Vol 17 No 5, May 2008
1674-1056/2008/17(05)/1640-05

Chinese Physics B

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Quantum optical $ABCD$ theorem in two-mode case*

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(Received 2 August 2007; revised manuscript received 10 September 2007)

By introducing the entangled Fresnel operator (EFO) this paper demonstrates that there exists $ABCD$ theorem for two-mode entangled case in quantum optics. The canonical operator method as mapping of ray-transfer $ABCD$ matrix is explicitly shown by EFO's normally ordered expansion through the coherent state representation and the technique of integration within an ordered product of operators.

Keywords: Fresnel operator, Quantum optical $ABCD$ theorem
PACC: 0425, 0450

In a recently published paper[1] we have shown that the multiplication rule of single-mode Fresnel operators (FO) directly leads to quantum optical $ABCD$ law which corresponds to the classical optical $ABCD$ law. In matrix optics, the $ABCD$ law tells us how the curvature of a centred spherical wavefront changes from one reference plane to the next. The multiplication rule of matrix optics guarantees that if the ray-transfer matrices of the n optical components are $M_1, M_2, M_3, \cdots, M_n$, respectively, then the whole ray transfer is determined by $M = M_1 M_2 M_3 \cdots M_n$. Remarkably, the propagation of Gaussian beam also obeys $ABCD$ law via optical diffraction integration, i.e. the input light field $f(x_1)$ and output light field $g(x_2)$ are related to each other by one-dimensional (1D) Fresnel integration[2,3] $g(x_2) = \int_{-\infty}^{\infty} \mathcal{K}(A, B, C; x_2, x_1) f(x_1) \mathrm{d}x_1$, where

$$\mathcal{K}(A, B, C; x_2, x_1) = \frac{1}{\sqrt{2\pi \mathrm{i} B}} \exp\left[\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_2x_1 + Dx_2^2\right)\right], \quad (1)$$

where $AD - BC = 1$. The $ABCD$ law for Gaussian beam passing through an optical system is[4]

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}, \quad (2)$$

where q_1 (q_2) represents the complex curvature of the input (output) Gaussian beam. While the single-mode FO corresponds to 1D classical Fresnel transform, the two-mode FO corresponds to entangled Fresnel transform which is not the direct product of two independent 1D Fresnel transforms. In this paper by introducing the entangled Fresnel operator (EFO) we demonstrate that there also exists $ABCD$ theorem for two-mode entangled case in quantum optics. In our derivation, the canonical operator methods as mapping of ray-transfer $ABCD$ matrix is explicitly shown by EFO's normally ordered expansion through the coherent state representation and the technique of integration within an ordered product (IWOP) of operators.[5,6]

As an extension of single-mode FO, instead of using the direct product of two independent single-mode FOs, which is a routine generalization, we introduce the following integration:

$$F_2(r, s) = s \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} |sz_1 + rz_2^*, rz_1^* + sz_2\rangle \langle z_1, z_2|, \quad (3)$$

where

$$|sz_1 + rz_2^*,\ rz_1^* + sz_2\rangle \equiv |sz_1 + rz_2^*\rangle_1 \otimes |rz_1^* + sz_2\rangle_2, \quad (4)$$

s and r are complex and satisfy the unimodularity condition $ss^* - rr^* = 1$, $|z_1\rangle = \exp\left(-\frac{1}{2}|z_1|^2 + z_1 a_1^\dagger\right)|0\rangle$ is the coherent state,[7,8] $a_i^\dagger$ is the Bose creation operator, $\left[a_i, a_j^\dagger\right] = \delta_{ij}$, $i = 1, 2$. Equation (3) indicates that $F_2(r, s)$ is an image of classical symplectic transform $(z_1, z_2) \to (sz_1 + rz_2^*,\ rz_1^* + sz_2)$ in two-dimensional complex phase space through two-mode coherent state representation. Using the normal ordering of vacuum projector $|00\rangle\langle 00| =: \exp\left(-a_1^\dagger a_1 - a_2^\dagger a_2\right):$

