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The multicomponent (2+1)-dimensional Glachette–Johnson (GJ) equation hierarchy and its super-integrable coupling system*

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This paper presents a set of multicomponent matrix Lie algebra, which is used to construct a new loop algebra $\tilde{A}_M$. By using the Tu scheme, a Liouville integrable multicomponent equation hierarchy is generated, which possesses the Hamiltonian structure. As its reduction cases, the multicomponent (2+1)-dimensional Glachette–Johnson (GJ) hierarchy is given. Finally, the super-integrable coupling system of multicomponent (2+1)-dimensional GJ hierarchy is established through enlarging the spectral problem.

Keywords: matrix Lie algebra, multicomponent GJ hierarchy, super-integrable coupling system
PACC: 0340K, 0220, 0365G

1. Introduction

Tu[1] proposed a simple and efficient method for obtaining integrable Hamiltonian hierarchies of soliton equations with infinite dimensions. The key idea of his work is to construct a loop algebra as follows:

$$\begin{cases} h(n)=\begin{pmatrix}\lambda^n & 0\\ 0 & -\lambda^n\end{pmatrix},\quad e(n)=\begin{pmatrix}0 & \lambda^n\\ 0 & 0\end{pmatrix},\quad f(n)=\begin{pmatrix}0 & 0\\ \lambda^n & 0\end{pmatrix},\\ [h(m),e(n)]=2e(m+n),[h(m),f(n)]=-2f(m+n),[e(m),f(n)]=h(m+n),\\ \deg h(n)=\deg e(n)=\deg f(n)=n, \end{cases} \tag{1}$$

where deg is the different gradation.

Under the frame of the zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0, \tag{2}$$

a good many integrable hierarchies were derived, such as the Ablowitz–Kaup–Newll–Segure (AKNS) hierarchy, the Kaup–Newll (KN) hierarchy, the Wadati–Konno–Ichika (WKI) hierarchy, and the Bopte–Pempinelli–Tu (BPT) hierarchy, etc.[2−7].

As we all know, Eq.(2) is compatible with the following Lax pairs:

$$\begin{aligned}&\varphi_x=U(u,\lambda)\varphi,\\ &\varphi_t=V^{(n)}(u,\lambda)\varphi,\quad \lambda_t=0.\end{aligned} \tag{3}$$

Recently, Zhou[8] introduced Lax pairs as follows:

$$\begin{aligned}&\varphi_y=A\varphi_x+U\varphi,\\ &\varphi_t=B\varphi_x+V^{(n)}\varphi,\quad \lambda_t=0,\end{aligned} \tag{4}$$

its compatibility is given by

$$\begin{aligned}&U_t-V_y^{(n)}+[U,V^{(n)}]+AV_x^{(n)}-BU_x=0,\\ &[A,V^{(n)}]=[B,U],\end{aligned} \tag{5}$$

where $A=\mathrm{diag}(a_1,\,a_2,\cdots,\,a_n)$, $B=\mathrm{diag}(b_1,\,b_2,\cdots,\,b_n)$.

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In this paper, we take A and B as the unit matrices and try to see for multicomponent (2+1)-dimensional integrable equation hierarchies by using Tu scheme and the generalized zero curvature equation(5). As an application example, we obtain a multicomponent (2+1)-dimensional Glachette–Johnson (GJ) hierarchy. Finally, the super-integrable coupling system of the multicomponent (2+1)-dimensional GJ hierarchy is presented through enlarging spectral problem.

2. The multicomponent (2+1)-dimensional GJ hierarchy

First, a few notations are defined.

Definition 1[9,10] Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_M)^{\mathrm{T}}$, $\beta = (\beta_1, \beta_2, \cdots, \beta_M)^{\mathrm{T}}$, their product is defined as

$$\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \cdots, \alpha_M\beta_M)^{\mathrm{T}}, \tag{6}$$

where M stands for an arbitrary positive integer.

