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Circuit realization of the fractional-order unified chaotic system

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This paper studies the chaotic behaviours of the fractional-order unified chaotic system. Based on the approximation method in frequency domain, it proposes an electronic circuit model of tree shape to realize the fractional-order operator. According to the tree shape model, an electronic circuit is designed to realize the 2.7-order unified chaotic system. Numerical simulations and circuit experiments have verified the existence of chaos in the fraction-order unified system.

Keywords: chaos, fractional-order unified chaotic system, circuit experiment
PACC: 0545

1. Introduction

Since Lorenz found the first chaotic attractor in a three-dimensional autonomous system in 1963,[1] chaos has been a focal point of study for many researchers in the past few decades.[2−13] Many chaotic systems have been found in recent years, such as Chen system,[2] Lü system,[3] Liu system,[4] Qi system[5] and so on. Recently, Lü *et al* introduced a new chaotic system, named unified chaotic system, which not only bridges the gap between the Lorenz and the Chen system but also represent the entire family of chaotic systems between them.[6] More recently, many authors began to study the chaotic dynamics of fractional-order dynamical system. In Ref.[7], it has been shown that the fractional-order Chua's system of order as low as 2.7 can produce a chaotic attractor. In Ref.[8], Li and Chen pointed out that chaos can exist in the fractional-order Rössler equation with order as low as 2.4, and hyperchaos can also exist in the fractional-order hyperchaotic Rössler system with order as low as 3.8. In Ref.[9], chaotic behaviours of a fractional-order mechanical chaotic 'jerk' model are studied. In Ref.[10], Lu and Chen investigated the fractional-order Chen system, and the simulation results show that the lowest order to have chaos in fractional-order Chen system is 0.3. In Ref.[11], Gao and Yu studied the chaotic behaviours in a fractional-order chaotic electronic oscillator. In Ref.[12], Wang and Liu studied a fractional-order critical chaotic system. All the above works develop this research direction greatly and take this research direction further and deeper.

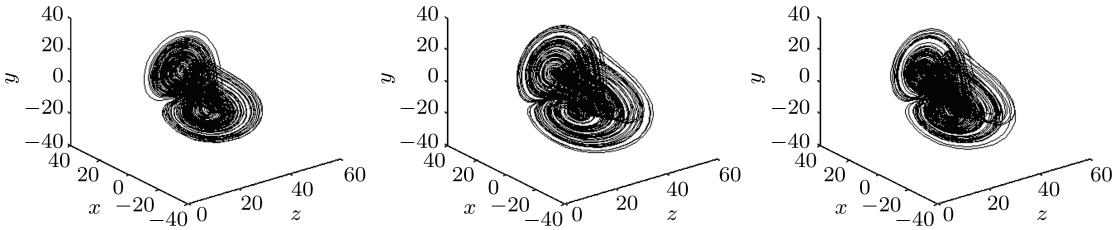
In this paper, the chaotic behaviours of the fractional-order unified chaotic system are studied. Based on the approximation method in frequency domain, an electronic circuit model of tree shape is designed to realize the fractional-order operator. The chaotic behaviours of 2.7-order unified chaotic system are studied by numerical simulations and circuit experiments. This is the first report of fractional-order unified system in electronic circuit.

2. Mathematical model of the unified chaotic system

The unified chaotic system[6] is described by

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= (25\alpha+10)(y-x),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (28-35\alpha)x - xz + (29\alpha-1)y,\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - \frac{\alpha+8}{3}z,\end{aligned} \tag{1}$$

where parameter $\alpha \in [0,1]$. According to Ref.[14], the linear part of system (1), a constant matrix $A = [a_{ij}]_{3\times3}$, provides a critical value $a_{12}a_{21}$. The whole family of chaotic system (1) can be classified as follows by the critical value: when $\alpha \in [0, 0.8)$, system (1) belongs to the generalized Lorenz system as $a_{12}a_{21} > 0$, when $\alpha = 0.8$, system (1) is the generalized Lü system as $a_{12}a_{21} = 0$, and when $\alpha \in (0.8, 1]$, system (1) belongs to the generalized Chen system as $a_{12}a_{21} < 0$. Take $\alpha = 0.5$, 0.8, 1 respectively, the chaotic attractors are shown in Fig.1.



