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Dynamical evolution processes of traffic flow and travel cost in urban transportation networks*

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Considering such a fact that travellers dynamically adjust their routes and the resultant link traffic flows in a network evolve over time, this paper proposes a dynamical evolutionary model of the traffic assignment problem with endogenous origin–destination (OD) demands. The model's stability is analysed and the resultant user equilibrium (UE) state is shown to be stable under certain conditions. Numerical results in a grid network indicate that the model can generate convergent flow patterns and finally terminates at the UE state. Impacts by the parameters associated with OD demand function and link cost function are also investigated.

Keywords: traffic assignment, user equilibrium, dynamical system, stability
PACC: 0547, 0580

1. Introduction

The fundamental aim of traffic assignment is to find link or path flow patterns, given the origin-destination (OD) trip rates, network topology, and link performance functions. The result of the assignment is an estimate of traffic volume on each link in the network, and the associated measures of system performance. To do so, we have to make some assumptions which describe how commuters make their travel decisions. A well received assumption in the literature is that all travellers have perfect information about traffic conditions over the entire network and make consistently correct decisions in route choice, leading to the deterministic assignment models. The user equilibrium (UE) model assumes that all travellers minimize their own individual travel costs on transportation networks, consequently all routes actually used for a specific OD pair have equal travel cost which is less than that on any of the unused paths.

Most existing studies centre on the final UE state while ignoring the evolution process of reaching the state.[1] In fact, travellers may adjust their routes every day and the resultant link flows evolve over time. Moreover, with the development of Intelligent Transportation Systems (ITS), travellers can more frequently update their routes under the help of traffic information provided by ITS. Obviously, the dynamical evolution process of traffic assignment is an important topic for better understanding various UE models, from point of either theoretical interest or practical purpose. Hence, the studies about the evolution of UE state gradually increase.[2−19]

In this paper, we propose a new dynamical evolution model of traffic assignment problem with endogenous OD demands. The model reasonably formulates travellers' behaviour on adjusting demands and routes. It is shown that the UE state or a partial UE state[7,8] can be approached along travellers' behaviour adjustment process over time. If the travel cost on each link increases with link flow and the demand between each OD pair decreases with OD cost, the UE state is then stable.

2. The model and stability analysis

Consider a transportation network $G = (N, A)$, where N and A denote the sets of nodes and links, respectively. Let K_{rs} denote the set of routes between OD pair (r, s), x_a the flow on link a, $x = (\cdots, x_a, \cdots)$,

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f_k^{rs} the flow on route k connecting OD pair (r,s), $f^{rs}=(\cdots,f_k^{rs},\cdots)$, $f=(\cdots,f^{rs},\cdots)$, and $\Gamma_a(x)$ the travel cost on link a, $\Gamma=(\cdots,\Gamma_a,\cdots)$. Let $\delta_{a,k}^{rs}$ be an element of the link-route incidence matrix, $\delta_{a,k}^{rs}=$ (1 if link a is on route k connecting OD pair (r,s) and 0 otherwise), $\Delta^{rs}=(\cdots,\delta_{a,k}^{rs},\cdots)$, $\Delta=(\cdots,\Delta^{rs},\cdots)$, w_k^{rs} an element of the route-OD pair incidence matrix, $w_k^{rs}=$ (1 if route k connects OD pair (r,s) and 0 otherwise), $W=(\cdots,w_k^{rs},\cdots)$, and $C_k^{rs}(f)$ the travel cost on route k between OD pair (r,s), $C^{rs}=(\cdots,C_k^{rs},\cdots)$, $C=(\cdots,C^{rs},\cdots)$. Let u_{rs} be the minimum travel cost between OD pair (r,s), $u=(\cdots,u_{rs},\cdots)$, and $D_{rs}(u)$ the travel demand between OD pair (r,s), $D=(\cdots,D_{rs},\cdots)$. Clearly, we have a relation $C(f)=\Delta^{\mathrm{T}}\Gamma(\Delta f)$.

