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A simple approach to solving double sinh–Gordon equation*

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By introducing an auxiliary ordinary differential equation and solving it by the method of variable separation, abundant types of explicit and exact solutions for the double sinh–Gordon equation are derived in a simple manner.

Keywords: auxiliary ordinary differential equation, double sinh-Gordon equation, explicit and exact solution

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1. Introduction

Finding the explicit and exact travelling wave solutions to nonlinear evolution equations (NEEs) plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluids, plasma and elastic media are often modelled by bell- and kink-shaped solitary wave solutions, and the vibration of masses in a lattice under exponential interaction force is often modelled by the travelling wave solutions of the Toda lattice. In the past decades, much effort has been devoted to the construction of exact solutions of NEEs and many powerful methods have been developed, such as the homogeneous balance method,[1,2] the hyperbolic tangent function expansion method,[3−6] the trial function method,[7−10] the sine–cosine method,[11] the Jacobi elliptic function expansion method,[12−17] the superposition method,[18] the auxiliary equation method,[19−25] the trial equation method,[26−27] the homotopy analysis method,[28−30] and so on. However, there is no general rule for solving NEEs. As a result, it is still a very challenging task to look for new and more powerful approaches to solve NEEs.

In the present paper, by applying the solutions of an auxiliary ordinary differential equation, we construct many explicit and exact solutions to double sinh–Gordon equation in a forthright way.

2. Solutions to double sinh–Gordon equation

The celebrated double sinh–Gordon equation in question reads

$$u_{\rm xt} = \sinh u + \lambda \sinh(2u), \tag{1}$$

which is an object of incessant studies in numerous applications of physics. Now we search for its travelling wave solutions in the form

$$u = u(\xi), \quad \xi = x - vt. \tag{2}$$

Inserting Eq.(2) into Eq.(1) yields the following ordinary differential equation (ODE) with respect to ξ,

$$-vu'' = \sinh u + \lambda \sinh(2u), \tag{3}$$

where primes denote the derivatives for ξ. In order to facilitate the solution of Eq.(3), here we introduce the following additional ODE which $u(\xi)$ in Eq.(3) satisfies,

$$u' = a_0 + \sum_{j=1}^{n} (a_j \cosh u + b_j \sinh u), \tag{4}$$

where a_j $(j = 0, 1, 2, \cdots)$ and b_j $(j = 1, 2, \cdots)$ are constants to be determined, and the positive integer n can be determined by the leading-order analysis method.

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Making use of the following formulae:

$$\sinh u = \frac{\mathrm{e}^u - \mathrm{e}^{-u}}{2},$$
$$\cosh u = \frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}, \tag{5}$$

and substituting Eq.(5) and Eq.(4) into Eq.(3) and applying the leading-order analysis method, we obtain $n = 1$, so that

$$u' = a_0 + a_1 \cosh u + b_1 \sinh u. \tag{6}$$

By means of variable separation and with the aid of mathematica, we can find the solutions to Eq.(6), which are listed as follows.

Case 1, when $a_1 = b_1 = 0, a_0 \neq 0$:

$$u(\xi) = a_0(\xi + \xi_0). \tag{7}$$

Case 2, when $a_0 = b_1 = 0, a_1 \neq 0$:

$$u(\xi) = \ln\left\{\tan\left[\frac{a_1}{2}(\xi + \xi_0)\right]\right\}, \tag{8}$$

$$u(\xi) = \ln\left\{-\cot\left[\frac{a_1}{2}(\xi + \xi_0)\right]\right\}, \tag{9}$$

$$u(\xi) = \ln\{\tan[a_1(\xi + \xi_0)] \pm \sec[a_1(\xi + \xi_0)]\}, \tag{10}$$

$$u(\xi) = \ln\{-\cot[a_1(\xi + \xi_0)] \mp \csc[a_1(\xi + \xi_0)]\}, \tag{11}$$

$$u(\xi) = \ln\left\{\frac{\tan[a_1(\xi + \xi_0)]}{1 \pm \sec[a_1(\xi + \xi_0)]}\right\}, \tag{12}$$

$$u(\xi) = \ln\{\frac{-\cot[a_1(\xi + \xi_0)]}{1 \pm \csc[a_1(\xi + \xi_0)]}\}, \tag{13}$$