(a) $\alpha = 0.5$ (b) $\alpha = 0.8$ (c) $\alpha = 1$

Fig.1. The chaotic attractors of unified chaotic system with different parameter α.

3. Fractional derivative and its approximation

There are many definitions of fractional derivative.[15] Among them, the Riemann–Liouville definition is more favoured than others, being the simplest definition and also the easiest one to use; the Riemann–Liouville definition is described by

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} \mathrm{d}\tau, \quad (2)$$

where $\Gamma(\cdot)$ is the gamma function, and $n-1 \leq q < n$. Considering all the initial values to be zero, the Laplace transform of the Riemann–Liouville fractional derivative is

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\left\{f(t)\right\}. \quad (3)$$

Therefore, the fractional integral operator of order "q" can be represented by the transfer function $F(s) = \dfrac{1}{s^q}$ in the frequency domain.

The standard definition of fractional differ-integral does not allow the direct engineering applications and circuit designs of the fractional operators. In Ref.[16], an efficient method to solve this problem is to approximate the fractional operators by using the standard integer-order operators. In Table 1 of Ref.[9], approximations of $\dfrac{1}{s^q}$ with $q = 0.1 - 0.9$ in a step of 0.1 were given with errors of around 2dB. We will use these approximations in the following simulations and electronic circuit experiments.

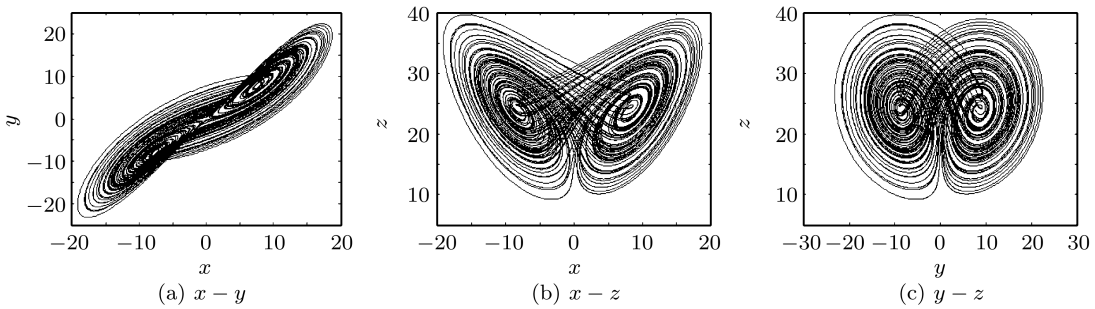
4. Fractional-order unified chaotic system and its circuit realization

4.1. Fractional-order unified chaotic system

Now, the fractional-order unified chaotic system can be described as follows:

$$\begin{aligned} \frac{\mathrm{d}^q x}{\mathrm{d}t^q} &= (25\alpha + 10)(y - x), \\ \frac{\mathrm{d}^q y}{\mathrm{d}t^q} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\ \frac{\mathrm{d}^q z}{\mathrm{d}t^q} &= xy - \frac{\alpha + 8}{3} z, \end{aligned} \quad (4)$$

where q is the fractional order satisfying $0 < q \leq 1$, and parameter $\alpha \in [0, 1]$, In this paper, we take $\alpha = 0.5, 0.8, 1$ respectively as example to study the fractional-order unified system. When $q = 0.9$, the numerical simulation phase portraits of 2.7-order unified chaotic system for different parameter α are shown in Fig.2, Fig.3, and Fig.4, respectively. And the largest Lyapunov exponents are 0.0214, 0.0302 and 0.0223, respectively.



(a) $x - y$ (b) $x - z$ (c) $y - z$

Fig.2. The numerical simulation phase portraits of 2.7-order unified chaotic system with $\alpha = 0.5$.

