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Unified symmetry of mechano-electrical systems with nonholonomic constraints

Li Yuan-Cheng(李元成), Xia Li-Li(夏丽莉), Liu Bing(刘 冰),
Jiao Zhi-Yong(焦志勇), and Wang Xiao-Ming(王小明)

College of Physics Science and Technology, China University of Petroleum, Dongying 257061, *China*

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The unified symmetry of mechano-electrical systems with nonholonomic constraints are studied in this paper, the definition and the criterion of unified symmetry of mechano-electrical systems with nonholonomic constraints are derived from the Lagrange–Maxwell equations. The Noether conserved quantity, Hojman conserved quantity and Mei conserved quantity are then deduced from the unified symmetry. An example is given to illustrate the application of the results.

Keywords: nonholonomic constraints, mechano-electrical systems, unified symmetry, conserved quantity

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1. Introduction

The study of symmetries is of importance not only in mathematics but also in physics. In general, there are three kinds of symmetries: the Noether symmetry,[1−3] the Lie symmetry[3−8] and the Mei symmetry.[9−13] And there are correspondingly three kinds of conserved quantities: the Noether conserved quantity, the Hojman conserved quantity[14] and the Mei conserved quantity.[15]

The study on symmetries and conserved quantities has been developing rapidly these years. Recently, Mei Feng-Xiang and Wu Hui-Bin *et al* have studied unified symmetry, Lie-form invariance, Noether–Lie symmetry or Noether-form invariance of many kinds of mechanical systems.[16−25] In this paper, on the basis of Refs.[26–30], we study the unified symmetry of mechano-electrical systems with nonholonomic constraints and obtain the three kinds of conserved quantities mentioned above from the definition and the criterion of unified symmetry of the system.

2. The Lagrange–Maxwell equation of the system

We consider a mechano-electrical system consisting of m loops, each loops is composed of linear conductors and circuits, the circuits are independent of each other, but the electromagnetic processes of the loops are not mutually independent. The current in the kth loop is i_k $(k = 1, \cdots, m)$, and u_k is the electromotive force in the kth loop. Supposed e_k is the electric charge in the capacitor, the relation between the charge and the current is $\dot{e}_k = i_k$. The resistance and the capacitance of the kth loop are R_k and C_k, respectively, and its motion is subject to g nonholonomic constraints

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (\beta = 1, \cdots, g). \tag{1}$$

The condition that the constraints (1) act on the virtual displacements is

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0. \tag{2}$$

Its configuration is determined by the n generalized coordinates $e_k (k = 1, \cdots, m)$, $q_s (s = 1, \cdots, n)$. The Lagrange–Maxwell function of this system is

$$L = T(q_s, \dot{q}_s) - V(q_s) + W_{\rm m}(q_s, \dot{e}_k) - W_{\rm e}(q_s, e_k). \tag{3}$$

Introduce the summation of the electrical and mechanical dissipation function

$$\psi = \psi_{\rm e}(i_k) + \psi_{\rm m}(q_s, \dot{q}_s),$$

then the Lagrange–Maxwell equation of the system is

$$\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial \psi}{\partial \dot{e}_k} = u_k \quad (k = 1, \cdots, m),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}+\frac{\partial \psi}{\partial \dot{q}_s}=Q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\quad(s=1,\cdots,n),\tag{4}$$

where W_{m} is the magnetic field energy, W_{e} is the electric field energy, ψ_{e} is the electrical dissipation function

$$W_{\mathrm{m}}=\frac{1}{2}\sum_{k=1}^{m}\sum_{r=1}^{m}L_{kr}i_k i_r,$$
$$W_{\mathrm{e}}=\frac{1}{2}\sum_{k=1}^{m}\frac{e_k^2}{C_k},$$
$$\psi_{\mathrm{e}}=\frac{1}{2}\sum_{k=1}^{m}R_k i_k^2.\tag{5}$$

T is the potential energy and V is the kinetic energy of the system:

$$T=\frac{1}{2}\sum_{s=1}^{n}\sum_{l=1}^{n}a_{sl}\dot{q}_s\dot{q}_l;\quad V=V(q_s),\tag{6}$$

where $a_{sl}=a_{sl}(q_s)$. Equations (4) consist of $n+m$ second order differential equations with q_s and e_k as the generalized coordinates.