Assume that all link costs $\Gamma(x)$ monotonically increase in link flows x and all OD demands $D(u)$ monotonically decrease in OD costs u.[4] We are now interested in articulating a continuous dynamical adjustment mechanism which finally leads to a UE state concerning the congestion externality and demand elasticity. Particularly, we will develop a model to formulate this mechanism, governing the demand and route adjustment from day to day.

Suppose travellers always choose routes with minimum travel costs. Hence, if the cost on route k, C_k^{rs}, is larger than the minimum OD cost, u_{rs}, some travellers on route k will shift to routes with smaller costs on the next day and as a result, the flow on route k declines and the flows on other links increase. Similarly, if the OD demand computed from OD cost, D_{rs}, is higher than the sum of all path flows between OD pair (r,s), the OD cost, u_{rs}, should increase on the next day. This mechanism can be formulated as

$$\begin{aligned} &f_k^{rs}(t+\Delta t)\\ &=f_k^{rs}(t)-\Delta t\kappa_k^{rs}f_k^{rs}(t)\left(C_k^{rs}(f(t))-u_{rs}(t)\right), \end{aligned} \tag{1a}$$

$$\begin{aligned} &u_{rs}(t+\Delta t)\\ &=u_{rs}(t)+\Delta t\eta_{rs}u_{rs}(t)\left(D_{rs}(u(t))-\sum_{k\in K_{rs}}f_k^{rs}(t)\right), \end{aligned} \tag{1b}$$

where Δt (time step) may represent one day, κ_k^{rs} and η_{rs} are positive parameters controlling the adjustment size every time step. Equation (1) in fact corresponds to a continuous dynamical system model:

$$\dot{f}_k^{rs}(t)=-\kappa_k^{rs}f_k^{rs}(t)\left(C_k^{rs}(f(t))-u_{rs}(t)\right), \tag{2a}$$

$$\dot{u}_{rs}(t)=\eta_{rs}u_{rs}(t)\left(D_{rs}(u(t))-\sum_{k\in K_{rs}}f_k^{rs}(t)\right). \tag{2b}$$

The above model is much simpler than those proposed in Refs.[4,5] and easy to understand. For the relationship between a specific initial state and hereafter evolution, the following properties hold: if $f_k^{rs}(0)=0$ for all routes and OD pairs, then $f_k^{rs}(t)=0$ for any $t>0$; if $f_k^{rs}(0)>0$, then $f_k^{rs}(t)\geq 0$ for any $t>0$; if $u_{rs}(0)>0$, then $u_{rs}(t)\geq 0$ for any $t>0$.

Undergoing a long process of adjustment, the system Eq.(2) will evolve to a point with $\dot{f}_k^{rs}(t)=0$ and $\dot{u}_{rs}(t)=0$. This point may have three types of flow patterns as follows.

(i) There exists one or more OD pair (r,s) with $u_{rs}=0$ and $f_k^{rs}=0$, $k\in K_{rs}$. In this case, if a small perturbation occurs on u_{rs}, leading to $u_{rs}>0$, then $D_{rs}(u)>0$ and $\dot{u}_{rs}(t)>0$. Hence, the evolution termination point is unstable.

(ii) For any OD pair (r,s), $u_{rs}>0$ and a partial UE flow pattern is reached.[8] The so-called partial UE refers to such a state that all actually used routes have identical cost but there exist some unused routes having smaller costs. Using the method similar to Ref.[8], we can prove that this flow pattern is unstable.

(iii) For any OD pair (r,s), $u_{rs}>0$ and an UE flow pattern is implemented. In the next, we briefly show that this point is stable under certain conditions.

It is well known in static traffic assignment that if link cost function and OD demand function monotonically increases and decreases, respectively, there exist such a UE flow f^* and corresponding OD cost u^* that the equations $f_k^{rs^*}\left(C_k^{rs}(f^*)-u_{rs}^*\right)=0$ and $C_k^{rs}(f^*)\geq u_{rs}^*$ hold for all $k\in K_{rs}$.