It follows from Definition 1 that set $a = (a_1, a_2, \cdots, a_M)$, $b = (b_1, b_2, \cdots, b_M)$, we have their product $a * b$ as

$$a * b = b * a = (a_1b_1, a_2b_2, \cdots, a_Mb_M).$$

Definition 2 If $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_M)^{\mathrm{T}}$, $a = (a_1, a_2, \cdots, a_M)$, a kind of product between α and a is given by

$$\alpha \bullet a = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_M \end{pmatrix} (a_1, a_2, \cdots, a_M) = \begin{pmatrix} \alpha_1 a_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_M a_M \end{pmatrix}. \tag{7}$$

In what follows, we construct a set of multicomponent matrix Lie algebra by using of the Definitions 1 and 2:

$$e_1 = \begin{pmatrix} -M & 0 \\ 0 & I_M \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & I_{1\times M} \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ I_{M\times 1} & 0, \end{pmatrix}$$

$$[e_1, e_2] = -(M+1)e_2, [e_1, e_3] = (M+1)e_3, [e_2, e_3] = -e_1, \tag{8}$$

where I_M is a $M \times M$ unit matrix, $I_{1\times M} = (1, 1, \cdots, 1)_{1\times M}$, $I_{M\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{M\times 1}$.

Definition 3 Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_M)^{\mathrm{T}}$, $a = (a_1, a_2, \cdots, a_M)$, c is a scalar, we define the following operation relations:

$$ce_1 = \begin{pmatrix} -cM & 0 \\ 0 & cI_M \end{pmatrix}, \quad ae_2 = \begin{pmatrix} 0 & a \bullet I_{1\times M} \\ 0 & 0 \end{pmatrix}, \quad \alpha e_3 = \begin{pmatrix} 0 & 0 \\ \alpha * I_{M\times 1} & 0 \end{pmatrix}.$$

In terms of Eq.(8), a loop algebra $\tilde{A}_M$ is presented as

$$\begin{aligned} &e_i(n) = e_i\lambda^n, \\ &[e_1(m), e_2(n)] = -(M+1)e_2(m+n), \\ &[e_1(m), e_3(n)] = (M+1)e_3(m+m), \\ &[e_2(m), e_3(n)] = -e_1(m+n), \\ &\deg e_i(n) = n, i = (1, 2, 3). \end{aligned} \tag{9}$$

Consider an isospectral problem

$$\varphi_x = U\varphi, \quad \lambda_t = 0,$$

$$U = -e_1(1) + se_1(0) + qe_2(0) + re_3(0), \tag{10}$$

where $q = (q_1, q_2, \cdots, q_M)$, $r = (r_1, r_2, \cdots, r_M)^{\mathrm{T}}$, s is a single function.

Set

$$V^{(n)} = \sum_{n\geq 0}(a(n)\lambda^n + b(n)\lambda^n + c(n)\lambda^n),$$

where $a(n)$ is a scalar smooth function, $b(n) = (b_1(n), b_2(n), \cdots, b_M(n))$, $c(n) = (c_1(n), c_2(n), \cdots, c_M(n))^{\mathrm{T}}$, $n = 1, 2, \cdots$.

Solving the generalized stationary zero curvature equation

$$V_y^{(n)} - V_x^{(n)} = [U, V^{(n)}], \tag{11}$$

we obtain

$$\begin{aligned} &a_y(n) - a_x(n) = \frac{1}{M}(qc(n) - rb(n)), \\ &b_y(n) - b_x(n) = (M+1)(-b(n+1) + sb(n)) - (M+1)qa(n), \\ &c_y(n) - c_x(n) = -(M+1)(-c(n+1) + sc(n)) + (M+1)ra(n). \end{aligned} \tag{12}$$

Taking

$$a(0) = \alpha, \quad b(0) = 0 = (0, 0, \cdots, 0)_{M\times 1}, \quad c(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{M\times 1},$$

we have

$$a(1) = 0, c(1) = -\alpha r = \begin{pmatrix} -\alpha r_1 \\ -\alpha r_2 \\ \vdots \\ -\alpha r_M \end{pmatrix}, \quad b(1) = (-\alpha q_1, -\alpha q_2, \cdots, -\alpha q_M),$$

$$b(2) = \left(-\alpha q_1 s_1 + \frac{\alpha(q_{1x} - q_{1y})}{M+1}, -\alpha q_2 s_2 + \frac{\alpha(q_{2x} - q_{2y})}{M+1}, \cdots, -\alpha q_M s_M + \frac{\alpha(q_{Mx} - q_{My})}{M+1} \right),$$

$$c(2) = \begin{pmatrix} -\alpha r_1 s_1 - \dfrac{\alpha(r_{1x} - r_{1y})}{M+1} \\ -\alpha r_2 s_2 - \dfrac{\alpha(r_{2x} - r_{2y})}{M+1} \\ \vdots \\ -\alpha r_M s_M - \dfrac{\alpha(r_{Mx} - r_{My})}{M+1} \end{pmatrix}, \quad a(2) = \frac{-\alpha}{M^2 + M} \sum_{i=1}^{M} q_i r_i. \tag{13}$$

