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Noether symmetry and Lie symmetry of discrete holonomic systems with dependent coordinates*

Shi Shen-Yang(施沈阳)[a)†] and Huang Xiao-Hong(黄晓虹)[b)]

[a)] *Department of Physics, Zhejiang Sci-Tech University, Hangzhou* 310018, *China*

[b)] *School of Physics and Electronic Information, Wenzhou University, Wenzhou* 325000, *China*

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The Noether symmetry, the Lie symmetry and the conserved quantity of discrete holonomic systems with dependent coordinates are investigated in this paper. The Noether symmetry provides a discrete Noether identity and a conserved quantity of the system. The invariance of discrete motion equations under infinitesimal transformation groups is defined as the Lie symmetry, and the condition of obtaining the Noether conserved quantity from the Lie symmetry is also presented. An example is discussed to show the applications of the results.

Keywords: discrete mechanics, Noether symmetry, Lie symmetry, discrete conserved quantity
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1. Introduction

The symmetries and conserved quantities of constrained mechanical systems have been systematically investigated in recent years.[1−15] There exist two symmetries in this subject: the Noether symmetry and the Lie symmetry. The Noether symmetry is the invariance of Hamiltonian action functional under infinitesimal transformation groups and has a close relationship with the invariant quantity of the system. The Lie symmetry is the invariance of differential equations of motion under infinitesimal transformation groups.

These two symmetry approaches have been extended to discrete systems in the last few years. There are many important developments in this field including Levi's generalization of Lie symmetry to discrete systems whose equations of motion have difference or differential-difference formalism,[16−18] and Dorodnitsyn's adaptation of Noether symmetry theory to discrete Lagrangian system.[19] These studies, however, are not only highly mathematical but have no consideration of special constrains in constrained mechanics.

The purpose of this paper is to extend the Noether symmetry and the Lie symmetry theory to discrete constrained mechanical systems. We deduce the Noether identity and conserved quantity from the Noether symmetry, and define the Lie symmetry for discrete holonomic systems with dependent coordinates. We also derive the criterion of determining when the Noether conserved quantity may be obtained from the Lie symmetry.

2. The discrete equations of motion of a system

We consider a mechanical system, which has coordinates $q_s(s = 1, \cdots, n)$ to determine its configuration and dependent coordinates $q_r(r = 1, \cdots, m)$, with m holonomic constrains related to these coordinates

$$f_\beta(t, q_u) = 0, \quad (u = 1, \cdots, n+m; \beta = 1, \cdots, m). \tag{1}$$

The differential equations of motion of the system are[1,2]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_u} - \frac{\partial L}{\partial q_u} = Q_u + \Lambda_u, \tag{2}$$

where $L(t, q_u, \dot{q}_u)$ and $Q_u(t, q_u, \dot{q}_u)$ are the Lagrangian and the generalized force of the system respectively.

$$\Lambda_u(t, q_u, \dot{q}_u) = \lambda_\beta(t, q_u, \dot{q}_u)\frac{\partial f_\beta}{\partial q_u} \tag{3}$$

is the constrained force, where λ_β is Lagrangian multiplier.

In discrete mechanics, continuous curve in configuration space is replaced with a discrete sequence $(q_{u,i}(t_i))$ $(u = 1, \cdots, n+m; i = 0, 1, \cdots, N)$. The discrete versions of L, Q_u, Λ_u are $L_\mathrm{d}(t_k, t_{k+1}, q_{u,k}, q_{u,k+1})$, $Q_{u,\mathrm{d}}(t_k, t_{k+1}, q_{u,k}, q_{u,k+1})$, and $\Lambda_{u,\mathrm{d}}$ $(t_k, t_{k+1},$

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†E-mail: hzshishenyang@sina.com