For verifying the stability of the system (2) under appropriate conditions, we adopt the Lyapunov function and the stability theorem.[4,20] Let (f^*,u^*) be the flow and OD cost in an UE state. If link cost function $\Gamma(x)$ monotonically increases in x, and $D(u)$ monotonically decreases in u, the following continuous differentiable function

$$\begin{aligned} V(f,u)=&\sum_{rs}\sum_{k}\left(f_k^{rs}(t)-f_k^{rs^*}(t)\right)^2\\ &+\sum_{rs}\left(u_{rs}(t)-u_{rs^*}(t)\right)^2 \end{aligned} \tag{3}$$

is a Lyapunov function of the dynamical system Eq.(2) around the UE state. This says, the above function has three properties: (i) $V(f,u)>0$ if $(f,u)\geq 0$ and $(f,u)\neq(f^*,u^*)$, (ii) $V(f^*,u^*)=0$ and (iii) $\dot{V}(f,u)\leq 0$ if $(f,u)\geq 0$.

According to the definition Eq.(3), we can see that the first two properties are natural. We have only to prove that the function Eq.(3) has the third

property. Let $y = (f,u)^{\mathrm{T}}$, $B = \begin{pmatrix} \Delta & 0 \\ 0 & I \end{pmatrix}$, $F(By) = \begin{pmatrix} \Gamma(\Delta f) \\ -D(u) \end{pmatrix}$, $G = \begin{pmatrix} 0 & -W \\ W^{\mathrm{T}} & 0 \end{pmatrix}$, $E = B^{\mathrm{T}}F(By) + Gy$, where I is a unit matrix. Then, the dynamical system Eq.(2) can be expressed as $\dot{y} = -bE$, where $b = \mathrm{diag}(\cdots, \kappa_k^{rs} f_k^{rs}, \cdots, \eta_{rs} u_{rs}, \cdots)$. Substituting $\dot{y}_i$ into $\dot{V}(f,u) = 2\sum_i (y_i - y_i^*)\dot{y}_i$, we obtain

$$\dot{V}(f,u) = 2\sum_i (y_i^* - y_i) b_{ii} E_i. \tag{4}$$

Since $\Gamma(\Delta f)$ and $-D(u)$ monotonically increase in $x = \Delta f$ and u respectively, then, $F(By)$ is monotonically increasing in By. We conclude that

$$[B^{\mathrm{T}}F(By)]^{\mathrm{T}}(y^* - y) \le [B^{\mathrm{T}}F(By^*)]^{\mathrm{T}}(y^* - y). \tag{5}$$

Furthermore, since y^* is a UE solution, we then have

$$[B^{\mathrm{T}}F(By^*) + Gy^*]^{\mathrm{T}}(y^* - y) \le 0. \tag{6}$$

Combining Eqs.(5) and (6) we have

$$[B^{\mathrm{T}}F(By)]^{\mathrm{T}}(y^* - y) \le [Gy^*]^{\mathrm{T}}(y - y^*). \tag{7}$$

Adding $(Gy)^{\mathrm{T}}(y^* - y)$ to both sides of Eq.(7), we obtain

$$[B^{\mathrm{T}}F(By) + Gy]^{\mathrm{T}}(y^* - y) \le [Gy^* - Gy]^{\mathrm{T}}(y - y^*). \tag{8}$$

Since $E = B^{\mathrm{T}}F(By) + Gy$ and $[Gy^* - Gy]^{\mathrm{T}}(y - y^*) = -(y - y^*)^{\mathrm{T}}G^{\mathrm{T}}(y - y^*)$, Eq.(8) becomes $E^{\mathrm{T}}(y^* - y) \le -(y - y^*)^{\mathrm{T}}G^{\mathrm{T}}(y - y^*)$, where the right-hand side is zero. This follows

$$E^{\mathrm{T}}(y^* - y) \le 0. \tag{9}$$

The above result holds for arbitrary $y(\ge 0)$. Combining Eqs.(4) and (9) and considering that b is a matrix with nonnegative elements on the diagonal line and zero elsewhere, we obtain

$$\dot{V}(f,u) = 2E^{\mathrm{T}}b(y^* - y) \le 0. \tag{10}$$

This completes the proof.

According to the stability theorem,[4,20] we conclude that if (f,u) are non-negative, link cost $\Gamma(x)$ monotonically increases and OD demand $D(u)$ monotonically decreases, then the UE state of dynamical system Eq.(2) is stable.