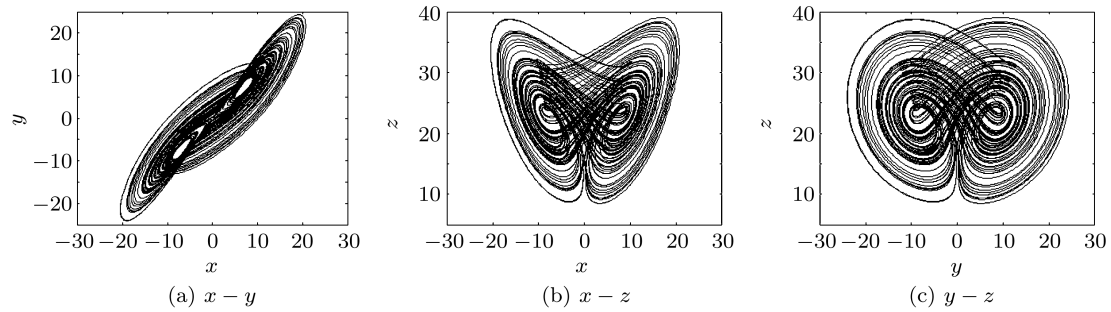


Fig.3. The numerical simulation phase portraits of 2.7-order unified chaotic system with $\alpha = 0.8$.

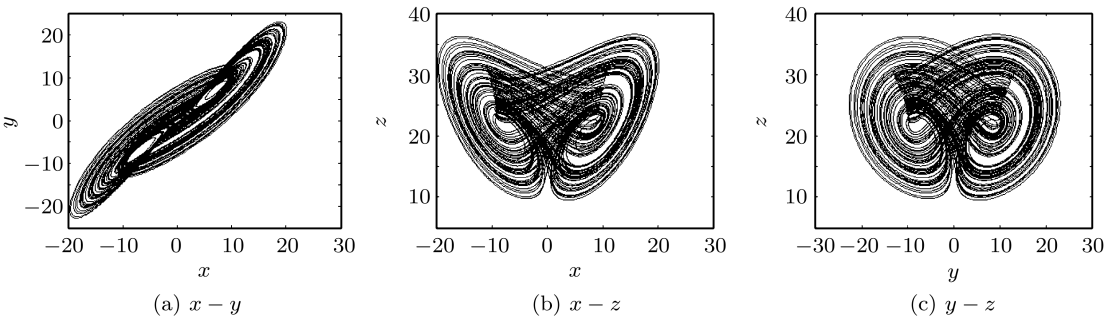


Fig.4. The numerical simulation phase portraits of 2.7-order unified chaotic system with $\alpha = 1$.

4.2. Circuit realization of unified chaotic system

In this section, firstly, an electronic circuit model of tree shape is designed to realize the fractional-order operator; secondly, an electronic circuit is designed to realize 2.7-order unified chaotic system. According to the circuit theory, in the Laplace domain, the corresponding circuit model between A and B in Fig.5 can be used to realize the approximations of $\frac{1}{s^q}$ with $q = 0.1 - 0.9$ in a step of 0.1 in Table 1 of Ref.[9], for convenience, we name it as the "tree shape model".

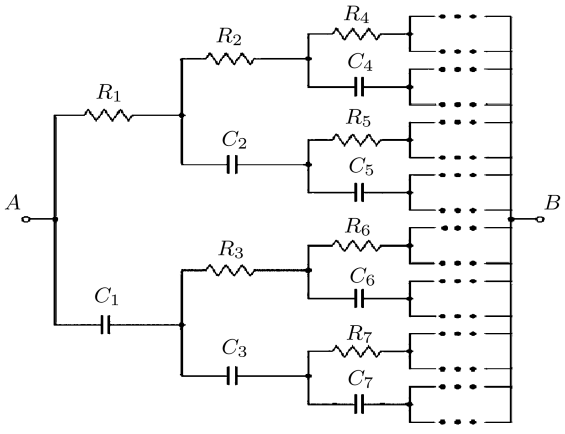


Fig.5. The tree shape model of $\frac{1}{s^q}$.

Taking $q = 0.9$ of Table 1 in Ref.[9] as an example to show the design process of the tree shape model, from Table 1 in Ref.[9], we can obtain approximation of $\frac{1}{s^{0.9}}$ with an error of 2dB as follows:

$$\frac{1}{s^{0.9}} \approx \frac{2.2675(s+1.292)(s+215.4)}{(s+0.01292)(s+2.154)(s+359.4)}, \quad (5)$$

and the corresponding tree shape model is shown in Fig.6.