The Eq.(11) can be written as

$$\begin{aligned} &(\lambda^n V^{(n)})_{+y} - (\lambda^n V^{(n)})_{+x} + [U, V^{(n)}]_+ \\ &= (\lambda^n V^{(n)})_{-x} - (\lambda^n V^{(n)})_{-y} - [U, V^{(n)}]_-. \end{aligned} \tag{14}$$

We find that the terms on the left-hand side in Eq.(14) are of degree ≥ 0, while the terms on the right-hand side in Eq.(14) are of degree ≤ 0. Therefore,

$$\begin{aligned} &(\lambda^n V^{(n)})_{+y} - (\lambda^n V^{(n)})_{+x} + [U, V^{(n)}]_+ \\ &= (M+1)b(n+1)e_2(0) - (M+1)c(n+1)e_3(0). \end{aligned}$$

Taking $\Delta_n = a(n+1)e_1(0), \tilde{V}^{(n)} = (\lambda^n V^{(n)})_+ + \Delta_n$, we find that the generalized zero curvature equation

$$U_t - \tilde{V}_y^{(n)} + \tilde{V}_x^{(n)} + [U, \tilde{V}^{(n)}] - U_x = 0 \tag{15}$$

leads to

$$\begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix}_t = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & 0 & \partial_x \end{pmatrix} \begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix} + \begin{pmatrix} -(M+1)b(n+1) + (M+1)qa(n+1) \\ (M+1)c(n+1)^{\mathrm{T}} - (M+1)r^{\mathrm{T}}a(n+1)^{\mathrm{T}} \\ \dfrac{1}{M}qc(n+1) - \dfrac{1}{M}rb(n+1) + (M+1)a_x(n+1) \end{pmatrix}$$

$$
\begin{aligned}
=&J_1\begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix} + \begin{pmatrix} 0 & (\partial_y-\partial_x)-(M+1)s & -\dfrac{1}{M}q \\ (\partial_y-\partial_x)-(M+1)s & 0 & -\dfrac{1}{M}r \\ \dfrac{1}{M}q^{\mathrm{T}} & \dfrac{1}{M}r^{\mathrm{T}} & -\dfrac{1}{M}(\partial_y-\partial_x) \end{pmatrix} \begin{pmatrix} c(n) \\ b^{\mathrm{T}}(n) \\ -(M^2+M)a(n) \end{pmatrix} \\
=&J_1\begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix} + J_2\begin{pmatrix} c(n) \\ b^{\mathrm{T}}(n) \\ -(M^2+M)a(n) \end{pmatrix}, \qquad (16)
\end{aligned}
$$

where J_2 is the Hamiltonian operator.

From Eq.(12), we obtain a recurrence operator

$$
\begin{pmatrix} c(n+1) \\ b^{\mathrm{T}}(n+1) \\ -(M^2+M)a(n+1) \end{pmatrix} = L\begin{pmatrix} c(n) \\ b^{\mathrm{T}}(n) \\ -(M^2+M)a(n) \end{pmatrix},
$$

where

$$
L = \begin{pmatrix} s+\dfrac{(\partial_y-\partial_x)}{M+1} & 0 & \dfrac{r}{M^2+M} \\ 0 & s+\dfrac{(\partial_x-\partial_y)}{M+1} & \dfrac{q}{M^2+M} \\ -(\partial_y^{-1}-\partial_x^{-1})q^{\mathrm{T}}(\partial_y-\partial_x)-\dfrac{1}{M+1}q^{\mathrm{T}}s & (\partial_y^{-1}-\partial_x^{-1})r(\partial_x-\partial_y)+\dfrac{1}{M}rs & 0 \end{pmatrix}.
$$