*Project supported by the National Natural Science Foundation of China (Grant Nos 10775097 and 10475056).
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and the IWOP technique we can calculate the integral in Eq.(3) and derive its normally ordered form

$$\begin{aligned} F_2(r,s) =& \exp\left(\frac{r}{s^*}a_1^\dagger a_2^\dagger\right) \\ & \times \exp\left[\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)\ln\left(s^*\right)^{-1}\right] \\ & \times \exp\left(-\frac{r^*}{s^*}a_1 a_2\right). \end{aligned} \tag{5}$$

Thus $F_2(r,s)$ induces the transform

$$\begin{aligned} F_2(r,s)\, a_1 F_2(r,s)^{-1} &= s^* a_1 - r a_2^\dagger, \\ F_2(r,s)\, a_2 F_2(r,s)^{-1} &= s^* a_2 - r a_1^\dagger, \end{aligned} \tag{6}$$

and F_2 is a 3-parameters two-mode squeezing operator. By introducing A, B, C, D through the relation

$$\begin{aligned} s =& \frac{1}{2}\left[A + D - \mathrm{i}\left(B - C\right)\right], \\ r =& -\frac{1}{2}\left[A - D + \mathrm{i}\left(B + C\right)\right], \end{aligned} \tag{7}$$

we see that the unimodularity $ss^* - rr^* = 1$ becomes $AD - BC = 1$. Using the overlap $\langle z\,|z'\rangle = \exp\Big[-\frac{1}{2}\left(|z|^2 + |z'|^2\right) + z^* z'\Big]$ and the IWOP technique we can obtain multiplication rule of $F_2(r,s)$,

$$\begin{aligned} & F_2(r,s)\, F_2(r',s') \\ =& ss' \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} \left|sz_1 + rz_2^*,\ rz_1^* + sz_2\right\rangle \left\langle z_1, z_2\right| \\ & \times \int \frac{\mathrm{d}^2 z_1' \mathrm{d}^2 z_2'}{\pi^2} \left|s'z_1 + r'z_2^*,\ r'z_1^* + s'z_2\right\rangle \left\langle z_1', z_2'\right| \\ =& F_2(r'', s''), \end{aligned} \tag{8}$$

where

$$\begin{aligned} & \begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix} \begin{pmatrix} s' & -r' \\ -r'^* & s'^* \end{pmatrix} = \begin{pmatrix} s'' & -r'' \\ -r''^* & s''^* \end{pmatrix}, \\ & |s''|^2 - |r''|^2 = 1, \end{aligned} \tag{9}$$

or $s'' = ss' + rr'^*,\ r'' = rs'^* + r's$. Equation (8) is equivalent to

$$F_2(A', B', C')\, F_2(A, B, C) = F_2(A'', B'', C''), \tag{10}$$

where

$$\begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{11}$$

On the other hand, we can prove that the matrix element of F_2 in the entangled state representation $|\eta\rangle$ [9−11]

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + a_1^\dagger \eta - a_2^\dagger \eta^* + a_1^\dagger a_2^\dagger\right]|00\rangle \tag{12}$$

is,[12,13]

$$\begin{aligned} & \langle\eta'|\, F_2(r,s)\, |\eta\rangle \\ =& \frac{1}{2\mathrm{i}B\pi} \exp\left[\frac{\mathrm{i}}{2B}\left(A|\eta|^2 - \left(\eta\eta'^* + \eta^*\eta'\right) + D|\eta'|^2\right)\right] \\ \equiv& \mathcal{K}_2^{(r,s)}(\eta', \eta), \end{aligned} \tag{13}$$

which is different from Eq.(1), so it can be considered as the integration kernel of entangled 2-dimensional optical Fresnel transform,

$$G(\eta') = \int \frac{\mathrm{d}^2\eta}{\pi} \mathcal{K}_2^{(r,s)}(\eta', \eta)\, F(\eta), \tag{14}$$

