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Squeezing via coupling of Bose–Einstein condensates in a double-well potential with a cavity light field*

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Squeezing via the interaction between the cavity light field and the Bose–Einstein Condensate (BEC) in a double-well potential is considered within the context of the two-mode approximation. For the cavity light field initially in a coherent state, it is shown that by choosing appropriate parameters, quadrature squeezing of the cavity light field can be achieved and it exhibits periodic oscillation. We also study the case in which BEC is tuned to resonance by periodically modulating the trapping potential, and the quadrature squeezing of the cavity field exhibits periodic collapse and revival effect. Both analytic and numerical calculations are performed, and they are found to be in good agreement with each other. The result shows that the quantum statistical properties of the cavity light field can be manipulated by its coupling with the condensates in the double-well potential. On the other hand, dynamical properties of the condensates in the double-well potential will be reflected by the quadrature squeezing of the light field.

Keywords: Bose–Einstein Condensate, double-well potential, squeezing
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Among various Bose–Einstein condensate (BEC) systems, the BEC in a double-well potential has attracted the intense interest in recent years.[1−13] Experimentally, this system was recently realized.[1] Theoretically, Milburn *et al* first proposed a two-mode model to investigate the system.[2] They also proposed a nondestructive measurement scheme for monitoring the Josephson oscillation between BECs in the double well[3] and investigated its application to weak force detection.[4] Other properties of the system such as entanglement[5] and dynamical behaviour[6−10] were investigated in detail. In a few very recent papers,[10−13] the periodically modulated trapping potential has received considerable attention for its potential application to manipulating the tunnelling between BECs in the double-well potential. Investigating BEC in the double-well potential can help us understand more complicate systems such as dynamics,[14] soliton[15] and phase diagram[16] of BEC in an optical lattice or a triple-well potential.[17]

Squeezing is of constant interest, for it offers the possibility of reducing quantum noise in accurate measurement. Kitagawa and Ueda[18] argued that nonlinear interaction could be used to create squeezed spin states (SSSs) in a spin system. In particular, Jääskeläinen *et al*[19] showed that a double-well potential with a time-dependent well separation could be used as a matter-wave beam splitter, and phase squeezing can be achieved. They assessed the phase resolution under the joint effects of collision-driven cubic nonlinearities and quantum tunnelling.[19]

As is known, quantum statistical properties of a BEC can be manipulated by its interaction with a light field.[20] On the other hand, the quantum statistical properties of the light field can also be affected by its interaction with an atomic system. Genes *et al*[21] showed that quadrature squeezing of a light field could be created via atom–cavity field coupling. Saito and Ueda[22] demonstrated that spin squeezing of an ensemble of atoms in a high-Q cavity could be enhanced by the continuous measuring of the photons in the cavity.

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In this article we study the quantum statistical properties of a cavity light field which interacts with a BEC in a double-well potential. The starting-point is that under an appropriate nonlinear interatomic interaction, which can be tuned through the Feshbach resonance, the BEC is prepared in an SSS. And then one of the wells is put into a high-Q cavity. Owing to the interaction between the cavity mode and BEC atoms, the squeezing can be generated in the cavity light field.

For the BEC in a symmetric double-well potential and the well that is put into a high-Q cavity, if the frequency of the cavity mode is far off-resonance with the BEC atoms, under the two-mode approximation,[2] the Hamiltonian of the total system can be written as ($\hbar = 1$)

$$\begin{aligned}\hat{H} =&\delta a^\dagger a + \omega_0(c_1^\dagger c_1 + c_2^\dagger c_2) + \frac{\Omega}{2}(c_1 c_2^\dagger + c_1^\dagger c_2)\\ &+ 2g[(c_1^\dagger)^2 c_1^2 + (c_2^\dagger)^2 c_2^2] - \chi a^\dagger a c_1^\dagger c_1, \end{aligned} \tag{1}$$

where c_i is the one-particle annihilation operator in the localized ground state of well i which obeys the Bose commutation relation $[c_i, c_j^\dagger] = \delta_{ij}$; a is the annihilation operator for the cavity field with the detuning δ; Ω is the tunnelling rate; g is the self-interaction strength; χ is the coupling strength of the BEC with the cavity mode.

