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Existences and stabilities of bright and dark breathers in a general one-dimensional discrete monatomic Chain*

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A general one-dimensional discrete monatomic model is investigated by using the multiple-method. It is proven that the discrete bright breathers (DBBs) and discrete dark breathers (DDBs) exist in this model at the anti-continuous limit, and then the concrete models of the DBBs and DDBs are also presented by the multiple-scale approach (MSA) and the quasi-discreteness approach (QDA). When the results are applied to some particular models, the same conclusions as those presented in corresponding references are achieved. In addition, we use the method of the linearization analysis to investigate this system without the high order terms of ε. It is found that the DBBs and DDBs are linearly stable only when coupling parameter χ is small, of which the limited value is obtained by using an analytical method.

Keywords: discrete bright breather, discrete dark breather, linear stability, anti-continuous limit
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1. Introduction

It is well known that discrete breathers (DBs), as intrinsic localized modes, are exact, periodic and localized solutions that can be obtained in a large variety of nonlinear discrete systems. They are usually understood as the time-periodic, spatially localized solution in a discrete system and have two kinds of modes, i.e. the discrete bright breather (DBB) and the discrete dark breather (DDB). In the former there is only one or a few oscillators vibrating with large amplitudes whereas the rest of them oscillate with small amplitudes; on the contrary, all the oscillators for the latter vibrate with large amplitudes except one or a few of them oscillating with very small amplitudes. The proofs of existences of bright one-site breathers and multi-breathers in a weakly coupled oscillator system and their stable properties were presented in Refs.[1–8], where the notion of the anti-continuous limit was used. The anti-continuous limit of a system, depending on at least one parameter χ, is the limit where the system becomes equivalent to a collection of uncoupled oscillators (e.g. $\chi = 0$). At the same time, quite a lot of other studies have been carried out to investigate the discrete localized models in monatomic or diatomic Fermi–Pasta–Ulam (FPU) and Klein–Gordon (KG) systems and others[9−16] by using the methods of the rotating-wave approximation (RWA) and the multiple-scale approach (MSA) combining with the quasi-discreteness approach (QDA). The intrinsic localized and periodic modes, the DBBs, were obtained. In this period, Xu and Tian also found other intrinsic localized modes, i.e. discrete-soliton trains (DSTs) in 2D discrete monatomic lattices,[17] and 3D envelop solitons and 3D vortices in 3D discrete monatomic lattices.[18] However, only a few papers about the DDBs can be found, where the DDBs in the KG lattice and the FPU lattice[19,20] are discussed, separately.

In this paper, we perform some analyses of the existences and the stabilities of the DBBs and the DDBs in a one-dimensional monatomic chain with on-site as well as inter-site potentials. This model almost covers all of one-dimensional monatomic systems, so we can catch the properties of the DBBs and the DDBs in a large variety of one-dimensional nonlinear discrete monatomic systems, if we master the properties of the DBBs and the DDBs in this model. In addition, we also calculate several concrete modes of the DBBs and the DDBs to reveal the properties of the DBBs and

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the DDBs in detail.

This paper is organized as follows. In Section 2, we describe the model, a one-dimensional monatomic chain with on-site as well as inter-site potentials, and the evolution equation. In Section 3, we first show the theorem of Mackay and Aubry. It gives a positive solution to the question about the existence of the DBBs and DDBs, and then we calculate their concrete modes and discuss several particular models. In Section 4, we analyse the linear stabilities of the DBBs and DDBs. The paper concludes with a short summary of the main results in Section 5.

2. The model

We study a one-dimensional monatomic chain with on-site and inter-site potentials. The Hamiltonian of the system is given by

$$H=\sum_n\left(\frac{1}{2}\dot{u}_n^2+V\left(u_n\right)\right)+\chi W\left(\boldsymbol{u}\right), \tag{1}$$

where u_n is the displacement of the nth atom; $V\left(u_n\right)$ is the on-site potential; $\boldsymbol{u}$ represents a set of variables $\{u_n\}$ and $\chi W(\boldsymbol{u})$ is the inter-site potential (the coupling potential) with χ being a parameter that describes the strength of the coupling. We suppose that all the atoms in the lattice have mass unity and initially χ is positive. The $W(\boldsymbol{u})$ can be defined as

$$\begin{aligned}W\left(u\right)=&\frac{K_2}{2}\sum_n\left(u_{n+1}-u_n\right)^2+\frac{K_3}{3}\sum_n\left(u_{n+1}-u_n\right)^3\\&+\frac{K_4}{4}\sum_n\left(u_{n+1}-u_n\right)^4+O\left(u^5\right).\end{aligned} \tag{2}$$