so $F_2(r,s)$ can be named as EFO, here $F(\eta) = \langle\eta|\, F\rangle, G(\eta') = \langle\eta'|\, G\rangle$, and we have used the completeness relation

$$\int \frac{\mathrm{d}^2\eta}{\pi} |\eta\rangle\langle\eta| = 1. \tag{15}$$

For the applications, in quantum teleportation, of entangled state representations with continuum variables, we refer to Ref.[14]. Using the relation in Eq.(7), we can put Eq.(5) into the following form:

$$\begin{aligned} F_2(A,B,C) =& \frac{2}{A + D + \mathrm{i}\left(B - C\right)} : \exp\left\{-\frac{A - D + \mathrm{i}\left(B + C\right)}{A + D + \mathrm{i}\left(B - C\right)} a_1^\dagger a_2^\dagger \right. \\ & \left. + \left(\frac{2}{A + D + \mathrm{i}\left(B - C\right)} - 1\right)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) + \frac{A - D - \mathrm{i}\left(B + C\right)}{A + D + \mathrm{i}\left(B - C\right)} a_1 a_2 \right\} : . \end{aligned} \tag{16}$$

Note

$$Q_i = \frac{a_i + a_i^\dagger}{\sqrt{2}}, \quad P_i = \frac{a_i - a_i^\dagger}{\sqrt{2}i} \tag{17}$$

and

$$\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta\,|\eta\rangle, \quad \left(a_2 - a_1^\dagger\right)|\eta\rangle = \eta^*\,|\eta\rangle, \tag{18}$$

which leads to

$$\begin{aligned} & \left(Q_1 - Q_2\right)|\eta\rangle = \sqrt{2}\eta_1\,|\eta\rangle, \\ & \left(P_1 + P_2\right)|\eta\rangle = \sqrt{2}\eta_2\,|\eta\rangle, \quad \eta = \eta_1 + \mathrm{i}\eta_2, \end{aligned} \tag{19}$$

we can prove the operator identity

$$\begin{aligned}\mathrm{e}^{\frac{\lambda}{2}\left[(Q_1-Q_2)^2+(P_1+P_2)^2\right]} &= \int \frac{\mathrm{d}^2\eta}{\pi}\mathrm{e}^{\frac{\lambda}{2}\left[(Q_1-Q_2)^2+(P_1+P_2)^2\right]}\left|\eta\right\rangle\left\langle\eta\right| \\ &= \int \frac{\mathrm{d}^2\eta}{\pi}\colon \exp\left\{(\lambda-1)\left|\eta\right|^2+a_1^\dagger\eta-a_2^\dagger\eta^*+a_1\eta^*-a_2\eta+\left(a_1-a_2^\dagger\right)\left(a_2-a_1^\dagger\right)\right\}\colon \\ &= \frac{1}{1-\lambda}\colon \exp\left[\frac{\lambda}{2\left(1-\lambda\right)}\left(\left(Q_1-Q_2\right)^2+\left(P_1+P_2\right)^2\right)\right]\colon . \end{aligned} \tag{20}$$

When $B=0$, $A=1$, $C\to C/A$, $D=1$, and using Eq.(20) we see that

$$\begin{aligned}F_2\left(1,0,C/A\right) &= \frac{2}{2-\mathrm{i}C/A}\colon \exp\left\{\frac{\mathrm{i}C/A}{2-\mathrm{i}C/A}\left(a_1^\dagger a_1+a_2^\dagger a_2-a_1^\dagger a_2^\dagger-a_1a_2\right)\right\}\colon \\ &= \exp\left\{\frac{\mathrm{i}C}{4A}\left[\left(Q_1-Q_2\right)^2+\left(P_1+P_2\right)^2\right]\right\}, \end{aligned} \tag{21}$$

which is in the form of a canonical operator. On the other hand, by introducing the $\left|\xi\right\rangle$ representation,

$$\left|\xi\right\rangle = \exp\left[-\frac{1}{2}\left|\xi\right|^2+a_1^\dagger\xi+a_2^\dagger\xi^*-a_1^\dagger a_2^\dagger\right]\left|00\right\rangle, \tag{22}$$