For convenience, we introduce the pseudo angular momentum operators

$$\hat{J}_z = \frac{1}{2}(c_1^\dagger c_1 - c_2^\dagger c_2), \tag{2}$$

$$\hat{J}_x = \frac{1}{2}(c_1^\dagger c_2 + c_1 c_2^\dagger), \tag{3}$$

$$\hat{J}_y = \frac{1}{2i}(c_1^\dagger c_2 - c_1 c_2^\dagger). \tag{4}$$

Then Hamiltonian (1) can be rewritten as

$$\hat{H} = \Omega \hat{J}_x + g\hat{J}_z^2 - \chi a^\dagger a \hat{J}_z, \tag{5}$$

where we have neglected the constant energy shifts which depend on the total atom number N, and for simplicity we choose $\delta = N\chi/2$ to remove the linear dispersion.

As the first step of our scheme, the BEC is prepared in an SSS by tuning the self-interaction strength g versus the tunnelling rate Ω. Here we use the dimensionless parameter $G = gN/\Omega$ to perform the analysis. In the absence of the interaction between the BEC and the cavity field, from expression (5) we can see that the ground state of the system can be expressed as

$$|\psi\rangle = \sum_{m=-J}^{J} c_m |J, m\rangle, \tag{6}$$

where $|J, m\rangle$ is the eigenstate of the operators $\hat{J}^2$ and $\hat{J}_z$. To demonstrate that the SSS is formed in the BEC, we perform the numerical calculation on the ground-state distributions in terms of both the probability amplitudes $|c_m|^2$ and $|c_{\theta_m}|^2$ in which $C_{\theta m}$ is the amplitude of the ground state expressed on the basis of the relative phase states[23]

$$|\psi\rangle = \sum_m c_{\theta_m} |\theta_m\rangle, \tag{7}$$

where

$$|\theta_m\rangle = \frac{1}{\sqrt{2J+1}} \sum_{m'=-J}^{J} \exp(\mathrm{i}m'\theta_m)|J, m'\rangle \tag{8}$$

with the relative phase θ_m being given by

$$\theta_m = \theta_0 + \frac{2\pi m}{2J+1} \tag{9}$$

with an arbitrary reference phase θ_0 chosen here to be zero.

The results are shown in Fig.1, where G is chosen to be equal to -1, which is the point discussed in Ref.[18] that is, high phase resolution can be achieved in a double-well potential BEC based interferometry. From Fig.1(b) we can see that the relative phase becomes well defined due to spin squeezing.

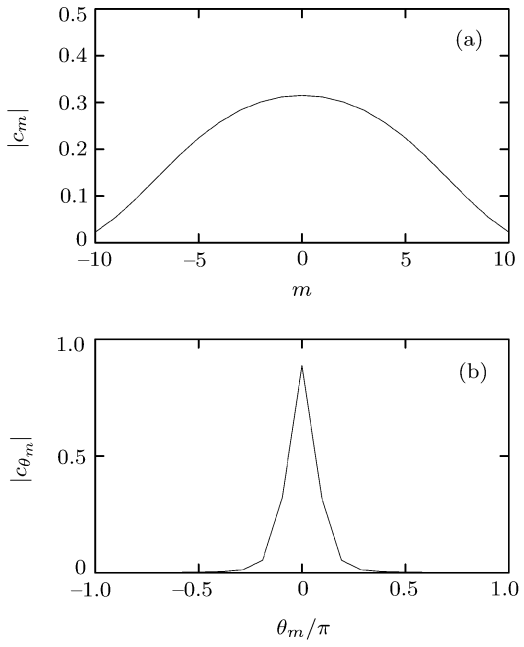


Fig.1. (a) Ground-state distribution $|c_m|$ for $N = 20$, $G = -1$; (b) The distribution of relative phase components $|c_{\theta_m}|$, with the parameters being the same as those in (a).