Thus, the system (16) can be written as

$$
U_t = \begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix}_t = J_1\begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix} + J_2L^n\begin{pmatrix} c(1) \\ b^{\mathrm{T}}(1) \\ -(M^2+M)a(1) \end{pmatrix}. \qquad (17)
$$

A direct calculation reads

$$
\begin{aligned}
&\left\langle V, \frac{\partial U}{\partial q}\right\rangle = c(0), \quad \left\langle V, \frac{\partial U}{\partial r}\right\rangle = b(0), \\
&\left\langle V, \frac{\partial U}{\partial s}\right\rangle = (M^2+M)a(0), \\
&\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = -(M^2+M)a(0).
\end{aligned}
$$

Substituting the above formulae into the trace identity we have

$$
\frac{\delta}{\delta u}\left(\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle\right) = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} \left\langle V, \dfrac{\partial U}{\partial q}\right\rangle \\ \left\langle V, \dfrac{\partial U}{\partial r}\right\rangle \\ \left\langle V, \dfrac{\partial U}{\partial s}\right\rangle \end{pmatrix}, \qquad (18)
$$

and obtain

$$
\begin{aligned}
&\left(\frac{\delta}{\delta q}, \frac{\delta}{\delta r}\right)(-(M^2+M)a(n+1)) \\
=&\left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)(c(n), b(n), (M^2+M)a(n)),
\end{aligned}
$$

or

$$
\begin{aligned}
&\left(\frac{\delta}{\delta u}\right)(-(M^2+M)a(n+1)) \\
=&(\gamma-n)(c(n), b(n), (M^2+M)a(n)). \qquad (19)
\end{aligned}
$$

To fix the constant γ, we simply set $n=0$ in Eq.(19) and find that $\gamma=0$. Therefore, we conclude that

$$
\begin{aligned}
&\frac{\delta}{\delta u}H_n = (c(n), b(n), (M^2+M)a(n)), \\
&H_n = \int \frac{-(M^2+M)a(n+1)}{n+1}\mathrm{d}x. \qquad (20)
\end{aligned}
$$

Hence, we obtain Hamiltonian structure of the multicomponent (2+1)-dimensional GJ hierarchy, when $n=1$,

$$
H_1 = \alpha qr, \quad \text{i.e.} \quad \begin{pmatrix} H_{11} \\ H_{12} \\ \vdots \\ H_{1M} \end{pmatrix} = \begin{pmatrix} \alpha q_1 r_1 \\ \alpha q_2 r_2 \\ \vdots \\ \alpha q_M r_M \end{pmatrix}. \qquad (21)
$$

Thus, the Hamiltonian structure of the system (16) is given by

$$U_t = \begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix}_t = J_1 \begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \end{pmatrix} + J_2 \frac{\delta H_1}{\delta u}. \tag{22}$$

It is easy to verify that $J_2 L = -L^* J_2$. Therefore, when $\partial_y = \partial_y^{-1} = 0$, the system (16) is Liouville integrable.

3. The super-integrable coupling system of multicomponent (2+1)-dimensional GJ hierarchy

To construct the super-integrable coupling system of the multicomponent (2+1)-dimensional GJ soliton equation hierarchy, we use an idea of enlarging spectral problem[11−18] to make an enlarged spectrum as follows:

$$\bar{\phi}_y - \bar{\phi}_x = \bar{U}\bar{\phi} = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \quad \bar{\phi} = (\phi^{\mathrm{T}}, \phi_a^{\mathrm{T}}) \tag{23}$$

with the enlarged spectral problem square matrix

$$\bar{U} = \bar{U}(\bar{u}) = \begin{pmatrix} U & U_a \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial U_a}{\partial \lambda} = 0. \tag{24}$$

This enlarged spectrum is equivalent to

$$\phi_y - \phi_x = U\phi + U_a\phi_a, \quad (\phi_a)_x = 0. \tag{25}$$

Choosing a solution for $\bar{V}$ as

$$\bar{V} = \bar{V}(\bar{u}) = \begin{pmatrix} V & V_a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} V(u) & V_a(\bar{u}) \\ 0 & 0 \end{pmatrix} \tag{26}$$

and then it leads to the additional equation

$$\begin{cases} V_y - V_x = [U, V], \\ (V_a)_y - (V_a)_x = UV_a - VU_a. \end{cases} \tag{27}$$