which is conjugate to $\left|\eta\right\rangle$, and is the common eigenvector of (Q_1+Q_2) and (P_1-P_2)

$$\begin{aligned}\left(Q_1+Q_2\right)\left|\xi\right\rangle &= \sqrt{2}\xi_1\left|\xi\right\rangle, \\ \left(P_1-P_2\right)\left|\xi\right\rangle &= \sqrt{2}\xi_2\left|\xi\right\rangle, \end{aligned} \tag{23}$$

and

$$\int \frac{\mathrm{d}^2\xi}{\pi}\left|\xi\right\rangle\left\langle\xi\right| = 1, \tag{24}$$

we can derive another operator identity

$$\begin{aligned}&\mathrm{e}^{\frac{\lambda}{2}\left[(Q_1+Q_2)^2+(P_1-P_2)^2\right]} \\ &= \int \frac{\mathrm{d}^2\xi}{\pi}\mathrm{e}^{\frac{\lambda}{2}\left[(Q_1+Q_2)^2+(P_1-P_2)^2\right]}\left|\xi\right\rangle\left\langle\xi\right| \\ &= \frac{1}{1-\lambda}\colon \exp\left[\frac{\lambda}{2\left(1-\lambda\right)}\left(Q_1+Q_2\right)^2+\left(P_1-P_2\right)^2\right]\colon . \end{aligned} \tag{25}$$

It then follows from Eqs.(16) and (25)

$$\begin{aligned}F_2\left(1,B/A,0\right) &= \frac{2}{2+\mathrm{i}B/A}\colon \exp\left\{-\frac{\mathrm{i}B/A}{2+\mathrm{i}B/A}\left(a_1^\dagger a_2^\dagger+a_2^\dagger a_2+a_1^\dagger a_1+a_1a_2\right)\right\}\colon \\ &= \frac{2}{2+\mathrm{i}B/A}\colon \exp\left\{-\frac{\mathrm{i}B/A}{2\left(2+\mathrm{i}B/A\right)}\left[\left(Q_1+Q_2\right)^2+\left(P_1-P_2\right)^2\right]\right\}\colon \\ &= \exp\left\{-\frac{\mathrm{i}B}{4A}\left[\left(Q_1+Q_2\right)^2+\left(P_1-P_2\right)^2\right]\right\}. \end{aligned} \tag{26}$$

In particular, when $B=C=0$, and $D=A^{-1}$, Eq.(16) becomes

$$\begin{aligned}F_2\left(A,0,0\right) =& \frac{2}{A+A^{-1}}\colon \exp\left\{-\frac{A-A^{-1}}{A+A^{-1}}a_1^\dagger a_2^\dagger\right. \\ &\left.+\left(\frac{2}{A+A^{-1}}-1\right)\left(a_1^\dagger a_1+a_2^\dagger a_2\right)+\frac{A-A^{-1}}{A+A^{-1}}a_1a_2\right\}\colon \\ =& \operatorname{sech}\lambda\colon \exp\left[-a_1^\dagger a_2^\dagger\tanh\lambda+\left(\operatorname{sech}\lambda-1\right)\left(a_1^\dagger a_1+a_2^\dagger a_2\right)+a_1a_2\tanh\lambda\right]\colon, \end{aligned} \tag{27}$$

where $\dfrac{A-A^{-1}}{A+A^{-1}}=\tanh\lambda$, $A=\mathrm{e}^\lambda$. Equation (27) is just the two-mode squeezing operator,[15,16]

$$F_2\left(A,0,0\right) = \exp\left[\mathrm{i}\left(Q_1P_2+Q_2P_1\right)\ln A\right], \tag{28}$$

which actually squeezes the entangled state $\left|\xi\right\rangle$ (its conjugate state is $\left|\eta\right\rangle$),

$$F_2\left(A,0,0\right)\left|\xi\right\rangle = \int \frac{\mathrm{d}^2\xi'}{\pi A}\left|\xi'/A\right\rangle\left\langle\xi'\right|\left.\xi\right\rangle = \frac{1}{A}\left|\xi/A\right\rangle. \tag{29}$$