We assume that the cavity field is initially in a coherent state, $|\alpha\rangle$, and the quadrature operators $\hat{P}$ and $\hat{Q}$ are defined as

$$\hat{Q} = \frac{1}{\sqrt{2}}(a + a^{\dagger}), \quad \hat{P} = -\frac{\mathrm{i}}{\sqrt{2}}(a - a^{\dagger}). \tag{10}$$

Squeezing of the field occurs if the variance of one of these two operators is smaller than the value it would have for the vacuum field. Initially the field is in a coherent state of real amplitude α, then the variance $(\Delta\hat{Q})^2 = (\Delta\hat{P})^2 = 1/2$. We define the following squeezing parameters

$$\xi_Q = \sqrt{2}\Delta\hat{Q}, \quad \text{and} \quad \xi_P = \sqrt{2}\Delta\hat{P}. \tag{11}$$

With this definition, the squeezing occurs for $\xi_Q < 1$ or $\xi_P < 1$.

We perform a numerical calculation based on Hamiltonian(5). The results are shown in Fig.2, where the squeezing parameters ξ_Q and ξ_P each are shown as a function of χt. Figures 2(a) and 2(b) are for atom number $N = 20$ while Figs.2(c) and 2(d) are for $N = 30$. From the figures we can see that as time goes on, the cavity field can exhibit periodic squeezing as a result of the interaction with the BEC which is prepared in the SSS. With the increase of atom number N, the oscillations of squeezing parameters ξ_Q and ξ_P become dense. This is because with a larger N, $\langle \hat{J}_z \rangle$ will have a bigger value with time evolution, which causes dense oscillations of cavity field squeezing parameters.

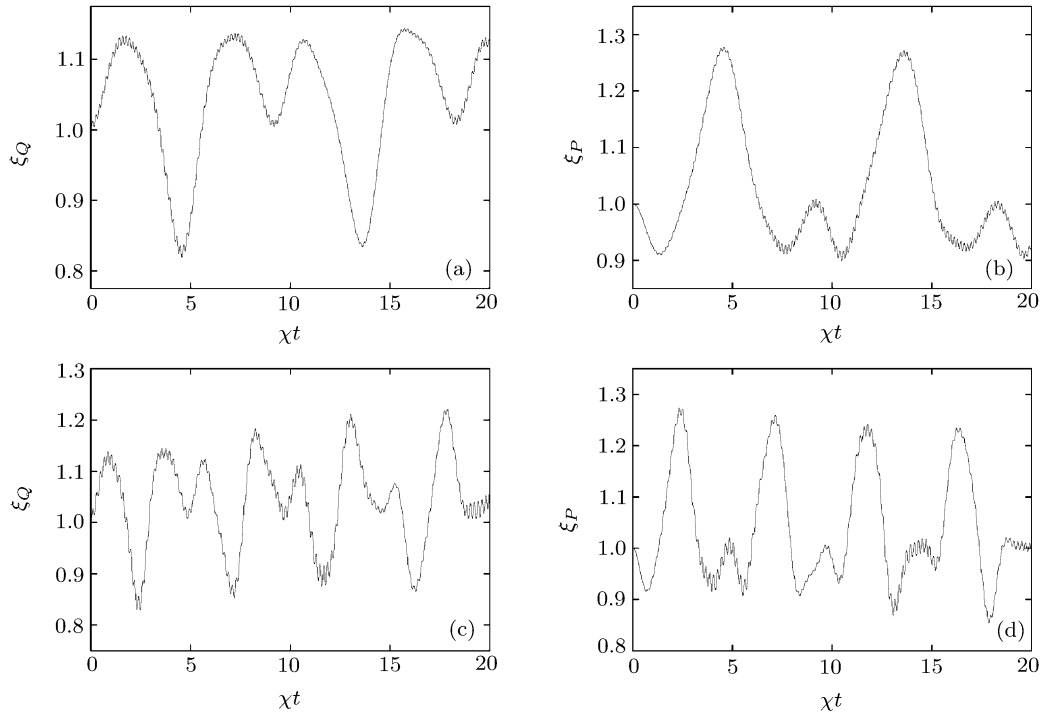


Fig.2. (a) and (b) Squeezing parameters ξ_Q and ξ_P each are shown as a function of χt for $N = 20$, $G = -1$, and $\alpha = 0.4$; (c) and (d) are the same as (a) and (b), except for $N = 30$.

