Vol 17 No 5, May 2008
1674-1056/2008/17(05)/1560-05

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Mei symmetry and Mei conserved quantity of nonholonomic systems with unilateral Chetaev type in Nielsen style*

Jia Li-Qun(贾利群)[a)†], Xie Jia-Fang(解加芳)[b)], and Luo Shao-Kai(罗绍凯)[c)]

[a)] *School of Science, Jiangnan University, Wuxi* 214122, *China*
[b)] *Faculty of Science, Beijing Institute of Technology, Beijing* 100081, *China*
[c)] *Institute of Mathematical Mechanics and Mathematical Physics, Zhejiang Sci-Tech University, Hangzhou* 310018, *China*

(Received 22 August 2007; revised manuscript received 7 September 2007)

This paper studies the Mei symmetry and Mei conserved quantity for nonholonomic systems of unilateral Chetaev type in Nielsen style. The differential equations of motion of the system above are established. The definition and the criteria of Mei symmetry, loosely Mei symmetry, strictly Mei symmetry for the system are given in this paper. The existence condition and the expression of Mei conserved quantity are deduced directly by using Mei symmetry. An example is given to illustrate the application of the results.

Keywords: Nielsen style, unilateral nonholonomic constrained system, Mei symmetry, Mei conserved quantity

PACC: 0320

1. Introduction

In 1918, Noether A E, a Germanic woman scientist, revealed the internal relation between the symmetry and the conserved quantity.[1] However, the scientific value of Noether theory was not found until the 1970s. From then on, the study on both symmetry and conserved quantity is flourished and therefore plentiful and substantial outcomes are achieved.[2−19]

The research on symmetry and conserved quantity of unilateral constraint systems has only an elementary development. In Ref.[20], Zhang studied form invariance for unilateral holonomic constrained systems, in Ref.[21] he presented symmetry and conserved quantity of unilateral holonomic systems in the phase space, and in Ref.[22] he studied non-Noether conserved quantity of non-holonomic constrained systems with unilateral non-Chetaev type. Li and Fang studied Mei symmetry of variable mass systems with unilateral holonomic constraints.[23]

Nielsen system is one of three famous mechanical systems (Lagrange system, Nielsen system, Appell system) in analytical systems and plays an important role in analytical theories.[24] There are many conclusions on symmetry and conserved quantity for Nielsen system, such as form invariance of Nielsen equation for non-conservative mechanical systems,[25] non-Noether conserved quantity led by form invariance of non-conservative Nielsen equation,[26] etc.

In this paper Mei symmetry and Mei conserved quantity for nonholonomic systems with unilateral Chetaev type in Nielsen style are studied. First, the differential equations of motion for the above system are established. Its definition and criterion of Mei symmetry by the total derivative of the function with respect to time t along the trajectory of a curve of the system are obtained. Next, the existence condition and the expression of Mei conserved quantity are deduced directly by using Mei symmetry. Finally, an example is given. The conclusions drawn from this paper are discussed.

2. Differential equations of motion of the system

Suppose that the position of the nonholonomic system in Nielsen style is determined by n generalized coordinates $q_s(s = 1, 2, \cdots, n)$, and it is subjected to the g ideal unilateral nonholonomic constraints of

*Project supported by the National Natural Science Foundation of China (Grant No 10572021).
†E-mail: jlq0000@163.com

Chetaev type

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})\geq 0 \quad (\beta=1,2,\cdots,g), \tag{1}$$

where the constraints are consistent and independent of each other. When the left of Eq.(1) is positive, the system will be deviated from the constraints; when it is nil, the system will be on the constraints. The restriction, namely Chetaev condition, under which constraints (1) impose on virtual displacements δq_s is

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s=0 \quad (\beta=1,2,\cdots,g). \tag{2}$$