3. Numerical analysis

We now apply the dynamical system model in an example network shown in Fig.1. This network consists of 4 OD pairs, 13 nodes, 19 links and 25 routes.[21]

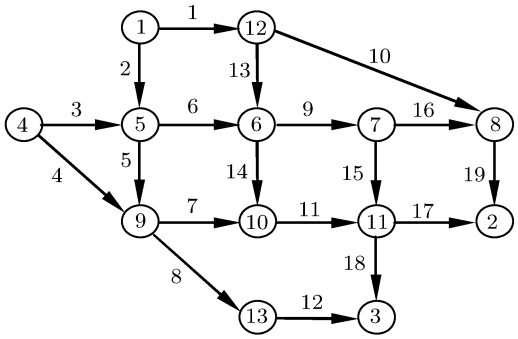


Fig.1. The example network.

The following BPR (Bureau of Public Roads) link cost functions are adopted

$$\Gamma_a(x_a) = U_a + V_a(x_a/K_a)^4, \tag{11}$$

where U_a is the free flow cost of link a, K_a is the link capacity and V_a is a positive constant. Table 1 lists the values of all parameters associated with link cost functions.

Table 1. Parameters of link cost functions.

links	U_a	V_a	K_a	links	U_a	V_a	K_a	links	U_a	V_a	K_a
1	1	0.060	30	8	4	0.030	50	15	3	0.045	30
2	2	0.090	30	9	3	0.075	30	16	1	0.080	30
3	1	0.030	50	10	3	0.055	30	17	2	0.090	40
4	2	0.075	30	11	2	0.060	30	18	4	0.040	50
5	2	0.045	30	12	4	0.050	30	19	2	0.075	30
6	3	0.060	40	13	2	0.030	50				
7	4	0.030	30	14	4	0.085	40				

The OD demand function is given by

$$D_{rs}(u_{rs}) = D^0_{rs}/(1 + \exp(\gamma(u_{rs} - \tilde{u}_{rs}))), \qquad (12)$$

where D^0_{rs} is the maximal demand between OD pair (r, s), $\tilde{u}_{rs}$ and γ are two positive parameters. Let $(D^0_{12}, D^0_{13}, D^0_{42}, D^0_{43}) = (240, 260, 250, 240)$, $(\tilde{u}_{12}, \tilde{u}_{13}, \tilde{u}_{42}, \tilde{u}_{43}) = (40, 30, 50, 60)$, and $\gamma = 0.001$. For each route, the initial flow is set to be 20 and the initial cost between each OD pair is set to be 50. The study time period [0, 15] is equally divided into 7500 intervals. All κ^{rs}_k and η_{rs} for controlling the adjustment size are set to be 0.1.

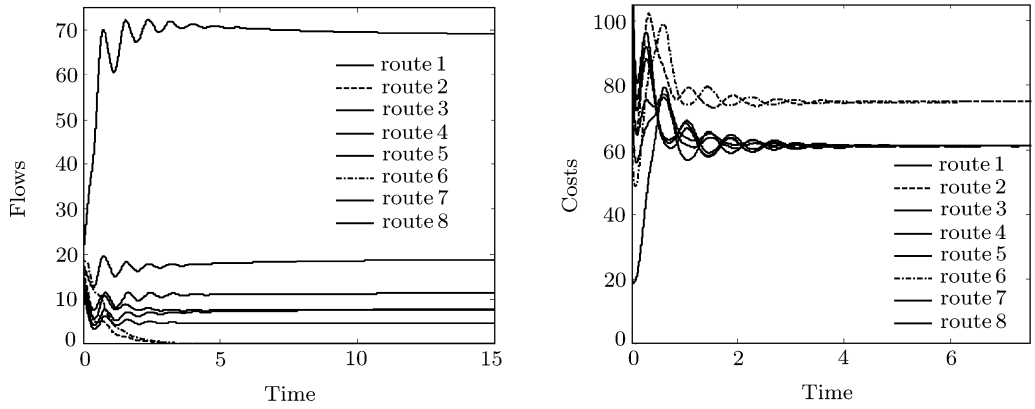


Fig.2. Evolutionary trajectories of route flows and route costs between OD pair (1, 2).