In general, it is supposed that before integrating Eqs.(4), we can find λ_β as a function of t, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\boldsymbol{e}$ and $\dot{\boldsymbol{e}}$ from the Eqs.(1) and Eqs.(4), while Eqs.(4) can be expressed as

$$E_k(L)=u_k-\frac{\partial \psi}{\partial \dot{e}_k},\quad(k=1,\cdots,m)$$
$$E_s(L)=Q_s+\Lambda_s-\frac{\partial \psi}{\partial \dot{q}_s},\quad(s=1,\cdots,n)\tag{7}$$

where $\Lambda_s=\lambda_\beta\dfrac{\partial f_\beta}{\partial \dot{q}_s}$, $E_s=\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial}{\partial \dot{q}_s}-\dfrac{\partial}{\partial q_s}$, $E_k=\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial}{\partial \dot{e}_s}-\dfrac{\partial}{\partial e_s}$.

It is supposed that the system (7) is nonsingular,

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_l}\right)\neq 0,\quad \det\left(\frac{\partial^2 L}{\partial \dot{e}_j\partial \dot{e}_k}\right)\neq 0.\tag{8}$$

From Eqs.(7), we can find all the generalized accelerations as

$$\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}),\quad \ddot{e}_k=\beta_k(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}).\tag{9}$$

3. Unified symmetry of the system

Definition If the symmetry of a mechano-electrical system with nonholonomic constraints is a Noether one as well as a Lie one and is also a Mei symmetry at one time, the symmetry is called a unified symmetry.

We first introduce the infinitesimal transformation:

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}),$$
$$q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}),$$
$$e_k^*(t^*)=e_k(t)+\varepsilon\eta_k(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}}),\tag{10}$$

where ε is an infinitesimal parameter and ξ_0, ξ_s, η_k are infinitesimal generators.

The constrain Eqs.(1) retain the form invariance:

$$X^{(1)}(f_\beta)=0,\quad(\beta=1,\cdots,g).\tag{11}$$

Limitations of the constraints (1) added on the virtual displacements δq are

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s-\dot{q}_s\xi_0)=0,\qquad(\beta=1,\cdots,g).\tag{12}$$

For the mechano-electrical system with nonholonomic constraints, the Noether identity can be written as

$$L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+X^{(1)}(L)+\left(Q_s+\Lambda_s-\frac{\partial \psi}{\partial \dot{q}_s}\right)(\xi_s-\dot{q}_s\xi_0)$$
$$+\left(u_k-\frac{\partial \psi}{\partial \dot{e}_k}\right)(\eta_k-\dot{e}_k\xi_0)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}}=0.\tag{13}$$

The determining equations of the Lie symmetry are

$$X^{(2)}\left\{\bar{E}_s(L)\right\}=X^{(1)}\left(Q_s+\Lambda_s-\frac{\partial \psi}{\partial \dot{q}_s}\right),$$
$$X^{(2)}\left\{\bar{E}'_k(L)\right\}=X^{(1)}\left(u_k-\frac{\partial \psi}{\partial \dot{e}_k}\right).\tag{14}$$

The criterion equations of the Mei symmetry are

$$\bar{E}_s\left\{X^{(1)}(L)\right\}=X^{(1)}\left(Q_s+\Lambda_s-\frac{\partial \psi}{\partial \dot{q}_s}\right),$$
$$\bar{E}'_k\left\{X^{(1)}(L)\right\}=X^{(1)}\left(u_k-\frac{\partial \psi}{\partial \dot{e}_k}\right),\tag{15}$$

where

$$X^{(1)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}+\eta_k\frac{\partial}{\partial e_k}+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta_k-\dot{e}_k\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{e}_k},$$

$$X^{(2)}=X^{(1)}+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-2\ddot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial\ddot{q}_s}+\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta_k-2\ddot{e}_k\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0-\dot{e}_k\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial\ddot{e}_k},$$
$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+\dot{e}_k\frac{\partial}{\partial e_k}+\alpha_s\frac{\partial}{\partial\dot{q}_s}+\beta_k\frac{\partial}{\partial\dot{e}_k},$$
$$\bar{E}_s=\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}-\frac{\partial}{\partial q_s},\qquad \bar{E}'_k=\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{e}_k}-\frac{\partial}{\partial e_k}.$$