$$u(\xi) = \mathrm{arcsinh}\{\tan[a_1(\xi + \xi_0)]\}, \tag{14}$$

$$u(\xi) = \mathrm{arcsinh}\{-\cot[a_1(\xi + \xi_0)]\}, \tag{15}$$

$$u(\xi) = \mathrm{arcsinh}\{\tan[2a_1(\xi + \xi_0)] \pm \sec[2a_1(\xi + \xi_0)]\}, \tag{16}$$

$$u(\xi) = \mathrm{arcsinh}\{-\cot[2a_1(\xi + \xi_0)] \mp \csc[2a_1(\xi + \xi_0)]\}, \tag{17}$$

$$u(\xi) = \mathrm{arcsinh}\left\{\frac{\tan[2a_1(\xi + \xi_0)]}{1 \pm \sec[2a_1(\xi + \xi_0)]}\right\}, \tag{18}$$

$$u(\xi) = \mathrm{arcsinh}\left\{\frac{-\cot[2a_1(\xi + \xi_0)]}{1 \pm \csc[2a_1(\xi + \xi_0)]}\right\}, \tag{19}$$

$$u(\xi) = \mathrm{arccosh}\{\pm \sec[a_1(\xi + \xi_0)]\}, \tag{20}$$

$$u(\xi) = \mathrm{arccosh}\{\pm \csc[a_1(\xi + \xi_0)]\}, \tag{21}$$

$$u(\xi) = 2\mathrm{arctanh}\left\{\tan\left[\frac{1}{2}a_1(\xi + \xi_0)\right]\right\}, \tag{22}$$

$$u(\xi) = 2\mathrm{arctanh}\left\{-\cot\left[\frac{1}{2}a_1(\xi + \xi_0)\right]\right\}, \tag{23}$$

$$u(\xi) = 2\mathrm{arctanh}\{\tan[a_1(\xi + \xi_0)] \pm \sec[a_1(\xi + \xi_0)]\}, \tag{24}$$

$$u(\xi) = 2\mathrm{arctanh}\{-\cot[a_1(\xi + \xi_0)] \mp \csc[a_1(\xi + \xi_0)]\}, \tag{25}$$

$$u(\xi) = 2\mathrm{arctanh}\left\{\frac{\tan[a_1(\xi + \xi_0)]}{1 \pm \sec[a_1(\xi + \xi_0)]}\right\}, \tag{26}$$

$$u(\xi) = 2\mathrm{arctanh}\left\{\frac{-\cot[a_1(\xi + \xi_0)]}{1 \pm \csc[a_1(\xi + \xi_0)]}\right\}. \tag{27}$$

Case 3, when $a_0 = a_1 = 0, b_1 \neq 0$:

$$u(\xi) = 2\mathrm{arctanh}\{\pm \exp[b_1(\xi + \xi_0)]\}, \tag{28}$$

$$u(\xi) = \ln\left\{-\tanh\left[\frac{b_1}{2}(\xi + \xi_0)\right]\right\}, \tag{29}$$

$$u(\xi) = \ln\left\{-\coth\left[\frac{b_1}{2}(\xi + \xi_0)\right]\right\}, \tag{30}$$

$$u(\xi) = \ln\{-\tanh[b_1(\xi + \xi_0)] \mp \mathrm{isech}[b_1(\xi + \xi_0)]\}, \tag{31}$$

$$u(\xi) = \ln\{-\coth[b_1(\xi + \xi_0)] \mp \mathrm{csch}[b_1(\xi + \xi_0)]\}, \tag{32}$$

$$u(\xi) = \ln\left\{\frac{-\tanh[b_1(\xi + \xi_0)]}{1 \pm \mathrm{sech}[b_1(\xi + \xi_0)]}\right\}, \tag{33}$$

$$u(\xi) = \ln\left\{\frac{-\coth[b_1(\xi + \xi_0)]}{1 \pm \mathrm{icsch}[b_1(\xi + \xi_0)]}\right\}, \tag{34}$$

$$u(\xi) = \mathrm{arcsinh}\{\pm \mathrm{csch}[b_1(\xi + \xi_0)]\}, \tag{35}$$

$$u(\xi) = \mathrm{arcsinh}\{\pm \mathrm{isech}[b_1(\xi + \xi_0)]\}, \tag{36}$$

$$u(\xi) = \mathrm{arccosh}\{\pm \tanh[b_1(\xi + \xi_0)]\}, \tag{37}$$

$$u(\xi) = \mathrm{arccosh}\{\pm \coth[b_1(\xi + \xi_0)]\}, \tag{38}$$

$$u(\xi) = \mathrm{arccosh}\{\pm \tanh[2b_1(\xi + \xi_0)] \mp \mathrm{isech}[2b_1(\xi + \xi_0)]\}, \tag{39}$$

