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Perturbation to Mei symmetry and adiabatic invariants for Hamilton systems

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Based on the concept of adiabatic invariant, this paper studies the perturbation to Mei symmetry and adiabatic invariants for Hamilton systems. The exact invariants of Mei symmetry for the system without perturbation are given. The perturbation to Mei symmetry is discussed and the adiabatic invariants induced from the perturbation to Mei symmetry of the system are obtained.

Keywords: Mei symmetry, perturbation, adiabatic invariant, Hamilton system
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1. Introduction

Studies on symmetries and conserved quantities of a mechanical system have very important significance in mathematics, mechanics and physics. So far many achievements have been obtained in studying on symmetries and conserved quantities of mechanical systems.[1−12]

As we know, even a small change, that we can call a perturbation, may influence the original symmetries and exact invariants of mechanical systems. Perturbation to symmetries and adiabatic invariants play a very important role in mechanical systems. An adiabatic invariant is a certain physical quantity which changes more slowly than a slowly-varying parameter of the system.[13] So the study on perturbation to symmetries and adiabatic invariants of mechanical systems has great significance. In recent years, some works have been done on this problem.[14−20]

However, most previous works have been on the perturbation to Noether symmetry or Lie symmetry of mechanical systems. There rarely are studies on the perturbation to Mei symmetry of mechanical systems. Recently, Ref.[21] discussed the perturbation to Mei symmetry and adiabatic invariants induced from the perturbation to Mei symmetry for nonholonomic mechanical systems. In this paper, the perturbation to Mei symmetry and adiabatic invariants induced from the perturbation to Mei symmetry for Hamilton systems are studied. The exact invariants of Mei symmetry for the system without perturbation are given. Based on the concept of adiabatic invariant, we discuss the perturbation to Mei symmetry and obtain the adiabatic invariants induced from the perturbation to Mei symmetry of the system.

2. Mei symmetry and exact invariants of system

Consider a Hamilton system whose configuration is determined by the n generalized coordinates $q_s(s = 1, 2, \cdots, n)$ and n generalized momenta $p_s(s = 1, 2, \cdots, n)$. The Hamilton canonical equations of the system have the form

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}, \quad (s = 1, 2, \cdots, n), \tag{1}$$

where $H = H(t, \boldsymbol{q}, \boldsymbol{p})$ is the Hamiltonian of the system. Equation (1) can be transformed into the form

$$\dot{q}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad \dot{p}_s = g_s(t, \boldsymbol{q}, \boldsymbol{p}). \tag{2}$$

Introduce the infinitesimal transformations

$$\begin{aligned} t^* &= t + \varepsilon\tau^0(t, \boldsymbol{q}, \boldsymbol{p}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s^0(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^*(t^*) &= q_s(t) + \varepsilon\eta_s^0(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \tag{3}$$

where ε is an infinitesimal parameter and τ^0, ξ_s^0, η_s^0 are infinitesimal generators.

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According to Mei symmetry theory, the criteria of Mei symmetry for the Hamilton system can be written as

$$\frac{\partial}{\partial p_s}[X_0^{(0)}(H)]=0,\quad \frac{\partial}{\partial q_s}[X_0^{(0)}(H)]=0, \tag{4}$$

where

$$X_0^{(0)}=\tau^0\frac{\partial}{\partial t}+\xi_s^0\frac{\partial}{\partial q_s}+\eta_s^0\frac{\partial}{\partial p_s}. \tag{5}$$

For the Hamilton system Eq.(1), if the infinitesimal generators τ^0, ξ_s^0 and η_s^0 satisfy Eq.(4), the symmetry is Mei symmetry of the system.[2]

Proposition 1 For the Hamilton system Eq.(1), if the infinitesimal generators τ^0, ξ_s^0 and η_s^0 of Mei symmetry and a gauge function $G^0=G^0(t,\boldsymbol{q},\boldsymbol{p})$ satisfy the following structure equation:

$$p_s\dot{\xi}_s^0-H\dot{\tau}^0-\frac{\partial H}{\partial q_s}\xi_s^0-\frac{\partial H}{\partial t}\tau^0+\dot{G}^0=0, \tag{6}$$

Mei symmetry of the system can lead to an exact invariant[2]

$$I_0=p_s\xi_s^0-H\tau^0+G^0=\text{const}. \tag{7}$$

3. Perturbation to Mei symmetry and adiabatic invariants of system

According to the concept of adiabatic invariant,[13] for the Hamilton system we have

Definition If $I_z(t,\boldsymbol{q},\boldsymbol{p},\varepsilon)$ is a physical quantity including ε in which the highest power is z in a Hamilton system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, I_z is called a zth order adiabatic invariant of the Hamilton system.