Using the decomposition of the matrix[17]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C/A & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 1 & B/A \\ 0 & 1 \end{pmatrix}, \tag{30}$$

we immediately see that $F_2(A,B,C)$ has its canonical operator (Q,P) representation

$$\begin{aligned} F_2(A,B,C) &= F_2(1,0,C/A)\, F_2(A,0,0)\, F_2(1,B/A,0) \\ &= \exp\left\{\frac{\mathrm{i}C}{4A}\left[(Q_1-Q_2)^2+(P_1+P_2)^2\right]\right\}\exp\left[\mathrm{i}\left(Q_1P_2+Q_2P_1\right)\ln A\right] \\ &\times \exp\left\{-\frac{\mathrm{i}B}{4A}\left[(Q_1+Q_2)^2+(P_1-P_2)^2\right]\right\}. \end{aligned} \tag{31}$$

Next we directly use the EFO to derive $ABCD$ rule in quantum optics for Gaussian beam in two-mode case. From Eq.(16) we see

$$\begin{aligned} F_2(A,B,C)\,|00\rangle = &\frac{2/(C+\mathrm{i}D)}{q_1-\mathrm{i}} \\ &\times \exp\left[-\frac{q_1+\mathrm{i}}{q_1-\mathrm{i}}a_1^\dagger a_2^\dagger\right]|00\rangle, \end{aligned} \tag{32}$$

where we have identified

$$q_1 = \frac{A+\mathrm{i}B}{C+\mathrm{i}D}. \tag{33}$$

Then according to the multiplication rule of two EFOs and Eq.(11) we obtain

$$\begin{aligned} &F_2(A',B',C')\,F_2(A,B,C)\,|00\rangle \\ =&\frac{2}{A''+D''+\mathrm{i}\left(B''-C''\right)}\exp\left[-\frac{A''-D''+\mathrm{i}\left(B''+C''\right)}{A''+D''+\mathrm{i}\left(B''-C''\right)}a_1^\dagger a_2^\dagger\right]|00\rangle \\ =&\frac{2}{A'A+B'C+C'B+D'D+\mathrm{i}\left(A'B+B'D-C'A-D'C\right)} \\ &\times\exp\left[-\frac{A'A+B'C-C'B-D'D+\mathrm{i}\left(A'B+B'D+C'A+D'C\right)}{A'A+B'C+C'B+D'D+\mathrm{i}\left(A'B+B'D-C'A-D'C\right)}a_1^\dagger a_2^\dagger\right]|00\rangle \\ =&\frac{2/(C+\mathrm{i}D)}{(A'q_1+B')-\mathrm{i}\left(C'q_1+D'\right)}\exp\left[-\frac{q_1\left(A'+\mathrm{i}C'\right)+B'+\mathrm{i}D'}{q_1\left(A'-\mathrm{i}C'\right)+B'-\mathrm{i}D'}a_1^\dagger a_2^\dagger\right]|00\rangle. \end{aligned} \tag{34}$$

Using Eq.(33), we can simplify Eq.(34) as

$$F_2(A',B',C')\,F_2(A,B,C)\,|00\rangle = \frac{2/(C+\mathrm{i}D)}{(q_2-\mathrm{i})\left(C'q_1+D'\right)}\exp\left[-\frac{q_2+\mathrm{i}}{q_2-\mathrm{i}}a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{35}$$

where

$$q_2 = \frac{A'q_1+B'}{C'q_1+D'}, \tag{36}$$

which resembles Eq.(2), therefore this is just the new $ABCD$ law for two-mode case in quantum optics.

In summary, the multiplication rule of EFO naturally leads to the quantum optical $ABCD$ law in two-mode case. The operator method as mapping of ray-transfer $ABCD$ matrix has been explicitly shown through EFO's normally ordered form and the coherent state representation.

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