$$u(\xi) = \mathrm{arccosh}\{\pm \coth[2b_1(\xi + \xi_0)] \mp \mathrm{csch}[2b_1(\xi + \xi_0)]\}, \tag{40}$$

$$u(\xi) = \mathrm{arccosh}\left\{\frac{\pm \tanh[2b_1(\xi + \xi_0)]}{1 \pm \mathrm{sech}[2b_1(\xi + \xi_0)]}\right\}, \tag{41}$$

$$u(\xi) = \mathrm{arccosh}\left\{\frac{\pm \coth[2b_1(\xi + \xi_0)]}{1 \pm \mathrm{icsch}[2b_1(\xi + \xi_0)]}\right\}. \tag{42}$$

Case 4, when $a_1 = 0, a_0 \neq 0, b_1 \neq 0$:

$$u(\xi) = \ln\left\{-\frac{1}{b_1}(a_0 + \sqrt{a_0^2 + b_1^2} \times \tanh\left[\frac{\sqrt{a_0^2 + b_1^2}}{2}(\xi + \xi_0)\right]\right\}, \tag{43}$$

$$u(\xi) = \ln\left\{-\frac{1}{b_1}(a_0 + \sqrt{a_0^2 + b_1^2} \times \coth\left[\frac{\sqrt{a_0^2 + b_1^2}}{2}(\xi + \xi_0)\right]\right\}, \tag{44}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{b_1^2-\left[a_0+\sqrt{a_0^2+b_1^2}\tanh\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2b_1\left[a_0+\sqrt{a_0^2+b_1^2}\tanh\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{45}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{b_1^2-\left[a_0+\sqrt{a_0^2+b_1^2}\coth\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2b_1\left[a_0+\sqrt{a_0^2+b_1^2}\coth\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{46}$$

$$u(\xi)=\operatorname{arccosh}\left\{\frac{-b_1^2-\left[a_0+\sqrt{a_0^2+b_1^2}\tanh\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2b_1\left[a_0+\sqrt{a_0^2+b_1^2}\tanh\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{47}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{-b_1^2-\left[a_0+\sqrt{a_0^2+b_1^2}\coth\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2b_1[a_0+\sqrt{a_0^2+b_1^2}\coth\left(\frac{\sqrt{a_0^2+b_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}. \tag{48}$$

Case 5, when $b_1=0, a_0\neq 0, a_1\neq 0$:
Case 5.1, for $a_0^2>a_1^2$:

$$u(\xi)=\ln\left\{\frac{1}{a_1}\left[-a_0-\sqrt{a_0^2-a_1^2}\tanh\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]\right\}, \tag{49}$$

$$u(\xi)=\ln\left\{\frac{1}{a_1}\left[-a_0-\sqrt{a_0^2-a_1^2}\coth\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]\right\}, \tag{50}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{a_1^2-\left[a_0+\sqrt{a_0^2-a_1^2}\tanh\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[a_0+\sqrt{a_0^2-a_1^2}\tanh\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{51}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{a_1^2-\left[a_0+\sqrt{a_0^2-a_1^2}\coth\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[a_0+\sqrt{a_0^2-a_1^2}\coth\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{52}$$

$$u(\xi)=\operatorname{arccosh}\left\{\frac{-a_1^2-\left[a_0+\sqrt{a_0^2-a_1^2}\tanh\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[a_0+\sqrt{a_0^2-a_1^2}\tanh\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{53}$$

$$u(\xi)=\operatorname{arccosh}\left\{\frac{-a_1^2-\left[a_0+\sqrt{a_0^2-a_1^2}\coth\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[a_0+\sqrt{a_0^2-a_1^2}\coth\left(\frac{\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right]}\right\}. \tag{54}$$