The differential equations of the system is

$$\begin{aligned}
&\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}-2\frac{\partial L}{\partial q_s}=Q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \quad (\text{when} \quad f_\beta=0),\\
&\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}-2\frac{\partial L}{\partial q_s}=Q_s \quad (\text{when} \quad f_\beta>0),\\
&(s=1,\cdots,n;\quad \beta=1,\cdots,g)
\end{aligned} \tag{3}$$

with $\lambda_\beta\geq 0, f_\beta\geq 0, \lambda_\beta f_\beta=0\,(\beta=1,2,\cdots,g)$. In Eq.(3), $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is Lagrangian, $Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is non-potential generalized force, λ_β is Lagrange multiplier. Equation (3) is also called Nielsen equation with multipliers for nonholonomic non-conservative systems of Chetaev type. Introduce Mei arithmetic operators

$$N_s=\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}-2\frac{\partial}{\partial q_s} \quad (s=1,2,\cdots,n), \tag{4}$$

then Eq.(3) can be written as

$$\begin{aligned}
&N_s(L)=Q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \quad (\text{when} \quad f_\beta=0),\\
&N_s(L)=Q_s \quad (\text{when} \quad f_\beta>0),\\
&(s=1,\cdots,n;\beta=1,\cdots,g).
\end{aligned} \tag{5}$$

Let the system be nonsingular and in a constrained state, we can derive multipliers $\lambda_\beta=\lambda_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ from Eqs.(1) and (5). So Eq.(5) can be rewritten as

$$\begin{aligned}
&N_s(L)=Q_s+\Lambda_s \quad (\text{when} \quad f_\beta=0),\\
&N_s(L)=Q_s \quad (\text{when} \quad f_\beta>0),\\
&(s=1,\cdots,n;\quad \beta=1,\cdots,g),
\end{aligned} \tag{6}$$

where

$$\Lambda_s=\Lambda_s(t,q,\dot{q})=\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s=1,2,\cdots,n) \tag{7}$$

is called generalized nonholonomic constrained counterforce. Equation (6) is the equation of holonomic nonconservative system with unilateral Chetaev type corresponding to Eqs.(1) and (3). The motion of nonholonomic nonconservative system with unilateral Chetaev type can be found from the solution to Eq.(6). As long as original condition of the motion satisfies the condition Eq.(1), then the solution to Eq.(6) would provide the motion of the system Eq.(1) and (3). From Eq.(6), one can solve all generalized accelerations:

$$\begin{aligned}
&\ddot{q}_s=A_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \quad (\text{when} \quad f_\beta=0),\\
&\ddot{q}_s=B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \quad (\text{when} \quad f_\beta>0),\\
&(s=1,\cdots,n;\quad \beta=1,\cdots,g).
\end{aligned} \tag{8}$$

3. The definition of Mei symmetry

Introduce the infinitesimal transformations of time and generalized coordinates as

$$\begin{aligned}
&t^*=t+\Delta t,\\
&q_s^*(t^*)=q_s(t)+\Delta q_s \quad (s=1,2,\cdots,n),
\end{aligned} \tag{9}$$

or

$$\begin{aligned}
&t^*=t+\varepsilon\xi_0(t,\boldsymbol{q}),\\
&q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q}) \quad (s=1,2,\cdots,n),
\end{aligned} \tag{10}$$

where ε is an infinitesimal parameter and ξ_0,ξ_s are infinitesimal generators, and the infinitesimal generators vector $X^{(0)}$ and its extension along the trajectory of a curve of the system as

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{11}$$

$$\tilde{X}^{(1)}=X^{(0)}+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}, \tag{12}$$

where

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+A_s\frac{\partial}{\partial \dot{q}_s} \quad (\text{when} \quad f_\beta=0),\\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+B_s\frac{\partial}{\partial \dot{q}_s} \quad (\text{when} \quad f_\beta>0),\\
&(s=1,\cdots,n;\quad \beta=1,\cdots,g).
\end{aligned} \tag{13}$$

The functions L, Q_s, Λ_s, f_β become L^*, Q_s^*, Λ_s^*, f_β^* under the transformations (10). Expanding f_β^*

with respect to $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, we have

$$\begin{aligned}
L^* =& L\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\
=& L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(1)}(L) + O(\varepsilon^2), \\
Q_s^* =& Q_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\
=& Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(1)}(Q_s) + O(\varepsilon^2) \\
& (s = 1, 2, \cdots, n), \\
\Lambda_s^* =& \Lambda_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\
=& \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(1)}(\Lambda_s) + O(\varepsilon^2) \\
& (s = 1, 2, \cdots, n), \\
f_\beta^* =& f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\
=& f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(\mathbf{1})}(f_\beta) + O(\varepsilon^2) \\
& (\beta = 1, 2, \cdots, g). \qquad (14)
\end{aligned}$$

Definition 1 If Eq.(6) keeps its form invariant when dynamic functions L, Q_s and Λ_s are substituted by the functions L^*, Q_s^* and Λ_s^* under the infinitesimal transformations, that is

$$\begin{aligned}
& N_s(L^*) = Q_s^* + \Lambda_s^* \quad (\text{when} \quad f_\beta = 0), \\
& N_s(L^*) = Q_s^* \quad (\text{when} \quad f_\beta > 0), \\
& (s = 1, \cdots, n; \quad \beta = 1, \cdots, g), \qquad (15)
\end{aligned}$$

then the invariance is called Mei symmetry of Nielsen equations for holonomic nonconservative systems with unilateral Chetaev type corresponding to the same nonholonomic systems.