$q_{u,k},\ q_{u,k+1})$ respectively $(k = 0, 1, \cdots, N-1)$. The constrained equations of the discrete version are

$$f_{\beta,\mathrm{d}}(t_k, t_{k+1}, q_{u,k}, q_{u,k+1}) = 0. \tag{4}$$

The discrete Euler–Lagrange equation and the energy evolution equation with a non-conservative generalized force, respectively, are[20]

$$D_3 L_{\mathrm{d}}(\phi_k)(t_{k+1} - t_k) + D_4 L_{\mathrm{d}}(\phi_{k-1})(t_k - t_{k-1}) + Q_{\mathrm{d}}(\phi_k)(t_{k+1} - t_k) = 0 \tag{5}$$

and

$$D_1 L_{\mathrm{d}}(\phi_k)(t_{k+1} - t_k) + D_2 L_{\mathrm{d}}(\phi_{k-1})(t_k - t_{k-1}) + L_{\mathrm{d}}(\phi_{k-1}) - L_{\mathrm{d}}(\phi_k) - Q_{\mathrm{d}}(\phi_k)(q_{k+1} - q_k) = 0, \tag{6}$$

where D_j is the partial derivative of the discrete function with respect to arguments j and $\phi_k = (t_k, t_{k+1}, q_k, q_{k+1})$, $(k = 0, 1, \cdots, N-1)$.

If we take coordinates q_u and discrete constrained force

$$\Lambda_{u,\mathrm{d}}(t_k, t_{k+1}, q_{u,k}, q_{u,k+1}) = \lambda_{\beta,d}(t_k, t_{k+1}, q_{u,k}, q_{u,k+1})\left(\frac{\partial f_{\beta,d}}{\partial q_{u,k}} + \frac{\partial f_{\beta,d}}{\partial q_{u,k+1}}\right) \tag{7}$$

into account, Eqs.(5) and (6) can be rewritten respectively as

$$D_3 L_{\mathrm{d}}(\phi_{u,k})(t_{k+1} - t_k) + D_4 L_{\mathrm{d}}(\phi_{u,k-1})(t_k - t_{k-1}) + [Q_{u,\mathrm{d}}(\phi_{u,k}) + \Lambda_{u,\mathrm{d}}(\phi_{u,k})](t_{k+1} - t_k) = 0 \tag{8}$$

and

$$D_1 L_{\mathrm{d}}(\phi_{u,k})(t_{k+1} - t_k) + D_2 L_{\mathrm{d}}(\phi_{u,k-1})(t_k - t_{k-1}) + L_{\mathrm{d}}(\phi_{u,k-1}) - L_{\mathrm{d}}(\phi_{u,k}) - [Q_{u,\mathrm{d}}(\phi_{u,k}) + \Lambda_{u,\mathrm{d}}(\phi_{u,k})](q_{u,k+1} - q_{u,k}) = 0, \tag{9}$$

where $\phi_{u,k} = (t_k, t_{k+1}, q_{u,k}, q_{u,k+1})$. Equations (8) and (9) are respectively the discrete Euler–Lagrange equation and the energy evolution equation of discrete holonomic system with dependent coordinates, which also are called the discrete equations of motion of the system.

3. The Noether symmetry of a discrete system

The recursive and derivative operators of a discrete system for any multi-dimensional discrete variable or function are represented as

$$R_{\pm} f(z_{u,k}) = f(z_{u,k\pm1}) \tag{10}$$

and

$$D_{\mathrm{d}} f(z_{u,k}) = \frac{R_+ f(z_{u,k}) - f(z_{u,k})}{t_{k+1} - t_k}. \tag{11}$$

Using representations (10) and (11), the discrete equations of motion, (8) and (9), can be rewritten respectively as

$$D_3 L_{\mathrm{d}}(\phi_{u,k}) + C_t D_4[R_- L_{\mathrm{d}}(\phi_{u,k})] + Q_{u,\mathrm{d}}(\phi_{u,k}) + \Lambda_{u,\mathrm{d}}(\phi_{u,k}) = 0 \tag{12}$$

and

$$D_1 L_{\mathrm{d}}(\phi_{u,k}) + C_t D_2[R_- L_{\mathrm{d}}(\phi_{u,k})] - D_{\mathrm{d}}[R_- L_{\mathrm{d}}(\phi_{u,k})] - [Q_{u,\mathrm{d}}(\phi_{u,k}) + \Lambda_{u,\mathrm{d}}(\phi_{u,k})] D_{\mathrm{d}}(q_{u,k}) = 0, \tag{13}$$

where $C_t = (t_k - t_{k-1})/(t_{k+1} - t_k)$.