This interaction is attractive because a nonzero value of a variable tends to increase the values of the neighbouring variables with the same signs. The on-site potential is given by

$$V\left(u_n\right)=\frac{1}{2}\omega_0^2u_n^2+\frac{1}{3}\alpha u_n^3+\frac{1}{4}\beta u_n^4+O\left(u_n^5\right), \tag{3}$$

where ω_0 is the linear frequency. Restricting to small-amplitude atoms in multi-potentials, the equation of motion for the displacement of the nth atom from its equilibrium (to $O(u^5)$) reads

$$\begin{aligned}\ddot{u}_n=&-\omega_0^2u_n-\alpha u_n^2-\beta u_n^3\\&+\chi\Big\{K_2\left(u_{n+1}+u_{n-1}-2u_n\right)\\&+K_3\left(u_{n+1}-u_{n-1}\right)\left(u_{n+1}+u_{n-1}-2u_n\right)\\&+K_4\left[\left(u_{n+1}-u_n\right)^3-\left(u_n-u_{n-1}\right)^3\right]\Big\}.\end{aligned} \tag{4}$$

This equation is a discrete equation, and its discrete modes can be obtained by virtue of the methods of the MSA and the QDA, or the RWA. The solutions depend obviously on the chosen potentials $V(u_n)$ and $W(\boldsymbol{u})$. We will analyse first the system for general potentials $V(u_n)$ and $W(\boldsymbol{u})$, and then the systems for several kinds of concrete potentials $V(u_n)$ and $W(\boldsymbol{u})$. These models appear in many physical systems.

3. The existences of DBBs and DDBs and the concrete modes

In this section we look for DBB solutions of Eq.(4). And the method of calculating DBBs and DDBs is based on the general methods of obtaining DBs.[1−4] The existence theorem by Aubry[1,4] states that every solution at the anti-continuous limit corresponding to a code sequence (CDS) can be continued up to a certain value of the coupling parameter $\chi_{\rm C}\neq 0$, and these continued solutions (CTSs) each will preserve their character. For example, if the CDS is periodic, the CTS is also periodic with the same period. If the CDS is chaotic with some topological entropy, the CTS is chaotic with the same topological entropy. If CDS is quasi-periodic, the CTS is quasi-periodic, etc.

3.1. The proofs of the existence of DBBs and DDBs

According to the above theorem, the solutions of the model with on-site and inter-site potentials will preserve the properties of the solutions of the model only with corresponding on-site potentials, if the conditions of the anti-continuous limit are satisfied. For example, if the solutions of the model only with on-site potentials are DBBs or DDBs, the solutions of the model with multi-potential are also DBBs or DDBs, as long as the following two hypotheses by MacKay and Aubry[1] are fulfilled.

The orbit of the uncoupled excited oscillator with the chosen frequency must satisfy $\dfrac{\partial\omega_b}{\partial I}\neq 0$, where $I=\int p\mathrm{d}q$ is the action variable of the oscillator; that is, the oscillator is truly nonlinear at that frequency.

The frequency of the orbit must satisfy $m\omega_b\neq\omega_0$ for any integer m; that is, none of the breather harmonics coincides with linear frequency ω_0.

The CTSs are determined by the CDS, so we must distinguish three types of solutions of the isolated oscillators that can be coded in the following way: In Ref.[4], $\sigma_n = 0$ for an oscillator at rest ($u_n(t) = 0, \forall t$); $\sigma_n = +1$ for an excited oscillator with frequency ω_b and $u_n(0) > 0$; $\sigma_n = -1$ for an excited oscillator with frequency ω_b and $u_n(0) < 0$. Time-reversible solutions of the whole system at $\chi = 0$ (anti-continuous limit) can be referred to by a coding sequence $\sigma = \{\sigma_n\}$. Hence, $\sigma = \{0, \ldots, 0, 1, 0, \ldots, 0\}$ corresponds to a single-site breather, and $\sigma = \{0, \ldots, 0, 1, 1, 0, \ldots, 0\}$ to a symmetric two-site breather (multi-breather), they are DBBs. In contrast, $\sigma = \{1, \ldots, 1, 0, 1, \ldots, 1\}$ corresponds to a one-site DDB. At the same time, the considered periodic solutions each correspond to a close trajectory in the phase space.