Figure 2 shows the evolutionary trajectories of route flows and route travel costs between OD pair (1, 2). There are 8 routes connecting the OD pair (1, 2). It can be seen that either route flows or route costs asymptotically converge to stable values. At the stable state, travel costs on routes 2 and 6 (dotted lines in the right-hand side) are larger than costs on other routes, thus these two routes have no flow (dotted lines in the left-hand side). Other routes are actually used by travellers, so they have identical and minimal travel costs. Obviously, the dynamical system can evolve to an UE state. For saving space, we hereinafter do not present the results between other OD pairs.

Figure 3 displays the evolution of travel cost and traffic demand between OD pair (1, 2) subjected to different values of the maximal demand. When D^0_{12} increases from 180 to 270 and other parameters remain unchanged, the OD travel cost u_{12} and the implemented OD demand D_{12} at stable state increase correspondingly. It can be seen that both u_{12} and D_{12} increase with equal sizes when D^0_{12} increases with 30 each time. However, oscillation can be observed at the earlier stage of the evolution.

Figure 4 shows the evolution of travel cost and traffic demand between OD pair (1, 2) subjected to different values of the capacity on link 1. It can be seen that the OD cost u_{12} and the implemented demand D_{12} at stable state decreases and increases, respectively, but with different sizes each time when the capacity K_1 increases equally. We can also observe that oscillation at earlier stage is negatively proportional to the link capacity.

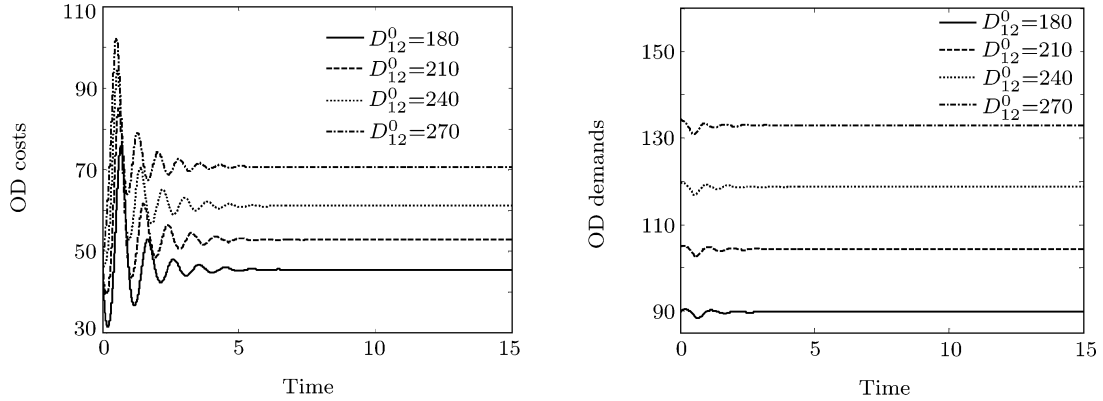


Fig.3. Evolution of travel cost and implemented demand between OD pair (1, 2) with different D^0_{12}.

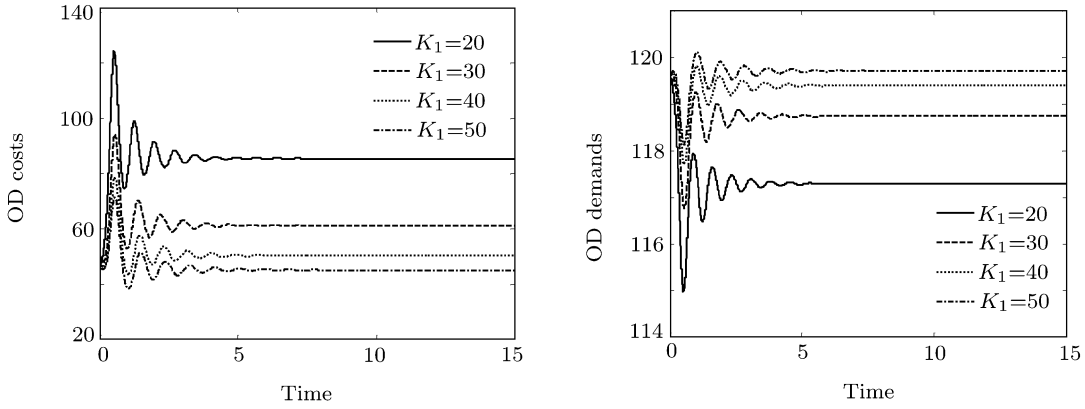


Fig.4. Evolution of travel cost and implemented demand between OD pair (1, 2) with different K_1.