The infinitesimal point transformations of discrete time and generalized coordinates are

$$t_k^* = t_k + \Delta t_k = t_k + \varepsilon \tau_k(t_k, q_{u,k})$$

and

$$q_{u,k}^* = q_{u,k} + \Delta q_{u,k} = q_{u,k} + \varepsilon \xi_{u,k}(t_k, q_{u,k}), \tag{14}$$

where τ_k and $\xi_{u,k}$ are the discrete generators. The vector field of generators is

$$X_{\mathrm{d}}^{(0)} = \tau_k \frac{\partial}{\partial t_k} + \xi_{u,k} \frac{\partial}{\partial q_{u,k}}, \tag{15}$$

which can be extended to the two- and three-point schemes

$$X_{\mathrm{d}}^{(1)} = X_{\mathrm{d}}^{(0)} + \tau_{k+1} \frac{\partial}{\partial t_{k+1}} + \xi_{u,k+1} \frac{\partial}{\partial q_{u,k+1}}, \tag{16}$$

and

$$X_{\mathrm{d}}^{(2)} = X_{\mathrm{d}}^{(1)} + \tau_{k-1} \frac{\partial}{\partial t_{k-1}} + \xi_{u,k-1} \frac{\partial}{\partial q_{u,k-1}}. \tag{17}$$

Criterion 1 If there exists a discrete gauge function $G_k(t_k, q_{u,k})$ such that the identities

$$L_{\mathrm{d}} D_{\mathrm{d}}(\tau_k) + X_{\mathrm{d}}^{(1)}(L_{\mathrm{d}}) + (Q_{u,\mathrm{d}} + \Lambda_{u,\mathrm{d}}) \times [\xi_{u,k} - D_{\mathrm{d}}(q_{u,k})\tau_k] + D_{\mathrm{d}}(G_k) = 0 \tag{18}$$

and

$$\left(\frac{\partial f_{\beta,\mathrm{d}}}{\partial q_{u,k}} + \frac{\partial f_{\beta,\mathrm{d}}}{\partial q_{u,k+1}}\right)[\xi_{u,k} - D_{\mathrm{d}}(q_{u,k})\tau_k] = 0 \tag{19}$$

hold, and in addition the Eqs.(12) and (13) also hold, then the discrete holonomic system with dependent coordinates has the discrete Noether conserved quantities

$$\tau_k R_- L_{\mathrm{d}}(\phi_{u,k}) + \tau_k(t_k - t_{k-1}) D_2[R_- L_{\mathrm{d}}(\phi_{u,k})] + \xi_{u,k}(t_k - t_{k-1}) D_4[R_- L_{\mathrm{d}}(\phi_{u,k})] + G_k = \mathrm{const} \tag{20}$$

and Eq.(18) is called the discrete Noether identity and Eq.(19) is called the restricted equation of the Noether symmetry.

Proof Using representations (10), (11) and (16), the extension of Noether identity (18) gives