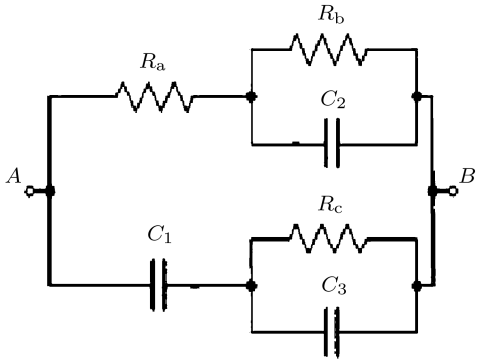


Fig.6. The tree shape model of $\frac{1}{s^{0.9}}$.

Applying circuit theory in Laplace domain, we can obtain the transfer function $H(s)$ between A and B in Fig.6 as follows:

$$
\begin{aligned}
&H(s)\\
&=\left[R_{\mathrm{a}}+\left(R_{\mathrm{b}}//\frac{1}{sC_2}\right)\right]//\left[\frac{1}{sC_1}+\left(R_{\mathrm{c}}//\frac{1}{sC_3}\right)\right]\\
&=\frac{1}{C_0}\frac{\left(\frac{C_0}{C_1}+\frac{C_0}{C_3}\right)\left(s+\frac{R_{\mathrm{a}}+R_{\mathrm{b}}}{R_{\mathrm{a}}C_2R_{\mathrm{b}}}\right)\left(s+\frac{1}{C_1R_{\mathrm{c}}+C_3R_{\mathrm{c}}}\right)}{s^3+\left(\frac{R_{\mathrm{a}}+R_{\mathrm{b}}}{R_{\mathrm{a}}C_2R_{\mathrm{b}}}+\frac{1}{C_3R_{\mathrm{c}}}+\frac{C_1+C_3}{C_1R_{\mathrm{a}}C_3}\right)s^2+\left(\frac{R_{\mathrm{a}}+R_{\mathrm{b}}}{R_{\mathrm{a}}C_2R_{\mathrm{b}}C_3R_{\mathrm{c}}}+\frac{1}{C_1R_{\mathrm{a}}C_3R_{\mathrm{c}}}+\frac{C_1+C_3}{C_1R_{\mathrm{a}}C_2R_{\mathrm{b}}C_3}\right)s+\frac{1}{C_1R_{\mathrm{a}}C_2R_{\mathrm{b}}C_3R_{\mathrm{c}}}}, \quad (6)
\end{aligned}
$$

where C_0 is unit parameter, take $C_0 = 1\,\mu\mathrm{F}$ and $F(s) = H(s)C_0 = \frac{1}{s^{0.9}}$. Comparing Eq.(6) with Eq.(5), we can obtain the values of resistances and capacitances in Fig.6 as follows: $R_{\mathrm{a}} = 1.55\,\mathrm{M\Omega}$, $R_{\mathrm{b}} = 61.54\,\mathrm{M\Omega}$, $R_{\mathrm{c}} = .526\,\mathrm{k\Omega}$, $C_1 = 0.7346\,\mu\mathrm{F}$, $C_2 = 0.5221\,\mu\mathrm{F}$, $C_3 = 1.103\,\mu\mathrm{F}$. In the same process, we can obtain the corresponding values of resistances and capacitances in Fig.5 to realize $\frac{1}{s^q}$ with $q = 0.1 - 0.8$ in a step of 0.1 in Table 1 of Ref.[9] in Laplace domain, which are shown in Tables 1 and 2 respectively.