Figure 1 shows the phase space of single oscillator in model (1), and suggests that each equation included in Eq.(4) has a periodic solution at the anti-continuous limit $\chi = 0$. They can be coded in the above way, so

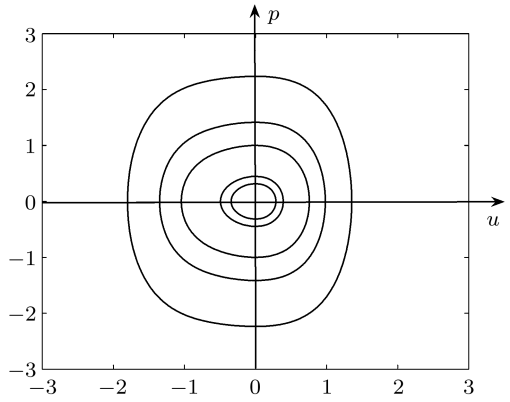


Fig.1. Nonlinear potential $V(u_n) = \frac{1}{2}u_n^2 + \frac{1}{3}u_n^3 + \frac{1}{4}u_n^4$ trajectories in the phase space.

the theorem gives an immediate solution to the question of the existences of the DBBs and DDBs. Because they are obtained by continuation of the configuration mentioned above $\sigma = \{0, \ldots, 0, 1, 0, \ldots, 0\}$, $\sigma = \{0, \ldots, 0, 1, 1, 0, \ldots, 0\}$, and $\sigma = \{1, \ldots, 1, 0, 1, \ldots, 1\}$, for any coupling potential which must satisfy the above two hypotheses.

3.2. The modes of the DBBs and the DDBs

In the following, we discuss Eq.(4) by virtue of the methods of the MSA and the QDA with $\chi \neq 0$. According to the MSA and the QDA, we assume

$$\begin{aligned} u_n(t) = {} & \varepsilon u_{n,n}^{(1)}(\xi_n, \tau, \phi_n) + \varepsilon^2 u_{n,n}^2(\xi_n, \tau, \phi_n) \\ & + \varepsilon^3 u_{n,n}^3(\xi_n, \tau, \phi_n) + \ldots, \end{aligned} \tag{5}$$

where ε is a small ordering parameter denoting the relative amplitude of the excitation; $\tau = \varepsilon^2 t$; $\xi_n = \varepsilon(na - \lambda t)$ is the multiple-scale slow variable; the fast variable, $\phi_n = nka - \omega t$, represents the phase of the carrier wave with k and ω being the number and the frequency of the carrier wave, respectively, a being the lattice constant, and λ being the parameter to be determined later. We perform the Taylor expansion

$$\begin{aligned} u_{n\pm1}^{(\gamma)} = {} & u_{n,n\pm1}^{(\gamma)} \pm \varepsilon a \frac{\partial u_{n,n\pm1}^{(\gamma)}}{\partial \xi_n} + \frac{\varepsilon^2 a^2}{2!} \frac{\partial^2 u_{n,n\pm1}^{(\gamma)}}{\partial \xi_n^2} \\ & \pm \frac{\varepsilon^3 a^3}{3!} \frac{\partial^3 u_{n,n\pm1}^{(\gamma)}}{\partial \xi_n^3} + \cdots, \\ u_{n\pm1} = {} & \sum_{\gamma=1}^{\infty} \varepsilon^\gamma \left[\sum_{q=0}^{\infty} \frac{1}{q!} \left(\pm \varepsilon a \frac{\partial}{\partial \xi_n} \right)^q u_{n,n\pm1}^{(\gamma)} \right]. \end{aligned} \tag{6}$$

Using the method of the chain, we make the transformation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \varepsilon^2 \frac{\partial}{\partial \tau} - \varepsilon\lambda \frac{\partial}{\partial \xi_n} - \omega \frac{\partial}{\partial \phi_n}, \tag{7a}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} = {} & \omega^2 \frac{\partial^2}{\partial \phi_n^2} + 2\varepsilon\omega\lambda \frac{\partial^2}{\partial \phi_n \partial \xi_n} + \varepsilon^2 \lambda^2 \frac{\partial^2}{\partial \xi_n^2} \\ & - 2\varepsilon^2 \omega \frac{\partial^2}{\partial \phi_n \partial \tau} - 2\varepsilon^3 \lambda \frac{\partial^2}{\partial \tau \partial \xi_n} + \varepsilon^4 \frac{\partial^2}{\partial \tau^2}. \end{aligned} \tag{7b}$$