The enlarged temporal spectral matrices read as

$$\begin{aligned} \bar{V}(n) &= \begin{pmatrix} V^{(n)} & V_a^{(n)} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} V^{(n)}(u) & V_a^{(n)}(\bar{u}) \\ 0 & 0 \end{pmatrix}, \quad n \geq 0, \end{aligned} \tag{28}$$

and thus enlarged associated temporal spectral problem,

$$\bar{\phi}_{t_n} = \bar{\phi}_x + \bar{V}^{(n)}\bar{\phi}, \quad \bar{\phi} = (\phi^{\mathrm{T}}, \phi_a^{\mathrm{T}})^{\mathrm{T}}, \quad n \geq 0 \tag{29}$$

is equivalent to

$$\phi_{t_n} = \phi_x + V^{(n)}\phi + V_a^{(n)}\phi_a, (\phi_a)_{t_n} = 0, n \geq 0. \tag{30}$$

Therefore, the enlarged generalized zero curvature equation

$$\bar{U}_{t_n} - \bar{V}_y^{(n)} + \bar{V}_x^{(n)} + [\bar{U}, \bar{V}^{(n)}] - \bar{U}_x = 0, n \geq 0 \tag{31}$$

equivalently yields the following triangular systems:

$$U_{t_n} - V_y^{(n)} + V_x^{(n)} + [U, V^{(n)}] - U_x = 0, n \geq 0, \tag{32a}$$

$$\begin{aligned} &(U_a)_{t_n} - (V_a^{(n)})_y + (V_a^{(n)})_x + UV_a^{(n)} \\ &-V^{(n)}U_a - (U_a)_x = 0, n \geq 0. \end{aligned} \tag{32b}$$

To present a realization of the above structure, let us introduce

$$V_a = \sum_{i \geq 0} V_{ai}\lambda^{-i}, \quad \frac{\partial V_{ai}}{\partial \lambda} = 0, \tag{33}$$

and then it implies that

$$U_0 V_{a,0} - V_0 U_a = 0, \tag{34}$$

$$(V_{ai})_y - (V_{ai})_x = U_0 V_{a,i+1} + U_1 V_{ai} - V_i U_a, \quad i \geq 0. \tag{35}$$

In order to determine V_a, then U_a is invertible, and so the matrix V_a in Eq.(33) is uniquely determined by Eqs.(34) and (35). We define the additional Lax matrices as

$$V_a^{(n)} = (\lambda^n V_a)_+, n \geq 0,$$

where the subscript + denotes taking the polynomial part in λ. Then by Eq.(31), we have

$$\begin{aligned} &(V_a^{(n)})_y - (V_a^{(n)})_x \\ =&(\lambda^n U V_a)_+ - (\lambda^n V)_+ U_a = (\lambda^n U V_a)_+ - V^{(n)}U_a, \end{aligned}$$

and further, we can compute that

$$\begin{aligned} &(V_a^{(n)})_y - (V_a^{(n)})_x + UV_a^{(n)} - V^{(n)}U_a \\ =& -(\lambda^n U V_a)_+ + UV_a^{(n)} \\ =& -U_0 V_a^{(n+1)} - U_1 V_a^{(n)} + (\lambda U_0 + U_1)V_a^{(n)} \\ =& -U_0(V_a^{(n)} + \lambda V_a^{(n)}) + \lambda U_0 V_a^{(n)} \\ =& -U_0 V_a^{(n)}. \end{aligned} \tag{36}$$

It follows that the additional equations determined by Eq.(32) become candidates of integrable couplings they are given by

$$U_{t_n} = K_n(u), \quad (U_a)_{t_n} = U_0 V_{a,n} + U_{ax}, \quad n \geq 0. \tag{37}$$