Criterion For the mechano-electrical system with nonholonomic constraints, if there exists a gauge function $G_{\mathrm{N}}=G_{\mathrm{N}}(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}})$ such that the infinitesimal generators ξ_0, ξ_s and η_k satisfy Eqs.(11), (12), and

$$\begin{aligned}&\left\{L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+X^{(1)}(L)+\left(Q_s+\Lambda_s-\frac{\partial\psi}{\partial\dot{q}_s}\right)(\xi_s-\dot{q}_s\xi_0)+\left(u_k-\frac{\partial\psi}{\partial\dot{e}_k}\right)(\eta_k-\dot{e}_k\xi_0)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}}\right\}^2\\&+\left\{X^{(2)}[\bar{E}_s(L)]-X^{(1)}\left(Q_s+\Lambda_s-\frac{\partial\psi}{\partial\dot{q}_s}\right)\right\}^2+\left\{X^{(2)}[\bar{E}'_k(L)]-X^{(1)}\left(u_k-\frac{\partial\psi}{\partial\dot{e}_k}\right)\right\}^2\\&+\left\{\bar{E}_s[X^{(1)}(L)]_s-X^{(1)}\left(Q_s+\Lambda_s-\frac{\partial\psi}{\partial\dot{q}_s}\right)\right\}^2+\left\{\bar{E}'_k[X^{(1)}(L)]-X^{(1)}\left(u_k-\frac{\partial\psi}{\partial\dot{e}_k}\right)\right\}^2=0,\end{aligned}\tag{16}$$

then the symmetry of the system is a unified symmetry.

4. Conserved quantities deduced from unified symmetry

The unified symmetry of the mechano-electrical systems with nonholonomic constraints can lead to all of the Noether conserved quantity, the Hojman conserved quantity and the Mei conserved quantity under certain conditions.

Proposition 1 For the mechano-electrical system with nonholonomic constraints, the unified symmetry will lead to the Noether conserved quantity

$$\begin{aligned}I_{\mathrm{N}}&=L\xi_0+\frac{\partial L}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+\frac{\partial L}{\partial\dot{e}_k}(\eta_k-\dot{q}_s\xi_0)+G_{\mathrm{N}}\\&=\text{const.}\end{aligned}\tag{17}$$

Proof Because the unified symmetry of the mechano-electrical system is certainly a Noether symmetry, there exists a gauge function $G_{\mathrm{N}}=G_{\mathrm{N}}(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}})$ such that the Noether identity (13) holds. According to the Noether theorem, this system has the conserved quantity (17).

Proposition 2 For the mechano-electrical systems with nonholonomic constraints, under the special infinitesimal transformation that have unchanging time ($\xi_0=0$), if there is a function $\mu=\mu(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}})$ such that

$$\sum_{s=1}^{n}\frac{\partial\alpha_s}{\partial\dot{q}_s}+\sum_{k=1}^{m}\frac{\partial\beta_k}{\partial\dot{e}_k}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0,\tag{18}$$

then the unified symmetry will lead to a Hojman conserved quantity

$$\begin{aligned}I_{\mathrm{H}}&=\frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s)+\frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)\\&\quad+\frac{1}{\mu}\frac{\partial}{\partial e_k}(\mu\eta_k)+\frac{1}{\mu}\frac{\partial}{\partial\dot{e}_k}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta_k\right)\\&=\text{const.}\end{aligned}\tag{19}$$

Proof Because the unified symmetry of the mechano-electrical systems is certainly a Lie symmetry, the determining equations (14) of the Lie symmetry hold. From expressions (14) and (18), using Hojman's proof method of the mechano-electrical systems, the system has the conserved quantity (19).

Proposition 3 For the mechano-electrical systems with nonholonomic constraints, if there is a function $G_{\mathrm{M}}=G_{\mathrm{M}}(t,\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}})$ such that

$$\begin{aligned}&X^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+X^{(1)}\left[X^{(1)}(L)\right]\\&+X^{(1)}\left(Q_s+\Lambda_s-\frac{\partial\psi}{\partial\dot{q}_s}\right)(\xi_s-\dot{q}_s\xi_0)\\&+X^{(1)}\left(u_k-\frac{\partial\psi}{\partial\dot{e}_k}\right)(\eta_k-\dot{e}_k\xi_0)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_M\\&=0,\end{aligned}\tag{20}$$

then the unified symmetry will lead to the Mei conserved quantity

$$\begin{aligned}I_{\mathrm{M}}&=X^{(1)}(L)\xi_0+\frac{\partial X^{(1)}(L)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)\\&\quad+\frac{\partial X^{(1)}(L)}{\partial\dot{e}_k}(\eta_k-\dot{e}_k\xi_0)+G_{\mathrm{M}}\\&=\text{const.}\end{aligned}\tag{21}$$

Proof Because the unified symmetry of the mechano-electrical systems is certainly a form invariance, so Eqs.(15) hold, from expressions (15) and (20), we have the conserved quantity (21).