Definition 2 If Eqs.(1) and (6) keep their forms invariant when dynamic functions L, Q_s, Λ_s and f_β are substituted by the functions L^*, Q_s^*, Λ_s^* and f_β^* under the infinitesimal transformations, that is,

$$f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \geq 0 \quad (\beta = 1, 2, \cdots, g), \qquad (16)$$

and Eq.(15) is also satisfied, then the invariance is called loosely Mei symmetry of the system (1) and (3).

Definition 3 If Eqs.(1) and (6) keep their forms invariant when dynamic functions L, Q_s, Λ_s and f_β are substituted by the functions L^*, Q_s^*, Λ_s^* and f_β^* under the infinitesimal transformations, and the additional restriction equation of the infinitesimal generators ξ_0, ξ_s

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) = 0 \qquad (\beta = 1, 2, \cdots, g) \qquad (17)$$

is satisfied, then the invariance is called strictly Mei symmetry of the system (1) and (3).

4. Criterion of Mei symmetry

Neglecting terms that are $O(\varepsilon^2)$ or higher and substituting Eq.(14) into Eq.(15) we have

$$\begin{aligned}
& N_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(\Lambda_s) \quad (\text{when} \quad f_\beta = 0), \\
& N_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s) \quad (\text{when} \quad f_\beta > 0), \\
& (s = 1, \cdots, n; \quad \beta = 1, \cdots, g).
\end{aligned}$$

Introduce generalized Mei arithmetic operators

$$\tilde{N}_s = \frac{\partial}{\partial \dot{q}_s}\frac{\overline{\mathrm{d}}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s} \quad (s = 1, 2, \cdots, n), \qquad (18)$$

we obtain

$$\begin{aligned}
& \tilde{N}_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s + \Lambda_s) \quad (\text{when} \quad f_\beta = 0), \\
& \tilde{N}_s[\tilde{X}^{(1)}(L)] = \tilde{X}^{(1)}(Q_s) \quad (\text{when} \quad f_\beta > 0), \\
& (s = 1, \cdots, n; \quad \beta = 1, \cdots, g). \qquad (19)
\end{aligned}$$

Similarly, we can also arrive at

$$\begin{aligned}
& \tilde{X}^{(1)}(f_\beta) = 0 \quad (\text{when} \quad f_\beta = 0), \\
& f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \tilde{X}^{(1)}(f_\beta) > 0 \quad (\text{when} \quad f_\beta > 0), \\
& (\beta = 1, \cdots, g). \qquad (20)
\end{aligned}$$

Consequently, the following criteria are presented as follows.

Criterion 1 For the holonomic nonconservative system (6) of unilateral Chetaev type, if infinitesimal generators ξ_0, ξ_s admit Eq.(19), then corresponding symmetry is called Mei symmetry of the system.

Equation (19) is the criterion equation of Mei symmetry for holonomic nonconservative systems with unilateral Chetaev type.

Criterion 2 For the nonholonomic nonconservative system (1), (3) of unilateral Chetaev type, if infinitesimal generators ξ_0, ξ_s admit Eq.(19) and Eq.(20), then corresponding symmetry is called loosely Mei symmetry of the system.

Equations (19) and (20) are the criterion equation, the restriction equation, respectively, of Mei symmetry for nonholonomic nonconservative systems with unilateral Chetaev type.

Criterion 3 For the nonholonomic nonconservative system (1), (3) of unilateral Chetaev type, if infinitesimal generators ξ_0, ξ_s admit Eqs.(17), (19) and (20), then corresponding symmetry is called strictly Mei symmetry of the system.

Equation (17) is the additional restriction equation of Mei symmetry for nonholonomic nonconservative systems with unilateral Chetaev type.

5. Mei conserved quantity led by Mei symmetry

In the following, we provide the proposition to give out the existence condition and the expression of Mei conserved quantity, which resulted from Mei symmetry directly.