To get the concrete expression of Eq.(31), we now denote U_a and $V_{a,i}$ as follows:

$$U_a = \begin{pmatrix} v_1, & \cdots, & v_l \\ w_1, & \cdots, & w_l \end{pmatrix}, \quad V_{a,i} = \begin{pmatrix} e_1(i), & \cdots, & e_l(i) \\ f_1(i), & \cdots, & f_l(i) \end{pmatrix}, \quad l \geq 0, \tag{38}$$

where l is an arbitrary natural number, v_l and $e_l(i)$ are scalars, but w_l and $f_l(i)$ are m-dimensional columns. Then, the recursion relation defined by Eqs.(34) and (35) yields

$$\begin{aligned}
&a_y(n) - a_x(n) = \frac{1}{M}(qc(n) - rb(n)),\\
&b_y(n) - b_x(n) = (M+1)(-b(n+1) + sb(n)) - (M+1)qa(n),\\
&c_y(n) - c_x(n) = -(M+1)(-c(n+1) + sc(n)) + (M+1)ra(n),\\
&e_y(n) - e_x(n) = Me(n+1) + Mse(n) + qf(n) - Mva(n) - wb(n),\\
&f_y(n) - f_x(n) = f(n+1) + re(n) - sf(n) - vc(n) + wa(n).
\end{aligned} \tag{39}$$

Taking

$$a(0) = \alpha, \quad b(0) = 0 = (0, 0, \cdots, 0)_{M\times 1},$$
$$e(0) = 0, \quad f(0) = 0, \quad c(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{M\times 1},$$

we obtain

$$a(1) = 0, c(1) = -\alpha r = \begin{pmatrix} -\alpha r_1 \\ -\alpha r_2 \\ \vdots \\ -\alpha r_M \end{pmatrix},$$
$$b(1) = (-\alpha q_1, -\alpha q_2, \cdots, -\alpha q_M),$$
$$e(1) = \alpha v_i = \alpha \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{pmatrix},$$
$$f(1) = \alpha w_i = \alpha \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{pmatrix}. \tag{40}$$

The additional equations determined by Eq.(31) become

$$U_0 V_{a,n} = \begin{pmatrix} e_1(i), & \cdots, & e_l(i) \\ f_1(i), & \cdots, & f_l(i) \end{pmatrix}, \quad n \geq 0. \tag{41}$$

There, the super-integrable hierarchy Eq.(31) of the multicomponent (2+1)-dimensional GJ hierarchy reads as

$$\begin{aligned}
\bar{U}_{t_n} = \begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \\ v_i^{\mathrm{T}} \\ w_i^{\mathrm{T}} \end{pmatrix}_{t_n} =& \bar{K}_n = \begin{pmatrix} \partial_x & 0 & 0 & 0 & 0 \\ 0 & \partial_x & 0 & 0 & 0 \\ 0 & 0 & \partial_x & 0 & 0 \\ 0 & 0 & 0 & \partial_x & 0 \\ 0 & 0 & 0 & 0 & \partial_x \end{pmatrix} \begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \\ v_i^{\mathrm{T}} \\ w_i^{\mathrm{T}} \end{pmatrix} + J_2 \begin{pmatrix} c(n+1) \\ b^{\mathrm{T}}(n+1) \\ -(M^2+M)a(n+1) \\ Me_i^{\mathrm{T}}(n+1) \\ f_i^{\mathrm{T}}(n+1) \end{pmatrix} \\
=& J_1 \begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \\ v_i^{\mathrm{T}} \\ w_i^{\mathrm{T}} \end{pmatrix} + J_2 L^n \begin{pmatrix} c(1) \\ b^{\mathrm{T}}(1) \\ -(M^2+M)a(1) \\ Me_i^{\mathrm{T}}(1) \\ f_i^{\mathrm{T}}(1) \end{pmatrix}, \quad n \geq 0,
\end{aligned} \tag{42}$$

with

$$J_2=\begin{pmatrix} 0 & (\partial_y-\partial_x)-(M+1)s & -\frac{1}{M}q & 0 & 0 \\ (\partial_y-\partial_x)-(M+1)s & 0 & -\frac{1}{M}r & 0 & 0 \\ \frac{1}{M}q^{\mathrm{T}} & \frac{1}{M}r^{\mathrm{T}} & -\frac{1}{M}(\partial_y-\partial_x) & 0 & 0 \\ 0 & \frac{1}{M}w_i^{\mathrm{T}} & \frac{-v_i^{\mathrm{T}}}{M+1} & \frac{1}{M}(\partial_x-\partial_y)+s & \frac{1}{M}q^{\mathrm{T}} \\ w_i^{\mathrm{T}} & 0 & \frac{w_i^{\mathrm{T}}}{M^2+M} & -r_i^{\mathrm{T}} & (\partial_y-\partial_x)+s \end{pmatrix},$$