5. Illustrative example

Suppose that the mechano-electrical system has a capacitor, the voltage of electrical source is U, and q, e are the generalized coordinates of this system, the Lagrangian function can be written as $L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}\dot{e}^2 - k\dot{q}_1 - \frac{1}{2}Ae^2$, the electrical dissipation function is $\psi = \frac{1}{2}R\dot{e}^2$, the capacitor's voltage drop is $u = R\dot{e}$, where R is the resistance, k and A are constants, the nonholonomic constraints $f = \dot{q}_1 + \dot{q}_2 - 2t(q_1 + q_2) = 0$. Let us study the unified symmetry of the system.

The differential equations of motion of the system are

$$\ddot{q}_1 = \lambda, \quad \ddot{q}_2 = \lambda, \quad \ddot{e} = -Ae + U. \tag{22}$$

We have

$$\lambda = t(\dot{q}_1 + \dot{q}_2) + (q_1 + q_2). \tag{23}$$

The differential equations of motion of the system can be written as

$$\begin{aligned} &\ddot{q}_1 = t(\dot{q}_1 + \dot{q}_2) + (q_1 + q_2), \\ &\ddot{q}_2 = t(\dot{q}_1 + \dot{q}_2) + (q_1 + q_2), \\ &\ddot{e} = -Ae + u. \end{aligned} \tag{24}$$

The generators are as follows:

$$\xi_0 = 0, \quad \xi_1 = t, \quad \xi_2 = -t, \quad \eta = 0. \tag{25}$$

After calculation, we have

$$\begin{aligned} &X^{(1)}(L) = \dot{q}_1 - k - \dot{q}_2, \quad X^{(1)}[X^{(1)}(L)] = 2, \\ &\bar{E}_1(L) = \ddot{q}_1, \quad \bar{E}_2(L) = \ddot{q}_2, \\ &\bar{E}'_k(L) = \ddot{e} + Ae, \quad \bar{E}_s(X^{(1)}(L)) = 0, \\ &\bar{E}'_k(X^{(1)}(L)) = 0, \quad X^{(2)}(\bar{E}_s(L)) = 0, \\ &X^{(2)}(\bar{E}'_k(L)) = 0. \end{aligned} \tag{26}$$

The Eqs.(11), (12) and (16) hold. It is easy to verify that generators (25) are unified symmetry of the mechano-electrical system with nonholonomic constraints.

Substituting relations (25) and (26) into the Noether identity (13), we obtain the gauge function

$$G_{\mathrm{N}} = q_2 - q_1 + kt. \tag{27}$$

The Noether conserved quantity deduced from generators (25) is

$$I_{\mathrm{N}} = t(\dot{q}_1 - \dot{q}_2) + q_2 - q_1 = \mathrm{const}. \tag{28}$$

which is the Noether conserved quantity deduced from the unified symmetry (25).

Substituting relations (25) and (26) into the equation (18), we obtain the gauge function

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\mu = -2t, \quad \mu = (\dot{q}_1 - \dot{q}_2)\mathrm{e}^{-t^2}. \tag{29}$$

From proposition 2, we have

$$I_{\mathrm{H}} = \frac{2}{\dot{q}_1 - \dot{q}_2} = \mathrm{const}, \tag{30}$$

which is a Hojman conserved quantity deduced from the unified symmetry (25).

Substituting relations (25) and (26) into Eqs.(20), we obtain

$$G_{\mathrm{M}} = \dot{q}_1 - \dot{q}_2 - 2t, \tag{31}$$

and proposition 3 gives

$$I_{\mathrm{M}} = \dot{q}_1 - \dot{q}_2 = \mathrm{const}, \tag{32}$$

which is the Mei conserved quantity deduced from the unified symmetry (25).

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