$$
\begin{aligned}
&L_{\rm d}D_{\rm d}(\tau_k)+X_{\rm d}^{(1)}(L_{\rm d})+(Q_{u,{\rm d}}+\Lambda_{u,{\rm d}})[\xi_{u,k}-D_{\rm d}(q_{u,k})\tau_k]+D_{\rm d}(G_k)\\
=&\tau_k\left\{D_1L_{\rm d}(\phi_{u,k})+C_tD_2[R_-L_{\rm d}(\phi_{u,k})]-D_{\rm d}[R_-L_{\rm d}(\phi_{u,k})]\right.\\
&-[Q_{u,{\rm d}}(\phi_{u,k})+\Lambda_{u,{\rm d}}(\phi_{u,k})]D_{\rm d}(q_{u,k})\}\\
&+\xi_{u,k}\left\{D_3L_{\rm d}(\phi_{u,k})+C_tD_4[R_-L_{\rm d}(\phi_{u,k})]+Q_{u,{\rm d}}(\phi_{u,k})+\Lambda_{u,{\rm d}}(\phi_{u,k})\right\}\\
&+D_{\rm d}\{\tau_kR_-L_{\rm d}(\phi_{u,k})+\tau_k(t_k-t_{k-1})D_2[R_-L_{\rm d}(\phi_{u,k})]\\
&+\xi_{u,k}(t_k-t_{k-1})D_4[R_-L_{\rm d}(\phi_{u,k})]+G_k\}=0. \qquad (21)
\end{aligned}
$$

According to Eqs.(12) and (13), the Eq.(21) becomes

$$
\begin{aligned}
&D_{\rm d}\{\tau_kR_-L_{\rm d}(\phi_{u,k})+\tau_k(t_k-t_{k-1})D_2[R_-L_{\rm d}(\phi_{u,k})]\\
&+\xi_{u,k}(t_k-t_{k-1})D_4[R_-L_{\rm d}(\phi_{u,k})]+G_k\}=0. \qquad (22)
\end{aligned}
$$

We therefore have the discrete conserved quantities

$$
\begin{aligned}
&\tau_kR_-L_{\rm d}(\phi_{u,k})+\tau_k(t_k-t_{k-1})D_2[R_-L_{\rm d}(\phi_{u,k})]\\
&+\xi_{u,k}(t_k-t_{k-1})D_4[R_-L_{\rm d}(\phi_{u,k})]+G_k=\text{const}. \qquad (23)
\end{aligned}
$$

4. The Lie symmetry and the Noether conserved quantities of a discrete system

We rewrite discrete Euler–Lagrange equation (8) and energy evolution equation (9) as

$$E(\varphi_{u,k})=0 \qquad (24)$$

and

$$F(\varphi_{u,k})=0, \qquad (25)$$

where $\varphi_{u,k}=(t_{k-1},t_k,t_{k+1},q_{u,k-1},q_{u,k},q_{u,k+1})$, $(k=1,\cdots,N-1)$. Applying extension (17) to Eqs.(24) and (25), we obtain

$$X_{\rm d}^{(2)}[E(\varphi_{u,k})]=0, \qquad (26)$$

and

$$X_{\rm d}^{(2)}[F(\varphi_{u,k})]=0. \qquad (27)$$

Also applying extension (16) to discrete constrained equation (4), we obtain

$$X_{\rm d}^{(1)}[f_{\beta,d}(\phi_{u,k})]=0. \qquad (28)$$

Criterion 2 If the infinitesimal generators τ_k and $\xi_{u,k}$ satisfy Eqs.(26)–(28), then the invariance of the discrete equations of motion (8) and (9) is the Lie symmetry of the discrete holonomic systems with dependent coordinates. Equations (26) and (27) are the determining equations of the Lie symmetry, and Eq.(28) is the restricted equation of the Lie symmetry.

By solving these equations, we can obtain generators for the infinitesimal Lie symmetry transformations. Considering criterion 1, we straightforward acquire the criterion for obtaining the Noether conserved quantities from the Lie symmetry.

Criterion 3 If the Lie symmetry generators τ_k and $\xi_{u,k}$ satisfy discrete Noether identity (18) and restricted equation (14), then the Lie symmetry of discrete holonomic system with dependent coordinates leads to discrete Noether conserved quantities (20).