Suppose the Hamilton system Eq.(1) is perturbed by small quantities $\varepsilon W_s=\varepsilon W_s(t,\boldsymbol{q},\boldsymbol{p})$, then the equations of motion for the system become

$$\dot{q}_s=\frac{\partial H}{\partial p_s},\quad \dot{p}_s=-\frac{\partial H}{\partial q_s}+\varepsilon W_s. \tag{8}$$

Due to the action of εW_s, the original symmetries and invariants of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformations of the original system without perturbation, then $\tau(t,\boldsymbol{q},\boldsymbol{p})$, $\xi_s(t,\boldsymbol{q},\boldsymbol{p})$ and $\eta_s(t,\boldsymbol{q},\boldsymbol{p})$ express the generators being perturbed, we have

$$\begin{aligned}\tau&=\tau^0+\varepsilon\tau^1+\varepsilon^2\tau^2+\cdots,\\ \xi_s&=\xi_s^0+\varepsilon\xi_s^1+\varepsilon^2\xi_s^2+\cdots,\\ \eta_s&=\eta_s^0+\varepsilon\eta_s^1+\varepsilon^2\eta_s^2+\cdots.\end{aligned} \tag{9}$$

After being perturbed, the original criteria of Mei symmetry for the Hamilton system become

$$\begin{aligned}&\frac{\partial}{\partial p_s}[X^{(0)}(H)]=0,\\ &\frac{\partial}{\partial q_s}[X^{(0)}(H)]-\varepsilon X^{(0)}(W_s)=0,\end{aligned} \tag{10}$$

where

$$X^{(0)}=\tau\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}+\eta_s\frac{\partial}{\partial p_s}. \tag{11}$$

Also, after being perturbed, the structure equation becomes

$$\begin{aligned}&p_s\dot{\xi}_s-H\dot{\tau}-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial t}\tau\\ &+\varepsilon W_s(\xi_s-\dot{q}_s\tau)+\dot{G}=0,\end{aligned} \tag{12}$$

here we have

$$G=G^0+\varepsilon G^1+\varepsilon^2G^2+\cdots. \tag{13}$$

Substituting Eq.(9) into Eq.(11), we obtain

$$X^{(0)}=\varepsilon^m X_m^{(0)}, \tag{14}$$

where

$$X_m^{(0)}=\tau^m\frac{\partial}{\partial t}+\xi_s^m\frac{\partial}{\partial q_s}+\eta_s^m\frac{\partial}{\partial p_s}. \tag{15}$$

Substituting Eqs.(9) into Eqs.(10) and (12), noticing Eqs.(13)–(15) and comparing the power of ε, we have

$$\begin{aligned}&\frac{\partial}{\partial p_s}[X_m^{(0)}(H)]=0,\\ &\frac{\partial}{\partial q_s}[X_m^{(0)}(H)]-X_{m-1}^{(0)}(W_s)=0,\end{aligned} \tag{16}$$

$$\begin{aligned}&p_s\dot{\xi}_s^m-H\dot{\tau}^m-\frac{\partial H}{\partial q_s}\xi_s^m-\frac{\partial H}{\partial t}\tau^m\\ &+W_s(\xi_s^{m-1}-\dot{q}_s\tau^{m-1})+\dot{G}^m=0.\end{aligned} \tag{17}$$

When $m=0$, we appoint that $\tau^{m-1}=\xi_s^{m-1}=\eta_s^{m-1}=0$ holds, then Eqs.(16) and (17) turn into Eqs.(4) and (6) respectively.

For the Hamilton system, when it is perturbed by small quantities εW_s, if the infinitesimal generators $\tau^m(t,\boldsymbol{q},\boldsymbol{p})$, $\xi_s^m(t,\boldsymbol{q},\boldsymbol{p})$ and $\eta_s^m(t,\boldsymbol{q},\boldsymbol{p})$ satisfy Eq.(16), the change of the symmetry is called the perturbation to Mei symmetry of the system.