and

$$L=\begin{pmatrix} s+\frac{(\partial_y-\partial_x)}{M+1} & 0 & \frac{r}{M^2+M} & 0 & 0 \\ 0 & s+\frac{(\partial_x-\partial_y)}{M+1} & \frac{q}{M^2+M} & 0 & 0 \\ -(\partial_y^{-1}-\partial_x^{-1})q^{\mathrm{T}}(\partial_y-\partial_x)-\frac{1}{M+1}q^{\mathrm{T}}s & (\partial_y^{-1}-\partial_x^{-1})r(\partial_x-\partial_y)+\frac{1}{M}rs & 0 & 0 & 0 \\ 0 & \frac{w_i^{\mathrm{T}}}{M} & \frac{-v_i^{\mathrm{T}}}{M^2+M} & \frac{\partial_y-\partial_x}{M}-s & \frac{-q_i^{\mathrm{T}}}{M} \\ 0 & -v_i^{\mathrm{T}} & \frac{w_i^{\mathrm{T}}}{M^2+M} & -r_i & (\partial_x-\partial_y)-s \end{pmatrix}.$$

The enlarged hierarchy(42) of the multicomponent (2+1)-dimensional GJ hierarchy reads as

$$\bar{U}_{t_n}=\begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \\ v_i^{\mathrm{T}} \\ w_i^{\mathrm{T}} \end{pmatrix}_{t_n}=J_1\begin{pmatrix} q^{\mathrm{T}} \\ r \\ s \\ v_i^{\mathrm{T}} \\ w_i^{\mathrm{T}} \end{pmatrix}+J_2L^n\begin{pmatrix} -\alpha r \\ -\alpha q^{\mathrm{T}} \\ 0 \\ -M\alpha v_i^{\mathrm{T}} \\ -\alpha w_i^{\mathrm{T}} \end{pmatrix}, \quad i=1,\cdots,M. \tag{43}$$

When $M=1$, the system (43) is the super-integrable coupling system of the (2+1)-dimension GJ hierarchy. When $M>1$ the system (43) is the super-integrable coupling system of the multicomponent (2+1)-dimensional GJ hierarchy.

4. Conclusions

We have obtained a multi-component Liouville integrable hierarchy Eq.(16) by using the loop algebra (9). It is worthwhile studying that how we expand the loop algebra (9) into a larger one and work out a type of Liouville integrable coupling system. Further, some linear combinations of the Lie algebra (8) can produce the other new Lie algebras. It follows that some multi-component integrable hierarchies might be generated by making use of generalized Tu scheme again. These problems are worth researching in the future.

References

[1] Tu G Z 1989 *J. Math. Phys.* **33** 330
[2] Ablowitz, Kaup D J, Newll A C and Segur H 1974 *The Incerse Scattering Transform-Fourier Analysis for Nonlinear Problems, Stud. Appl. Math.* **53** 249
[3] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)
[4] Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798
[5] Wadati M, Konno K and Ichikawa Y H 1979 *J. Phys. Soc. Jpn.* **47** 1698
[6] Boiti M, Laddomada C, Pempinelli F and Tu G Z 1983 *Prog. Theor. Phys.* **69** 48
[7] Tu G Z 1988 *Sci. Sin.* A **31** 28
[8] Zhou Z N 2002 *J. Phys. Jpn.* **71** 1857
[9] Guo F K and Zhang Y F 2003 *J. Math. Phys.* **44** 5793
[10] Zhang Y F 2004 *Chaos, Solitons and Fractals* **21** 305
[11] Zhang Y F 2003 *Chin. Phys.* **12** 1194
[12] Ma W X 2005 *J. Math. Phys.* **46** 033507
[13] Yu F J, Li L and Zhang H Q 2006 *Phys. Lett.* A **353** 307
[14] Yu F J and Zhang H Q 2006 *Chaos, Solitons and Fractals* **29** 1173
[15] Zhang Y F and Zhang Y S 2004 *Chin. Phys.* **13** 1183
[16] Yang H X and Xu X X 2005 *Chin. Phys.* **14** 869
[17] Xia T C and You F C 2007 *Chin. Phys.* **16** 605
[18] Xia T C, Wang H and Zhang Y F 2005 *Chin. Phys.* **14** 247