5. An example

Now we us take an example (the motion of a top on a plane) to show this procedure. The discrete Lagrangian of the system and its recursive operator, respectively, are

$$
\begin{aligned}
L_{\rm d}(\phi_{u,k})=&\frac{1}{2}M\left[\left(\frac{q_{4,k+1}-q_{4,k}}{t_{k+1}-t_k}\right)^2+\left(\frac{q_{5,k+1}-q_{5,k}}{t_{k+1}-t_k}\right)^2+\left(\frac{q_{6,k+1}-q_{6,k}}{t_{k+1}-t_k}\right)^2\right]\\
&-\frac{1}{2}Mg(q_{6,k+1}+q_{6,k})+\frac{1}{2}A\left[\left(\frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right)^2+\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\right)^2\sin^2\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\\
&+\frac{1}{2}B\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]^2, \qquad (29)
\end{aligned}
$$

and

$$\begin{aligned}R_-[L_{\rm d}(\phi_{u,k})] =&L_{\rm d}(\phi_{u,k-1}) = \frac{1}{2}M\left[\left(\frac{q_{4,k}-q_{4,k-1}}{t_k-t_{k-1}}\right)^2+\left(\frac{q_{5,k}-q_{5,k-1}}{t_k-t_{k-1}}\right)^2+\left(\frac{q_{6,k}-q_{6,k-1}}{t_k-t_{k-1}}\right)^2\right]\\ &-\frac{1}{2}Mg(q_{6,k}+q_{6,k-1})+\frac{1}{2}A\left[\left(\frac{q_{2,k}-q_{2,k-1}}{t_k-t_{k-1}}\right)^2+\left(\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\right)^2\sin^2\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]\\ &+\frac{1}{2}B\left[\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}+\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]^2. \end{aligned}\tag{30}$$

The discrete constrained equation is

$$f_{\rm d}(\phi_{u,k}) = \frac{1}{2}(q_{6,k+1}+q_{6,k})-N\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right).\tag{31}$$

According to restricted equations (19) and (28), together with constrained equation (31), we straightforward have

$$\tau_k(t_k,q_{u,k}) = 0,\tag{32}$$

and

$$\xi_{2,k}(t_k,q_{u,k}) = \xi_{6,k}(t_k,q_{u,k}) = 0.\tag{33}$$

Applying Eqs.(7), (8) and (9) to Eqs.(29)–(31), we obtain a group of discrete Euler–Lagrange equations

$$\begin{aligned}E(\varphi_{1,k}) =& -A\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\right)\sin^2\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)+A\left(\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\right)\sin^2\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\\ &-B\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\\ &+B\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\left[\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}+\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right] = 0, \end{aligned}\tag{34}$$

$$\begin{aligned}E(\varphi_{3,k}) =& -B\left(\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}\right)+B\left(\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}\right)\\ &+B\left[\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)-\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right] = 0, \end{aligned}\tag{35}$$

$$E(\varphi_{4,k}) = -M\left(\frac{q_{4,k+1}-q_{4,k}}{t_{k+1}-t_k}\right)+M\left(\frac{q_{4,k}-q_{4,k-1}}{t_k-t_{k-1}}\right) = 0\tag{36}$$

and

$$E(\varphi_{5,k}) = -M\left(\frac{q_{5,k+1}-q_{5,k}}{t_{k+1}-t_k}\right)+M\left(\frac{q_{5,k}-q_{5,k-1}}{t_k-t_{k-1}}\right) = 0,\tag{37}$$

and a group of discrete energy evolution equations

$$\begin{aligned}F(\varphi_{1,k}) =&\frac{1}{2}A\left[\left(\frac{q_{2,k+1}-q_{2,k}}{t_{k+1}-t_k}\right)^2+\left(\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\right)^2\sin^2\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\\ &-\frac{1}{2}A\left[\left(\frac{q_{2,k}-q_{2,k-1}}{t_k-t_{k-1}}\right)^2+\left(\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\right)^2\sin^2\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]\\ &+B\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\\ &-B\left[\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}+\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\\ &-\frac{1}{2}B\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]^2\\ &+\frac{1}{2}B\left[\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}+\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]^2 = 0, \end{aligned}\tag{38}$$