As is shown in Refs.[10–13], the properties of BEC can be manipulated by the periodic modulation of the potential parameters. Here we will study the effects arising from modulating the difference in well depth. At certain modulation frequencies the system exhibits resonances, and then it can be approximated by a two-state model. The validity of the two-mode approximation under the periodic modulation is addressed in Ref.[12].

For a difference in time-dependent energy $\epsilon(t)$ between the two wells, the Hamiltonian of the system can be written as

$$\hat{H} = \epsilon(t)\hat{J}_z + \Omega\hat{J}_x + g\hat{J}_z^2. \tag{12}$$

For periodic modulation, we assume $\epsilon(t) = \epsilon\cos(\omega t)$. Then we perform the unitary transformation with the unitary operator

$$\hat{T}(t) = \exp[-\mathrm{i}\theta(t)\hat{J}_z - \mathrm{i}gt\hat{J}_z^2], \tag{13}$$

where $\theta(t)=\int_0^t \mathrm{d}t'\epsilon(t')=\epsilon\sin(\omega t)/\omega$. Then we have

$$\begin{aligned}\hat{H}' =&\Omega T^{\dagger}\hat{J}_x T\\ =&\frac{\Omega}{2}[\hat{J}_+\mathrm{e}^{-\mathrm{i}\theta(t)+\mathrm{i}gt(2\hat{J}_z+1)}+\mathrm{H.c.}]\\ =&\frac{\Omega}{2}\sum_{n=-\infty}^{\infty}J_n(\epsilon/\omega)\{\hat{J}_+\mathrm{e}^{\mathrm{i}t[g(2\hat{J}_z+1)+n\omega]}+\mathrm{H.c.}\},\end{aligned} \tag{14}$$

where we have performed the Fourier–Bessel expansions and $J_n(x)$ is the nth-order Bessel function. If all BEC atoms are initially in the 2nd well ($|\Psi(t=0)\rangle = |-J\rangle$), then from expression (14) we can see that the coupling between $|-J\rangle$ and $|-J+1\rangle$ is in resonance once the following conditions are satisfied:

$$n\omega+g(-2J+1)=0, \tag{15}$$

$$\left|\frac{\Omega}{2}\frac{\sqrt{(J-m)(J+m+1)}}{n\omega+g(2m+1)}J_n(\epsilon/\omega)\right|\ll 1, \tag{16}$$

where m is an integer lying between $[-J,J]$ but not equal to $-J$. Then the Hamiltonian H' can be approximately written as

$$\hat{H}'=\Omega'(|-J\rangle\langle -J+1|+\mathrm{H.c.}), \tag{17}$$

where $\Omega'=\dfrac{\sqrt{2J}}{2}\Omega J_n(\epsilon/\omega)$. To demonstrate that the above approximation is correct, we perform a numerical calculation on the populations P_{-J} and P_{-J+1} based on Hamiltonian (12). The results are shown in Fig.3. From the figures we can see that Rabi oscillation exists between the states $|-J\rangle$ and $|-J+1\rangle$, but $P_{-J}+P_{-J+1}$ is not strictly equal to 1. This is because there is still a possibility with which the population transfers to the states besides $|-J\rangle$ and $|-J+1\rangle$ although the states $|-J\rangle$ and $|-J+1\rangle$ are in resonance. But for a not very long time evolution as we have discussed here, the above approximation is correct.