Table 1. The values of resistances in tree shape model of $\frac{1}{s^q}$(2dB).

q	$R_1/\mathrm{M\Omega}$	$R_2/\mathrm{M\Omega}$	$R_3/\mathrm{M\Omega}$	$R_4/\mathrm{M\Omega}$	$R_5/\mathrm{M\Omega}$	$R_6/\mathrm{M\Omega}$
0.1	0.9517	0.6332	1.363			
0.2	0.5454	0.7553	3.307	1.211		
0.3	0.3476	1.296	4.854	2.339	0.2196	
0.4	0.1845	0.4294	2.080	5.698	0.7819	1.961
0.5	0.1749	0.4417	2.480	9.410	0.6957	2.354
0.6	0.08291	0.3202	2.594	15.45	0.5436	2.853
0.7	0.02169	0.1205	2.016	24.99	0.2806	3.247
0.8	0.008586	0.09269	6.325	39.69	0.4900	

Table 2. The values of capacitances in tree shape model of $\frac{1}{s^q}$(2dB).

q	$C_1/\mu\mathrm{F}$	$C_2/\mu\mathrm{F}$	$C_3/\mu\mathrm{F}$	$C_4/\mu\mathrm{F}$	$C_5/\mu\mathrm{F}$	$C_6/\mu\mathrm{F}$
0.1	0.02572	15.77	0.0006468			
0.2	5.366	0.1814	0.01262	1.560		
0.3	4.927	0.5417	0.02525	2.752	0.3098	
0.4	5.023	0.4836	0.02834	0.1456	0.1428	4.854
0.5	3.793	0.5827	0.06416	0.1751	0.2292	4.441
0.6	2.741	0.5489	0.09585	0.1692	0.2442	3.247
0.7	1.917	0.4657	0.1131	0.1500	0.1964	2.019
0.8	0.9503	0.6139	0.2349	0.2337	0.2391	

Now, according to the tree shape model, we can design an electronic circuit to realize the 2.7-order unified chaotic system. As voltage values directly used by the state variables of the 2.7-order unified chaotic system would exceed the power supply range of operational amplifiers and analogue multipliers, a simple transformation should be made before the electronic circuit design. To facilitate the circuitry design, the 2.7-order unified chaotic system is described by

$$
\begin{aligned}
\frac{\mathrm{d}^{0.9}x}{\mathrm{d}t^{0.9}} &= (25\alpha+10)(y-x),\\
\frac{\mathrm{d}^{0.9}y}{\mathrm{d}t^{0.9}} &= 28x-y+29\alpha y-35\alpha x-10xz,\\
\frac{\mathrm{d}^{0.9}z}{\mathrm{d}t^{0.9}} &= 10xy-\frac{\alpha+8}{3}z, \quad (7)
\end{aligned}
$$

where parameter $\alpha = 0.5, 0.8, 1$ respectively.

Applying circuit theory and circuitry design method in Refs.[17,18,19], we present the designed circuitry to realize Eq.(7) in Fig.7, where AD633 is the multiplier with an output coefficient of 0.1 and LM741 is the operational amplifier.