$$
\begin{aligned}
F(\varphi_{3,k}) =& B\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}\\
&-B\left[\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}+\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}\\
&-\frac{1}{2}B\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]^2\\
&+\frac{1}{2}B\left[\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}+\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]=0, \qquad (39)
\end{aligned}
$$

$$
F(\varphi_{4,k})=\frac{1}{2}M\left(\frac{q_{4,k+1}-q_{4,k}}{t_{k+1}-t_k}\right)^2-\frac{1}{2}M\left(\frac{q_{4,k}-q_{4,k-1}}{t_k-t_{k-1}}\right)^2=0, \qquad (40)
$$

$$
F(\varphi_{5,k})=\frac{1}{2}M\left(\frac{q_{5,k+1}-q_{5,k}}{t_{k+1}-t_k}\right)^2-\frac{1}{2}M\left(\frac{q_{5,k}-q_{5,k-1}}{t_k-t_{k-1}}\right)^2=0, \qquad (41)
$$

where we omit the $E(\varphi_{2,k})$, $E(\varphi_{6,k})$, $F(\varphi_{2,k})$, and $F(\varphi_{6,k})$ due to conditions (32) and (33).

Using expression (29) and Eqs.(32)–(33), discrete Noether identity (18) becomes

$$
\begin{aligned}
&-\frac{1}{2}M\left(\frac{q_{4,k+1}-q_{4,k}}{t_{k+1}-t_k}\right)^2D_{\rm d}(\tau_k)+M\frac{q_{4,k+1}-q_{4,k}}{t_{k+1}-t_k}D_{\rm d}(\xi_{4,k})\\
&-\frac{1}{2}M\left(\frac{q_{5,k+1}-q_{5,k}}{t_{k+1}-t_k}\right)^2D_{\rm d}(\tau_k)+M\frac{q_{5,k+1}-q_{5,k}}{t_{k+1}-t_k}D_{\rm d}(\xi_{5,k})\\
&-\xi_{1,k}A\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2}\sin^2\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)+\xi_{1,k+1}A\frac{q_{1,k+1}-q_{1,k}}{(t_{k+1}-t_k)^2}\sin^2\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\\
&-\xi_{1,k}B\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\frac{\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)}{t_{k+1}-t_k}\\
&+\xi_{1,k+1}B\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\frac{\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)}{t_{k+1}-t_k}\\
&-\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\frac{\xi_{3,k}B}{t_{k+1}-t_k}\\
&+\left[\frac{q_{3,k+1}-q_{3,k}}{t_{k+1}-t_k}+\frac{q_{1,k+1}-q_{1,k}}{t_{k+1}-t_k}\cos\left(\frac{1}{2}(q_{2,k+1}+q_{2,k})\right)\right]\frac{\xi_{3,k+1}B}{t_{k+1}-t_k}+D_{\rm d}(G_k)=0. \qquad (42)
\end{aligned}
$$

Solving Eq.(42), we obtain the generators for the infinitesimal Noether symmetry transformations as follows:

$$
\xi_{1,k}=C_{11},\quad \tau_k=\xi_{2,k}=\xi_{3,k}=\xi_{4,k}=\xi_{5,k}=\xi_{6,k}=0,\quad G_k=0, \qquad (43)
$$

$$
\xi_{3,k}=C_{31},\quad \tau_k=\xi_{1,k}=\xi_{2,k}=\xi_{4,k}=\xi_{5,k}=\xi_{6,k}=0,\quad G_k=0, \qquad (44)
$$

$$
\xi_{4,k}=C_{41}t_k,\quad \tau_k=\xi_{1,k}=\xi_{2,k}=\xi_{3,k}=\xi_{5,k}=\xi_{6,k}=0,\quad G_k=-MC_{41}q_{4,k}, \qquad (45)
$$

and

$$
\xi_{5,k}=C_{51}t_k,\quad \tau_k=\xi_{1,k}=\xi_{2,k}=\xi_{3,k}=\xi_{4,k}=\xi_{6,k}=0,\quad G_k=-MC_{51}q_{5,k}, \qquad (46)
$$

where C_{11}, C_{31}, C_{41}, C_{51} are constants.