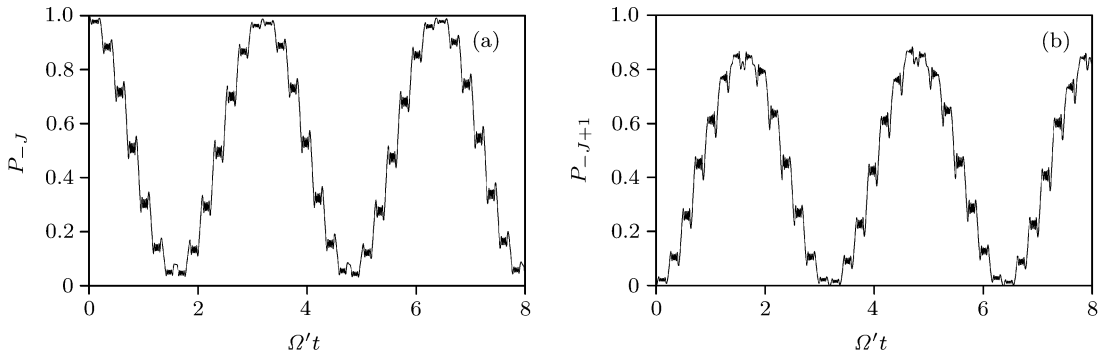


Fig.3. Time dependence of the populations P_m for the states $|m>=|-J>$ and $|m>=|-J+1>$. The parameters are taken as $\Omega=0.25g$, $\epsilon=14g$, $\omega=3g$ and $N=16$.

According to Hamiltonian (17), the state vector of the system can be written as

$$|\psi(t)\rangle=\cos(\Omega' t)|-J\rangle-\mathrm{i}\sin(\Omega' t)|-J+1\rangle. \tag{18}$$

Then as we have done previously, one well is put into a high-Q cavity, and the frequency of the cavity mode is far off-resonance with the BEC atoms. The system Hamiltonian can be written as

$$\hat{H}=\Omega'(|-J\rangle\langle -J+1|+\mathrm{H.c.})-\chi a^{\dagger}a\hat{J}_z. \tag{19}$$

We note that the transformation operator $\hat{T}(t)$ Eq.(13) is only $\hat{J}_z$ dependent and will not cause any transformations of the 2nd term in expression (19).

Just as what we have done previously, we assume that the cavity mode is initially in the coherent state $|\alpha\rangle=\sum_{n=0}^{\infty}c_n|n\rangle$, and the BEC is in the state $\cos\phi|-J\rangle-\mathrm{i}\sin\phi|-J+1\rangle$ (where ϕ can be adjusted), the state vector of the whole system can be written as

$$\begin{aligned}|\psi(t)\rangle=\sum_{n=0}^{\infty}&[c_{1n}(t)|-J\rangle\otimes|n\rangle\\ &+c_{2n}(t)|-J+1\rangle\otimes|n\rangle],\end{aligned} \tag{20}$$

where

$$\begin{aligned}c_{1n}(t)=-\frac{1}{\Omega'}&[D_{1n}(A_n+B_n+\lambda_{2n})\mathrm{e}^{\mathrm{i}(A_n+B_n)t}\\ &+D_{2n}(A_n-B_n+\lambda_{2n})\mathrm{e}^{\mathrm{i}(A_n-B_n)t}],\end{aligned} \tag{21}$$

$$c_{2n}(t)=D_{1n}\mathrm{e}^{\mathrm{i}(A_n+B_n)t}+D_{2n}\mathrm{e}^{\mathrm{i}(A_n-B_n)t} \tag{22}$$

with $\lambda_{1n}=n\chi J$, $\lambda_{2n}=n\chi(J-1)$, $A_n=-(\lambda_{1n}+$

$\lambda_{2n})/2$, $B_n = \sqrt{(\lambda_{1n} - \lambda_{2n})^2 + 4\Omega'^2}$,

$$D_{1n} = -\frac{c_n}{2B_n}[\Omega' \cos\phi - \mathrm{i}(A_n - B_n + \lambda_{2n}) \sin\phi],$$
$$D_{2n} = \frac{c_n}{2B_n}[\Omega' \cos\phi - \mathrm{i}(A_n + B_n + \lambda_{2n}) \sin\phi].$$