4. Conclusions

How do the route flows and travel costs in a traffic network evolve to an UE state? By proposing a dynamical system model of traffic assignment, we have basically answered this question in this paper. The model is simple in the form and easy to understand, but can well reproduce the evolution process of travel behaviour adjustment from disequilibrium to equilibrium. We have shown that the UE state of the dynamical system is stable under certain conditions, i.e., all link costs increase in link flows and all OD demands decrease in OD costs. Numerical results in a grid network verify our theoretical analyses. Furthermore, we illustrate that the model can be used to investigate the dynamical impacts on evolution by the parameters associated with OD demand function and link cost function.

Our on-going researches include the day-to-day evolution of route choice behaviour under different road toll schemes and different market penetration levels of advanced traveller information systems.[22,23] It is also interesting to investigate various disequilibrium congestion pricing policies toward system optimum.[24]

References

[1] Patriksson M 1994 *The Traffic Assignment Problem—Models and Methods* (Amsterdam: VSP)
[2] Xu M and Gao Z Y 2007 *Chin. Phys.* **16** 1608
[3] Xu M and Gao Z Y 2007 *Chaos, Solitons Fractals* doi: 10.1016/j.chaos.2007.01.077
[4] Friesz T L, Berstein D H, Mehta N J, Tobin R L and Ganjalizadeh S 1994 *Oper. Res.* **42** 1120
[5] Friesz T L, Berstein D H and Stough R 1996 *Trans. Sci.* **30** 14
[6] Balijepalli N C and Watling D P 2005 *Transportation and Traffic Theory* (Oxford: Elsevier) p741
[7] Jin W L 2006 *Proc 11th Int. Con. of Hong Kong Society for Transportation Studies* (Hong Kong) p421
[8] Jin W L 2007 *Trans. Res.* Part B **41** 32
[9] Davis G A and Nihan N L 1993 *Oper. Res.* **41** 169
[10] Hazelton M L and Watling D P 2004 *Trans. Sci.* **38** 331
[11] Peeta S 1998 *Trans. Res.* Part C **6** 189
[12] Arentze T A and Timmermans H J P 2004 *Trans. Res.* Part B **38** 613
[13] Cantarella G E and Cascetta E 1995 *Trans. Sci.* **29** 305
[14] Li K P, Gao Z Y and Chen T L 2003 *Chin. Phys.* **12** 1213
[15] Wang T, Gao Z Y and Zhao X M 2006 *Acta Phys. Sin.* **55** 634 (in Chinese)
[16] Xu W, Jin Y F, Li W and Ma S J 2005 *Chin. Phys.* **14** 1077
[17] Yang W S, Wang B H, He P, Wang W N, Quan H J and Xie Y B 2003 *Chin. Phys.* **12** 931
[18] Jin Y F, Xu W, Ma S J and Li W 2005 *Acta Phys. Sin.* **54** 3480 (in Chinese)
[19] Li K P and Gao Z Y 2005 *Chin. Phys.* **14** 930
[20] Hahn W 1963 *Theory and Application of Lyapunov's Direct Method* (Englewood Cliffs, NJ: Prentice-Hall)
[21] Nguyen S and Dupius C 1984 *Trans. Sci.* **18** 185
[22] Huang H J, Liu T L and Yang H 2008 *J. Advanced Transportation* **42** 111
[23] Yang H and Huang H J 2004 *Trans. Res.* Part C **12** 93
[24] Yang H and Huang H J 2005 *Mathematical and Economic Theory of Road Pricing* (Oxford: Elsevier)