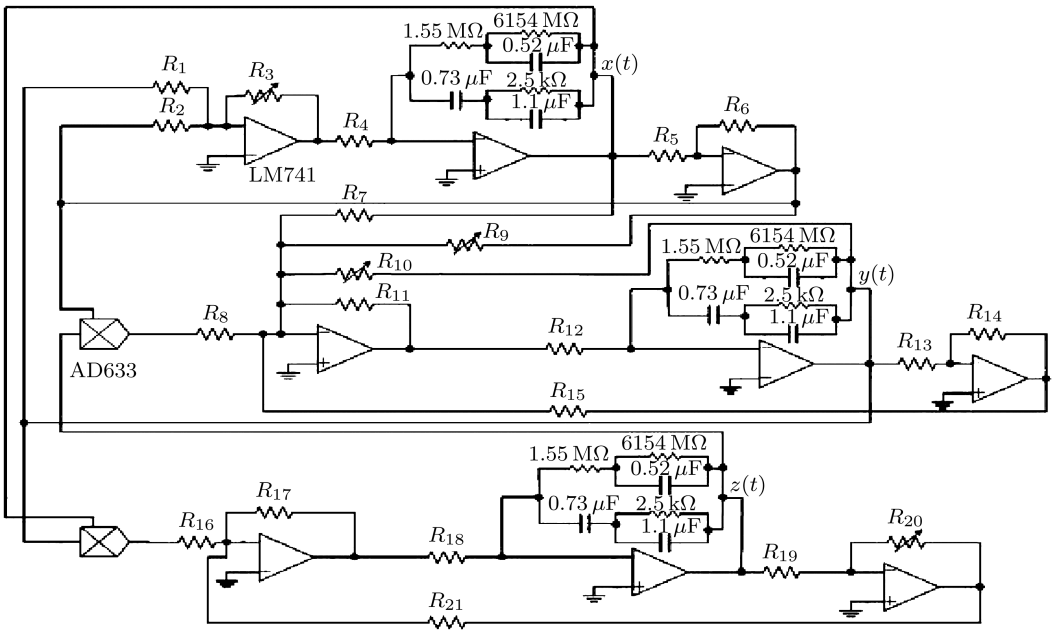


Fig.7. Circuitry realization of 2.7-order unified chaotic system.

The circuitry equations in Laplace domain are described by

$$\begin{aligned}
\frac{X(s)}{F(s)} &= \frac{R_3}{C_0 R_4}\left[\frac{Y(s)}{R_1} - \frac{R_6 X(s)}{R_2 R_5}\right],\\
\frac{Y(s)}{F(s)} &= \frac{R_{11}}{C_0 R_{12}}\left[\frac{X(s)}{R_7} - \frac{R_{14} Y(s)}{R_{13} R_{15}} + \frac{Y(s)}{R_{10}}\right.\\
&\quad \left. - \frac{R_6 X(s)}{R_5 R_9} - \frac{k R_6 L\{x(t)z(t)\}}{R_5 R_8}\right],\\
\frac{Z(s)}{F(s)} &= \frac{R_{17}}{C_0 R_{18}}\left[\frac{kL\{x(t)y(t)\}}{R_{16}} - \frac{R_{20} Z(s)}{R_{19} R_{21}}\right], \quad (8)
\end{aligned}$$

where k is the output coefficient of AD633, $F(s) = \dfrac{1}{s^{0.9}}$. Let $C_0 = 1\,\mu\text{F}$, R_1, R_5, R_{14}, $R_{16} = 10\,\text{k}\Omega$, R_2, $R_6 = 35\,\text{k}\Omega$, R_4, R_{12}, $R_{13}R_{17}$, R_{18}, $R_{19} = 100\,\text{k}\Omega$, $R_7 = 10.35\,\text{k}\Omega$, $R_8 = 10.15\,\text{k}\Omega$, R_{11}, $R_{15} = 29\,\text{k}\Omega$, $R_{21} = 30\,\text{k}\Omega$, then we can obtain $R_3 = (25\alpha + 10)\,\text{k}\Omega$, $R_9 = \dfrac{29}{\alpha}\,\text{k}\Omega$, $R_{10} = \dfrac{10}{\alpha}\,\text{k}\Omega$, $R_{20} = (\alpha + 8)\,\text{k}\Omega$. Table 3 shows the corresponding relationship between different parameter α and R_3, R_9, R_{10}, R_{20}.

Table 3. The corresponding relationship of different α with R_3, R_9, R_{10}, R_{20}.

α	$R_3/\text{k}\Omega$	$R_9/\text{k}\Omega$	$R_{10}/\text{k}\Omega$	$R_{20}/\text{k}\Omega$
0.5	22.5	58	20	8.5
0.8	30	36.25	12.5	8.8
1	35	29	10	9

Considering the accuracy of resistances and capacitances in circuitry experiment, we take $R_\text{a} = 1.55\,\text{M}\Omega$, $R_\text{b} = 61.5\,\text{M}\Omega$, $R_\text{c} = 2.5\,\text{k}\Omega$, $C_1 = 0.7\,\mu\text{F}$, $C_2 = 0.5\,\mu\text{F}$, $C_3 = 1.1\,\mu\text{F}$ to realize the tree shape model in Fig.7. The observed chaotic phase portraits of different parameter α in circuit experiment is shown in Fig.8, Fig.9, and Fig.10 respectively. Comparing Figs.2–4 with Figs.8–10 respectively, we observe a good qualitative agreement between the numerical simulations and the experimental measurements.