According to definition (10), we have

$$\langle \hat{Q} \rangle = \sum_{i=1,2} \sum_{n=0}^{\infty} [\sqrt{n} c^*_{i,n-1}(t) + \sqrt{n+1} c^*_{i,n+1}(t)] c_{i,n}(t)/\sqrt{2},$$

and

$$\langle \hat{Q}^2 \rangle = \frac{1}{2} + \frac{1}{2} \sum_{i=1,2} \sum_{n=0}^{\infty} [\sqrt{n(n-1)} c^*_{i,n-2}(t) + \sqrt{(n+1)(n+2)} c^*_{i,n+2}(t) + 2n c^*_{in}(t)] c_{in}(t).$$

In the limit $|\alpha| \ll 1$, only the first few states are of importance. We can count only the states with $n \leq 2$ by retaining the terms until being proportional to α^2, we have $c_0 \approx 1 - \alpha^2/2$, $c_1 \approx \alpha$, and $c_2 \approx \alpha^2/\sqrt{2}$. For simplicity, we also take the limit $\chi \gg \Omega'$. After some lengthy but straightforward calculation, we can derive the following result with the terms until α^2 terms retained

$$\begin{aligned}(\Delta Q)^2 =& \langle Q^2 \rangle - \langle Q \rangle^2 \\ =& \frac{1}{2} + \alpha^2 \{1 + \cos\phi \cos(\Omega' t + \phi) \cos(\lambda_{12} t) \\ &+ \sin\phi \sin(\Omega' t + \phi) \cos(\lambda_{22} t) \\ &- 2[\cos\phi \cos(\Omega' t + \phi) \cos(\lambda_{11} t) \\ &+ \sin\phi \sin(\Omega' t + \phi) \cos(\lambda_{21} t)]^2\}. \end{aligned} \quad (23)$$

If $\phi = 0$, we have

$$\begin{aligned}(\Delta Q)^2 =& \frac{1}{2} + \alpha^2 [1 + \cos(\Omega' t) \cos(\lambda_{12} t) \\ &- 2\cos^2(\Omega' t) \cos^2(\lambda_{11} t)]. \end{aligned} \quad (24)$$

We can see from the equation that the squeezing parameter will oscillate with time and at some point it will exhibit squeezing. We have also performed numerical calculation from the system Hamiltonian, and the results are shown in Fig.4. From the figure we can see that the squeezing parameter experiences collapse and revival with time going, the numerical results are in good agreement with the analytic results (24).

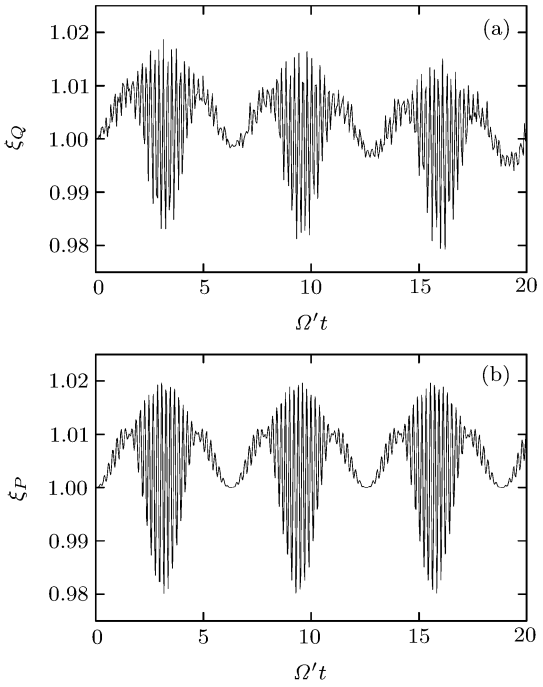


Fig.4. Squeezing parameter ξ_Q is shown as a function of $\Omega' t$, where the parameters are the same as those in Fig.3 with $\alpha = 0.4$. (a) numerical solution; (b) analytical solution.