Case 5.2, for $a_0^2<a_1^2$:

$$u(\xi)=\ln\left\{\frac{1}{a_1}\left[-a_0+\sqrt{a_1^2-a_0^2}\tan\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]\right\}, \tag{55}$$

$$u(\xi)=\ln\left\{\frac{1}{a_1}\left[-a_0-\sqrt{a_1^2-a_0^2}\cot\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]\right\}, \tag{56}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{-a_1^2+\left[-a_0+\sqrt{a_1^2-a_0^2}\tan\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[-a_0+\sqrt{a_1^2-a_0^2}\tan\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{57}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{a_1^2-\left[a_0+\sqrt{a_1^2-a_0^2}\cot\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[a_0+\sqrt{a_1^2-a_0^2}\cot\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]}\right\}, \tag{58}$$

$$u(\xi)=\operatorname{arccosh}\left\{\frac{a_1^2+\left[-a_0+\sqrt{a_1^2-a_0^2}\tan\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[-a_0+\sqrt{a_1^2-a_0^2}\tan\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]}\right\},, \tag{59}$$

$$u(\xi)=\operatorname{arccosh}\left\{\frac{-a_1^2-\left[a_0+\sqrt{a_1^2-a_0^2}\cot\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]^2}{2a_1\left[a_0+\sqrt{a_1^2-a_0^2}\cot\left(\frac{\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right]}\right\}. \tag{60}$$

Case 5.3, for $a_1=a_0$:

$$u(\xi)=\ln\left[-1-\frac{2}{a_0(\xi+\xi_0)}\right], \tag{61}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{-2(1+a_0(\xi+\xi_0))}{a_0(\xi+\xi_0)[2+a_0(\xi+\xi_0)]}\right\}, \tag{62}$$

$$u(\xi)=\operatorname{arccosh}\left\{\frac{-a_0^2(\xi+\xi_0)^2-2a_0(\xi+\xi_0)-2}{a_0(\xi+\xi_0)[2+a_0(\xi+\xi_0)]}\right\}. \tag{63}$$

Case 5.4, for $a_1=-a_0$:

$$u(\xi)=\ln\left[1+\frac{2}{a_0(\xi+\xi_0)}\right], \tag{64}$$

$$u(\xi)=\operatorname{arcsinh}\left\{\frac{2(1+a_0(\xi+\xi_0))}{a_0(\xi+\xi_0)[2+a_0(\xi+\xi_0)]}\right\}, \tag{65}$$

$$u(\xi)=\operatorname{arccosh}\left\{\frac{a_0^2(\xi+\xi_0)^2+2a_0(\xi+\xi_0)+2}{a_0(\xi+\xi_0)[2+a_0(\xi+\xi_0)]}\right\}. \tag{66}$$

Putting Eq.(6) into Eq.(3) and equating the coefficients of sinh and cosh functions yields a set of algebraic equations with respect to the undetermined constants a_0, a_1, b_1 and v, as follows:

$$\begin{aligned}&a_0b_1v=0,\\&a_1b_1v=0,\\&a_0a_1v+1=0,\\&\frac{v}{2}(a_1^2+b_1^2)+\lambda=0.\end{aligned} \tag{67}$$

After solving the above system of algebraic equations, we obtain

$$a_1=2\lambda a_0,\quad b_1=0,\quad a_0\neq 0,\quad v=-\frac{1}{2\lambda a_0^2}. \tag{68}$$

Employing Eq.(68) and considering case 5, we acquire the following solutions of the double sinh–Gordon equation (1):

$$u(x,t)=\ln\left\{\frac{1}{2\lambda a_0}\left[-a_0-|a_0|\sqrt{1-4\lambda^2}\tanh\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]\right\}\quad\text{for}\quad|\lambda|<\frac{1}{2}, \tag{69}$$

$$u(x,t)=\ln\left\{\frac{1}{2\lambda a_0}\left[-a_0-|a_0|\sqrt{1-4\lambda^2}\coth\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]\right\}\quad\text{for}\quad|\lambda|<\frac{1}{2}, \tag{70}$$

$$u(x,t)=\operatorname{arcsinh}\left\{\frac{4\lambda^2a_0^2-\left[a_0+|a_0|\sqrt{1-4\lambda^2}\tanh\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[a_0+|a_0|\sqrt{1-4\lambda^2}\tanh\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|<\frac{1}{2}, \tag{71}$$

$$u(x,t)=\operatorname{arcsinh}\left\{\frac{4\lambda^2a_0^2-\left[a_0+|a_0|\sqrt{1-4\lambda^2}\coth\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[a_0+|a_0|\sqrt{1-4\lambda^2}\coth\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|<\frac{1}{2},\tag{72}$$

$$u(x,t)=\operatorname{arccosh}\left\{\frac{-4\lambda^2a_0^2-\left[a_0+|a_0|\sqrt{1-4\lambda^2}\tanh\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[a_0+|a_0|\sqrt{1-4\lambda^2}\tanh\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|<\frac{1}{2},\tag{73}$$