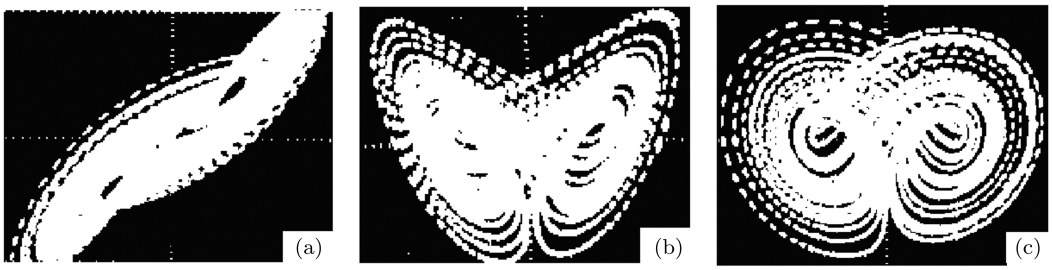


Fig.8. The observed phase portraits of 2.7-order unified chaotic system with $\alpha = 0.5$. (a) $x - y$ (500 mV/div, 500 mV/div), (b) $x - z$ (500 mV/div, 500 mV/div), (c) $y - z$ (500 mV/div, 500mV/div).

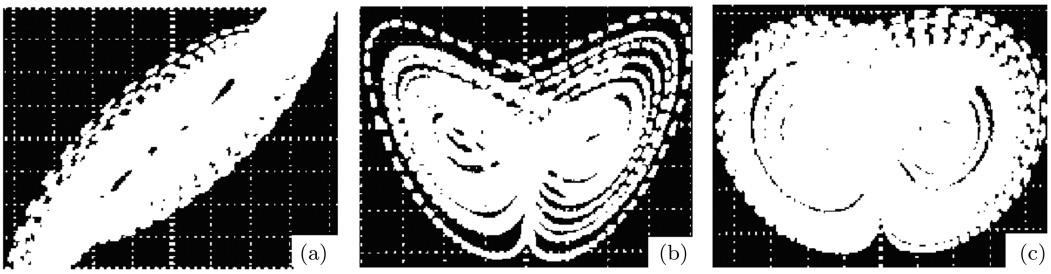


Fig.9. The observed phase portraits of 2.7-order unified chaotic system with $\alpha = 0.8$. (a) $x - y$ (500 mV/div, 500 mV/div), (b) $x - z$ (500 mV/div, 500 mV/div), (c) $y - z$ (500 mV/div, 500 mV/div).

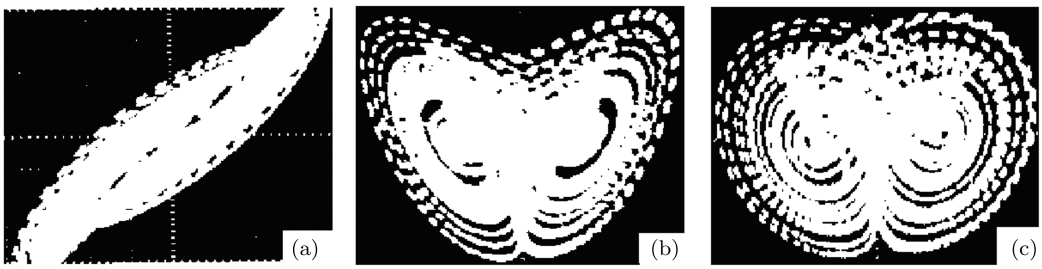


Fig.10. The observed phase portraits of 2.7-order unified chaotic system with $\alpha = 1$. (a) $x - y$ (500 mV/div, 500 mV/div), (b) $x - z$ (500 mV/div, 500 mV/div), (c) $y - z$ (500 mV/div, 500 mV/div).

5. Conclusion

In this paper, we have studied the chaotic behaviours of the fractional-order unified chaotic system. Based on the approximation method in frequency domain, an electronic circuit model of tree shape is designed to realize the fractional-order operator. The chaotic behaviours of 2.7-order fractional-order unified chaotic system have been studied not only by numerical simulations, but also by electronic circuit experiments. To the best of our knowledge, it is the first time to realize the fractional-order unified chaotic system in electronic circuit among all the relative reports in the literature. Future works regarding this topic include the study of some types of control and synchronization of fractional-order chaotic systems.

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