In conclusion, we have studied squeezing via the interaction between the cavity light field and the BEC in a double-well potential within the context of the two-mode approximation. We show that spin squeezing can be generated in the BEC in a double-well potential by appropriately tuning the system parameter. For the cavity light field initially in a coherent state, quadrature squeezing can be achieved and it exhibits periodic oscillation. With the increase of the BEC atom number, the oscillation becomes dense. Finally we show that the BEC in resonance can be approximated by a two-state model, and the squeezing of the cavity mode can also be generated by their interaction and exhibits periodic collapse and revival. Both analytic and numerical calculations are performed, and we found the results to be in good agreement with each other. The result shows that the quantum statistical properties of the cavity light field can be greatly manipulated by its coupling with the condensates in the double-well potential and it can serve as an estimation of the dynamical properties of the condensates in the double-well potential.

References

[1] Shin Y, Saba M, Pasquini T A, Ketterle W, Pritchard D E and Leanhardt A E 2004 *Phys. Rev. Lett.* **92** 050405
Albiez M, Gati R, Folling J, Hunsmann S, Cristiani M and Oberthaler1 M K 2005 *Phys. Rev. Lett.* **95** 010402

[2] Milburn G J, Corney J F, Wright E M and Walls D F 1997 *Phys. Rev.* A **55** 4318

[3] Corney J F and Milburn G J 1998 *Phys. Rev.* A **58** 2399

[4] Corney J F, Milburn G J and Zhang W P 1999 *Phys. Rev.* A **59** 4630

[5] Ng H T, Law C K and Leung P T 2003 *Phys. Rev.* A **68** 013604
Ng H T and Leung P T 2005 *Phys. Rev.* A **71** 013601

[6] Jääskeläinen M and Meystre P 2005 *Phys. Rev.* A **71** 043603

[7] Steel M J and Collett M J 1998 *Phys. Rev.* A **57** 2920

[8] Jack M W, Collett M J and Walls D F 1996 *Phys. Rev.* A **54** R4625
Ruostekoski J and Walls D F 1998 *Phys. Rev.* A **58** R50

[9] Salmond G L, Holmes C A and Milburn G J 2002 *Phys. Rev.* A **65** 033623

[10] Haroutyunyan H L and Nienhuis G 2004 *Phys. Rev.* A **70** 063603

[11] Choi S and Bigelow N P 2005 *Phys. Rev.* A **72** 033612

[12] Eckardt A, Jinasundera T, Weiss C and Holthaus M 2005 *Phys. Rev. Lett* **95** 200401
Weiss C and Jinasundera T 2005 *Phys. Rev.* A **72** 053626

[13] Wang G F, Fu L B and Liu J 2006 *Phys. Rev.* A **73** 013619

[14] Liu W M, Fan W B, Zheng W M, Liang J Q and Chui S T 2002 *Phys. Rev. Lett.* **88** 170408
Xie Z W, Cao Z X, Kats E I and Liu W M 2005 *Phys. Rev.* A **71** 025601

[15] Xie Z W, Zhang W P, Chui S T and Liu W M 2004 *Phys. Rev.* A **69** 053609
Li Z D, He P B, Li L, Liang J Q and Liu W M 2005 *Phys. Rev.* A **71** 053611

[16] Zheng G P, Liang J Q and W. M. Liu 2005 *Phys. Rev.* A **71** 053608

[17] Liu B, Fu L B, Yang S P and Liu J 2007 *Phys. Rev.* A **75** 033601
Wang G F, Liu B, Fu L B and Zhao H 2007 *Acta Phys. Sin.* **56** 3733 (in Chinese)

[18] Kitagawa M and Ueda M 1993 *Phys. Rev.* A **47** 5138

[19] Jääskeläinen M, Zhang W P and Meystre P 2004 *Phys. Rev.* A **70** 063612

[20] Zeng H P, Zhang W P and Lin F C 1995 *Phys. Rev.* A **52** 2155

[21] Genes C, Berman P R and Rojo A G 2003 *Phys. Rev.* A **68** 043809

[22] Saito H and Ueda M 2003 *Phys. Rev.* A **68** 043820

[23] Pegg D T and Barnett S M 1989 *Phys. Rev.* A **39** 1665; Luis A and Sanchez-Soto L L 1993 *Phys. Rev.* A **48** 4702