$$u(x,t)=\operatorname{arccosh}\left\{\frac{-4\lambda^2a_0^2-\left[a_0+|a_0|\sqrt{1-4\lambda^2}\coth\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[a_0+|a_0|\sqrt{1-4\lambda^2}\coth\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|<\frac{1}{2},\tag{74}$$

$$u(x,t)=\ln\left\{\frac{1}{2\lambda a_0}\left[-a_0+|a_0|\sqrt{4\lambda^2-1}\tan\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]\right\}\quad\text{for}\quad|\lambda|>\frac{1}{2},\tag{75}$$

$$u(x,t)=\ln\left\{\frac{1}{2\lambda a_0}\left[-a_0-|a_0|\sqrt{4\lambda^2-1}\cot\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]\right\}\quad\text{for}\quad|\lambda|>\frac{1}{2},\tag{76}$$

$$u(x,t)=\operatorname{arcsinh}\left\{\frac{-4\lambda^2a_0^2+\left[-a_0+|a_0|\sqrt{4\lambda^2-1}\tan\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[-a_0+|a_0|\sqrt{4\lambda^2-1}\tan\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|>\frac{1}{2},\tag{77}$$

$$u(x,t)=\operatorname{arcsinh}\left\{\frac{4\lambda^2a_0^2-\left[a_0+|a_0|\sqrt{4\lambda^2-1}\cot\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[a_0+|a_0|\sqrt{4\lambda^2-1}\cot\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|>\frac{1}{2},\tag{78}$$

$$u(x,t)=\operatorname{arccosh}\left\{\frac{4\lambda^2a_0^2+\left[-a_0+|a_0|\sqrt{4\lambda^2-1}\tan\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[-a_0+|a_0|\sqrt{4\lambda^2-1}\tan\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|>\frac{1}{2},\tag{79}$$

$$u(x,t)=\operatorname{arccosh}\left\{\frac{-4\lambda^2a_0^2-\left[a_0+|a_0|\sqrt{4\lambda^2-1}\cot\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]^2}{4\lambda a_0\left[a_0+|a_0|\sqrt{4\lambda^2-1}\cot\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}(x+\frac{1}{2\lambda a_0^2}t+\xi_0)\right)\right]}\right\}\quad\text{for}\quad|\lambda|>\frac{1}{2},\tag{80}$$

$$u(x,t)=\ln\left[-1-\frac{2}{a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)}\right]\quad\text{for}\quad\lambda=\frac{1}{2},\tag{81}$$

$$u(x,t)=\operatorname{arcsinh}\left\{\frac{-2\left[1+a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\right]}{a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\left[2+a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\right]}\right\}\quad\text{for}\quad\lambda=\frac{1}{2},\tag{82}$$

$$u(x,t)=\operatorname{arccosh}\left\{\frac{-a_0^2\left(x+\frac{1}{a_0^2}t+\xi_0\right)^2-2a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)-2}{a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\left[2+a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\right]}\right\}\quad\text{for}\quad\lambda=\frac{1}{2},\tag{83}$$

$$u(x,t)=\ln\left[1+\frac{2}{a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)}\right] \quad \text{for} \quad \lambda=-\frac{1}{2}, \tag{84}$$

$$u(x,t)=\operatorname{arcsinh}\left\{\frac{2\left[1+a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\right]}{a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\left[2+a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\right]}\right\} \quad \text{for} \quad \lambda=-\frac{1}{2}, \tag{85}$$

$$u(x,t)=\operatorname{arccosh}\left\{\frac{a_0^2\left(x+\frac{1}{a_0^2}t+\xi_0\right)^2+2a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)+2}{a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\left[2+a_0\left(x+\frac{1}{a_0^2}t+\xi_0\right)\right]}\right\} \quad \text{for} \quad \lambda=-\frac{1}{2}. \tag{86}$$

In the above equations, a_0 and ξ_0 are arbitrary real constants with $a_0 \neq 0$.

Finally, it is worthwhile to mention that we have verified all solutions obtained by putting them back into the original equation with the help of Mathematica.

3. Conclusions

In summary, by introducing an additional ordinary differential equation and solving it by means of the method of variable separation, we find many explicit and exact solutions of double sinh–Gordon equation in a forthright way. The key idea of our approach is to take full advantage of manifold types of solutions for an auxiliary ordinary differential equation introduced in this paper. Its advantage is to transform the problem for solving a NPDE to that of solving an ODE whose solutions are easily found. Therefore, the problem under study is significantly simplified. How to generalize this technique to solve other NEEs in mathematical physics deserves further investigation.

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