The next step is to calculate the generators for the infinitesimal Lie symmetry transformations. Substituting Eqs.(34)–(41) into Eqs.(26) and (27) leads to the determining equations

$$
\begin{aligned}
&\tau_{k-1}D_1E(\varphi_{u,k})+\tau_kD_2E(\varphi_{u,k})+\tau_{k+1}D_3E(\varphi_{u,k})\\
&+\xi_{u,k-1}D_4E(\varphi_{u,k})+\xi_{u,k}D_5E(\varphi_{u,k})\\
&+\xi_{u,k+1}D_6E(\varphi_{u,k})=0, \qquad (47)\\
&\tau_{k-1}D_1F(\varphi_{u,k})+\tau_kD_2F(\varphi_{u,k})+\tau_{k+1}D_3F(\varphi_{u,k})\\
&+\xi_{u,k-1}D_4F(\varphi_{u,k})+\xi_{u,k}D_5F(\varphi_{u,k})\\
&+\xi_{u,k+1}D_6F(\varphi_{u,k})=0, \qquad (48)
\end{aligned}
$$

where the representations of $D_jE(\varphi_{u,k})$ and $D_jF(\varphi_{u,k})$ in Eqs.(47) and (48) can be calculated from Eqs.(34)–(41), $(u = 1, 3, 4, 5)$. Solving Eqs.(47) and (48), we obtain the generators for the infinitesimal Lie symmetry transformations as follows:

$$\tau_k = 0, \quad \xi_{1,k} = C_{11}, \tag{49}$$

$$\tau_k = 0, \quad \xi_{3,k} = C_{31}, \tag{50}$$

$$\tau_k = 0, \quad \xi_{4,k} = C_{41}t_k + C_{42}q_{4,k} + C_{43}, \tag{51}$$

$$\tau_k = 0, \quad \xi_{5,k} = C_{51}t_k + C_{52}q_{5,k} + C_{53}, \tag{52}$$

where C_{11}, C_{31}, C_{41}, C_{42}, C_{43}, C_{51}, C_{52}, C_{53} are constants.

Comparing the generators (49)–(52) with (43)–(46), we have a group of generators (43)–(46), which are not only the Lie infinitesimal symmetry transformation but also the Noether infinitesimal symmetry transformation.

According to criterion 1, we obtain the discrete Noether conserved quantities from the Lie symmetry

$$\begin{aligned}
&C_{11}A\left(\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\right)\sin^2\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\\
&+B\left[C_{31}+C_{11}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]\\
&\times\left[\frac{q_{3,k}-q_{3,k-1}}{t_k-t_{k-1}}+\frac{q_{1,k}-q_{1,k-1}}{t_k-t_{k-1}}\cos\left(\frac{1}{2}(q_{2,k}+q_{2,k-1})\right)\right]\\
&+t_k\left(C_{41}\frac{q_{4,k}-q_{4,k-1}}{t_k-t_{k-1}}+C_{51}\frac{q_{5,k}-q_{5,k-1}}{t_k-t_{k-1}}\right)\\
&-M(C_{41}q_{4,k}+C_{51}q_{5,k})=\text{const.}
\end{aligned} \tag{53}$$

The representations of $D_2[R_-L_{\rm d}(\phi_{u,k})]$, and $D_4[R_-L_{\rm d}(\phi_{u,k})]$ in Eq.(20) can be calculated from Eq.(30).

6. Conclusions

There are three main results in this paper, i.e. (i) discrete Euler–Lagrange equation (8) and energy evolution equation (9) for a discrete holonomic system with dependent coordinates, (ii) restricted equations of the Noether and Lie symmetry, (19) and (28), (iii) discrete Noether identity (18) and Noether conserved quantities (20) of a discrete holonomic system with dependent coordinates. These results can be generalized to other discrete constrained mechanical systems such as those with non-holonomic constraints.

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