Proposition 2 For the Hamilton system perturbed by small quantities εW_s, if the infinitesimal generators $\tau^m(t,\boldsymbol{q},\boldsymbol{p})$, $\xi_s^m(t,\boldsymbol{q},\boldsymbol{p})$, $\eta_s^m(t,\boldsymbol{q},\boldsymbol{p})$ and a gauge function $G^m=G^m(t,\boldsymbol{q},\boldsymbol{p})$ satisfy Eqs.(16) and (17), then the perturbation to Mei symmetry of the system will lead to a zth order adiabatic invariant

$$I_z=\varepsilon^m[p_s\xi_s^m-H\tau^m+G^m],\ \ (m=0,1,\cdots,z). \tag{18}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t} =&\varepsilon^m[\dot{p}_s\xi_s^m + p_s\dot{\xi}_s^m - \dot{H}\tau^m - H\dot{\tau}^m + \dot{G}^m]\\ =&\varepsilon^m\Bigg[\dot{p}_s\xi_s^m - \dot{H}\tau^m + \frac{\partial H}{\partial q_s}\xi_s^m + \frac{\partial H}{\partial t}\tau^m\\ &- W_s(\xi_s^{m-1} - \dot{q}_s\tau^{m-1})\Bigg]\\ =&\varepsilon^m\Bigg[\left(\dot{p}_s + \frac{\partial H}{\partial q_s}\right)(\xi_s^m - \dot{q}_s\tau^m)\\ &- W_s(\xi_s^{m-1} - \dot{q}_s\tau^{m-1})\Bigg]\\ =&\varepsilon^m[\varepsilon W_s(\xi_s^m - \dot{q}_s\tau^m) - W_s(\xi_s^{m-1} - \dot{q}_s\tau^{m-1})]\\ =&\varepsilon^{z+1}W_s(\xi_s^z - \dot{q}_s\tau^z). \end{aligned} \tag{19}$$

It shows that $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to ε^{z+1}.

4. An example

The Hamiltonian for the Toda crystal lattice of three particles is

$$\begin{aligned}H =&\frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \exp(q_1 - q_2)\\ &+ \exp(q_2 - q_3) + \exp(q_3 - q_1),\end{aligned} \tag{20}$$

and the system is perturbed by the small quantities

$$\varepsilon W_1 = -\varepsilon \mathrm{e}^{q_1}, \quad \varepsilon W_2 = \varepsilon W_3 = 0. \tag{21}$$

Let us study the first order adiabatic invariants induced from the perturbation to Mei symmetry of the system.

The equations of motion for the system without perturbation are

$$\begin{aligned}&\dot{q}_1 = p_1, \quad \dot{p}_1 = -\exp(q_1 - q_2) + \exp(q_3 - q_1),\\ &\dot{q}_2 = p_2, \quad \dot{p}_2 = -\exp(q_2 - q_3) + \exp(q_1 - q_2),\\ &\dot{q}_3 = p_3, \quad \dot{p}_3 = -\exp(q_3 - q_1) + \exp(q_2 - q_3).\end{aligned} \tag{22}$$

Firstly, we seek the exact invariants. Choose infinitesimal generators as

$$\begin{aligned}&\tau^0 = 1, \quad \xi_1^0 = \xi_2^0 = \xi_3^0 = 0,\\ &\eta_1^0 = \eta_2^0 = \eta_3^0 = 0.\end{aligned} \tag{23}$$

It is easy to verify that they satisfy Eqs.(4), so the infinitesimal generators (23) are Mei symmetrical for the system Eq.(20). Also, they satisfy Eq.(6).

From Eq.(6), we obtain the gauge function

$$G^0 = 0. \tag{24}$$

According to Proposition 1, Mei symmetry of the system Eq.(20) can lead to an exact invariant

$$\begin{aligned}I_0 =& -\Bigg[\frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \exp(q_1 - q_2)\\ &+ \exp(q_2 - q_3) + \exp(q_3 - q_1)\Bigg]\\ =&\text{const.}\end{aligned} \tag{25}$$

Then we seek the first order adiabatic invariants. Choose infinitesimal generators as

$$\begin{aligned}&\tau^1 = 0, \quad \xi_1^1 = \xi_2^1 = \xi_3^1 = 0,\\ &\eta_1^1 = \eta_2^1 = \eta_3^1 = 0.\end{aligned} \tag{26}$$

First of all, the infinitesimal generators (26) should satisfy the perturbed criteria of Mei symmetry Eq.(16), then it should also satisfy Eq.(17). From Eq.(17), we obtain

$$G^1 = -\mathrm{e}^{q_1}. \tag{27}$$

According to Proposition 2, we can get the first order adiabatic invariant induced from the perturbation to Mei symmetry of the system

$$\begin{aligned}I_1 =& -\Bigg[\frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \exp(q_1 - q_2)\\ &+ \exp(q_2 - q_3) + \exp(q_3 - q_1)\Bigg]\\ &- \varepsilon\Bigg[\frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \exp(q_1 - q_2)\\ &+ \exp(q_2 - q_3) + \exp(q_3 - q_1) + \mathrm{e}^{q_1}\Bigg].\end{aligned} \tag{28}$$

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