Proposition If the infinitesimal generators ξ_0, ξ_s of Mei symmetry, or loosely Mei symmetry, or strictly Mei symmetry admit

$$\begin{aligned}&\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)]+\tilde{X}^{(1)}(Q_s+\Lambda_s)(\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_M=0\quad(\text{when}\quad f_\beta=0)\,,\\&\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)]+\tilde{X}^{(1)}(Q_s)(\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_M=0\quad(\text{when}\quad f_\beta>0)\,,\\&(s=1,\cdots,n;\quad\beta=1,\cdots,g).\end{aligned}\tag{21}$$

in which $G_M(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the gauge function, then the system possesses Mei conserved quantity caused by Mei symmetry as follows:

$$I_{\mathrm{M}}=\tilde{X}^{(1)}(L)\xi_0+\frac{\partial\tilde{X}^{(1)}(L)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\mathrm{M}}=\text{const.}\tag{22}$$

Proof

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{M}}=&\left[\frac{\partial}{\partial t}\tilde{X}^{(1)}(L)+\dot{q}_s\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)+A_s\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right]\xi_0+\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\\&+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right](\xi_s-\dot{q}_s\xi_0)+\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-A_s\xi_0-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right]+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{M}}\\=&\tilde{X}^{(1)}[\tilde{X}^{(1)}(L)]-\xi_s\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)+\dot{q}_s\xi_0\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)+\tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\\&+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{M}}+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)\right](\xi_s-\dot{q}_s\xi_0)\\=&\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\tilde{X}^{(1)}(L)-\frac{\partial}{\partial q_s}\tilde{X}^{(1)}(L)-\tilde{X}^{(1)}(Q_s+\Lambda_s)\right](\xi_s-\dot{q}_s\xi_0)=0.\end{aligned}$$

In a similar way, we can also prove the other case that is not in a constrained state.

6. Example

Consider an unilateral nonholonomic system

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2),\tag{23}$$

$$f=\dot{q}_1+bt\dot{q}_2-bq_2+t\geq 0,\tag{24}$$

$$Q_1=Q_2=0.\tag{25}$$

where b is arbitrary constant.

Let us now turn our attention to the differential equations (3) of the system by which we have

$$\begin{aligned}&\ddot{q}_1=\Lambda_1=\lambda,\quad\ddot{q}_2=\Lambda_2=\lambda bt\quad(\text{when}\quad f=0)\,,\\&\ddot{q}_1=0,\quad\ddot{q}_2=0\quad(\text{when}\quad f>0).\end{aligned}\tag{26}$$

If the system breaks away constraint, from Eqs.(24) and (26), we obtain

$$\lambda=-\frac{1}{1+b^2t^2}.\tag{27}$$

When Eq.(8) is taken into account, substituting Eq.(27) into Eq.(26) we obtain

$$\begin{aligned}&\ddot{q}_1=A_1=\Lambda_1=-\frac{1}{1+b^2t^2},\\&\ddot{q}_2=A_2=\Lambda_2=-\frac{bt}{1+b^2t^2}\quad(\text{when}\quad f=0)\,,\\&\ddot{q}_1=B_1=0,\quad\ddot{q}_2=B_2=0\quad(\text{when}\quad f>0)\,.\end{aligned}\tag{28}$$

Taking

$$\xi_0=0,\quad\xi_1=-bt\dot{q}_1+\dot{q}_2+bq_1+bt,\quad\xi_2=1,\tag{29}$$

we achieve

$$\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_1 = b, \quad \tilde{X}^{(1)}(L) = b\dot{q}_1, \quad \tilde{X}^{(1)}[\tilde{X}^{(1)}(L)] = b^2,$$
$$\tilde{X}^{(1)}(Q_1 + \Lambda_1) = \tilde{X}^{(1)}(Q_2 + \Lambda_2) = \tilde{X}^{(1)}(f) = 0,$$
$$\tilde{N}_1[\tilde{X}^{(1)}(L)] = \tilde{N}_2[\tilde{X}^{(1)}(L)] = 0,$$
$$\tilde{X}^{(1)}(Q_1) = \tilde{X}^{(1)}(Q_2) = 0. \tag{30}$$

It is not difficult to verify by Eq.(30) that criterion equation (19) and restriction equation (20) are both satisfied, however, the additional restriction equation (17) is not existent. Due to Criterion 2 we know that the infinitesimal generators in Eq.(29) are ones of loosely Mei symmetry for nonholonomic systems of unilateral Chetaev type in Nielsen style.

