this paper can also be applied to other quantum mechanics problems on NC space, which will be reported in our further studies.

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