From Eq.(21), we obtain

$$G_{\mathrm{M}} = -b^2 t.$$

In the last, we can find non-Noether conserved quantity, Mei conserved quantity, of the system expressed as

$$I_{\mathrm{M}} = b(-bt\dot{q}_1 + \dot{q}_2 + bq_1) = \mathrm{const}. \tag{31}$$

7. Concluding remarks

This paper presents the definition and the criterion of Mei symmetry for nonholonomic systems of unilateral Chetaev type in Nielsen style. In addition, Mei conserved quantity is also studied and the existence condition, its expression for the system are provided. These conclusions include Criteria 1, 2, 3 and the proposition.

Through Eqs.(3), (5) and (6) etc., it shows that if non-potential generalized force $Q_s = 0$, the conclusions of this paper will become corresponding ones for nonholonomic conservative systems of unilateral Chetaev type in Nielsen style (the example in this paper is just so); if generalized nonholonomic constrained counterforce $\Lambda_s = 0$, then they will be ones corresponding to holonomic nonconservative systems; if $Q_s = \Lambda_s = 0$, and then they will be ones corresponding to holonomic conservative systems; if unilateral constraints are turned into double-faced constraints, they will be the conclusions of nonholonomic nonconservative systems with double-faced Chetaev type in Nielsen style. Therefore, the study of this paper has universal significance.

References

[1] Noether A E 1918 *Nachr. Akad. Wiss. Göttingen. Math. Phys.* **KI** 235

[2] Santilli R M 1978 *Foundations of Theoretical Mechanics* I (New York: Springer) p141

[3] Vujanović B 1986 *Acta Mech.* **65** 63

[4] Mei F X 2000 *Acta Phys. Sin.* **49** 1207 (in Chinese)

[5] Mei F X and Shang M 2000 *Acta Phys. Sin.* **49** 1901 (in Chinese)

[6] Wang S Y and Mei F X 2002 *Chin. Phys.* **11** 5

[7] Ge W K 2002 *Acta Phys. Sin.* **51** 1156(in Chinese)

[8] Zhang H B 2002 *Chin. Phys.* **11** 765

[9] Luo S K 2003 *Chin. Phys.* **12** 140

[10] Mei F X 2003 *Acta Phys. Sin.* **52** 1048(in Chinese)

[11] Fu J L and Chen L Q 2003 *Chin. Phys.* **12** 695

[12] Chen X W and Li Y M 2003 *Chin. Phys.* **12** 936

[13] Mei F X, Xu X J and Zhang Y F 2004 *Acta Mech. Sin.* **20** 668

[14] Jia L Q and Zheng S W 2006 *Acta Phys. Sin.* **55** 3829 (in Chinese)

[15] Jia L Q, Zhang Y Y and Zheng S W 2007 *Acta Phys. Sin.* **56** 649 (in Chinese)

[16] Jia L Q, Zheng S W and Zhang Y Y 2007 *Acta Phys. Sin.* **56** 5575 (in Chinese)

[17] Jia L Q, Luo S K and Zhang Y Y 2007 *Acta Phys. Sin.* **56** 6188 (in Chinese)

[18] Jia L Q, Zhang Y Y and Luo S K 2007 *Chin. Phys.* **16** 3168

[19] Jia L Q, Xie J F and Zheng S W 2008 *Chin. Phys.* **17** 17

[20] Zhang Y 2004 *Acta Phys. Sin.* **53** 331 (in Chinese)

[21] Zhang Y 2005 *Acta Phys. Sin.* **54** 4488 (in Chinese)

[22] Zhang Y 2006 *Acta Phys. Sin.* **55** 504 (in Chinese)

[23] Li H and Fang J H 2004 *Acta Phys. Sin.* **53** 2807 (in Chinese)

[24] Mei F X 1985 *Foundations of Mechanics of Nonholonomic Systems* (Beijing: Beijing Institute of Technology Press) p214 (in Chinese)

[25] Fang J H, Xue Q Z and Zhao S Q 2002 *Acta Phys. Sin.* **51** 2183 (in Chinese)

[26] Xu X J, Mei F X and Qin M C 2004 *Acta Phys. Sin.* **53** 4021 (in Chinese)