Substituting Eqs.(5), (6) and (7) into Eq.(4), we have

$$\begin{aligned} & \varepsilon\omega^2 \frac{\partial^2 u_{n,n}^{(1)}}{\partial \phi_n^2} + 2\varepsilon^2 \omega\lambda \frac{\partial^2 u_{n,n}^{(1)}}{\partial \phi_n \partial \xi_n} + \varepsilon^3 \lambda^2 \frac{\partial^2 u_{n,n}^{(1)}}{\partial \xi_n^2} - 2\varepsilon^3 \omega \frac{\partial^2 u_{n,n}^{(1)}}{\partial \phi_n \partial \tau} - 2\varepsilon^4 \lambda \frac{\partial^2 u_{n,n}^{(1)}}{\partial \tau \partial \xi_n} + \varepsilon^5 \frac{\partial^2 u_{n,n}^{(1)}}{\partial \tau^2} \\ & + \varepsilon^2 \omega^2 \frac{\partial^2 u_{n,n}^{(2)}}{\partial \phi_n^2} + 2\varepsilon^3 \omega\lambda \frac{\partial^2 u_{n,n}^{(2)}}{\partial \phi_n \partial \xi_n} + \varepsilon^4 \lambda^2 \frac{\partial^2 u_{n,n}^{(2)}}{\partial \xi_n^2} - 2\varepsilon^4 \omega \frac{\partial^2 u_{n,n}^{(2)}}{\partial \phi_n \partial \tau} - 2\varepsilon^5 \lambda \frac{\partial^2 u_{n,n}^{(2)}}{\partial \tau \partial \xi_n} + \varepsilon^6 \frac{\partial^2 u_{n,n}^{(2)}}{\partial \tau^2} \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^3\omega^2\frac{\partial^2 u_{n,n}^{(3)}}{\partial\phi_n^2}+2\varepsilon^4\omega\lambda\frac{\partial^2 u_{n,n}^{(3)}}{\partial\phi_n\partial\xi_n}+\varepsilon^5\lambda^2\frac{\partial^2 u_{n,n}^{(3)}}{\partial\xi_n^2}-2\varepsilon^5\omega\frac{\partial^2 u_{n,n}^{(3)}}{\partial\phi_n\partial\tau}-2\varepsilon^6\lambda\frac{\partial^2 u_{n,n}^{(3)}}{\partial\tau\partial\xi_n}+\varepsilon^7\frac{\partial^2 u_{n,n}^{(3)}}{\partial\tau^2}+\cdots\\
=&-\omega_0^2\left(\varepsilon u_{n,n}^{(1)}+\varepsilon^2 u_{n,n}^{(2)}+\varepsilon^3 u_{n,n}^{(3)}+\cdots\right)-\alpha\left(\varepsilon u_{n,n}^{(1)}+\varepsilon^2 u_{n,n}^{(2)}+\cdots\right)^2-\beta\left(\varepsilon u_{n,n}^{(1)}+\varepsilon^2 u_{n,n}^{(2)}+\cdots\right)^3\\
&+\chi\left\{K_2\left[\varepsilon\left(u_{n,n+1}^{(1)}+u_{n,n-1}^{(1)}-2u_{n,n}^{(1)}\right)\pm\varepsilon^2 a\frac{\partial u_{n,n\pm1}^{(1)}}{\partial\xi_n}+\varepsilon^2\left(u_{n,n+1}^{(2)}+u_{n,n-1}^{(2)}-2u_{n,n}^{(2)}\right)+\frac{\varepsilon^3 a^2}{2}\frac{\partial^2 u_{n,n\pm1}^{(1)}}{\partial\xi_n^2}\right.\right.\\
&\left.\pm\varepsilon^3 a\frac{\partial u_{n,n\pm1}^{(2)}}{\partial\xi_n}+\varepsilon^3\left(u_{n,n+1}^{(3)}+u_{n,n-1}^{(3)}-2u_{n,n}^{(3)}\right)+\cdots\right]+K_3\left[\varepsilon\left(u_{n,n+1}^{(1)}-u_{n,n-1}^{(1)}\right)+\varepsilon^2 a\left(\frac{\partial u_{n,n+1}^{(1)}}{\partial\xi_n}\right.\right.\\
&\left.+\frac{\partial u_{n,n-1}^{(1)}}{\partial\xi_n}\right)+\varepsilon^2\left(u_{n,n+1}^{(2)}-u_{n,n-1}^{(2)}\right)+\cdots\Big]\left[\varepsilon\left(u_{n,n+1}^{(1)}+u_{n,n-1}^{(1)}-2u_{n,n}^{(1)}\right)+\varepsilon^2 a\left(\frac{\partial u_{n,n+1}^{(1)}}{\partial\xi_n}-\frac{\partial u_{n,n-1}^{(1)}}{\partial\xi_n}\right)\right.\\
&\left.+\varepsilon^2\left(u_{n,n+1}^{(2)}+u_{n,n-1}^{(2)}-2u_{n,n}^{(2)}\right)+\cdots\Big]+K_4\left[\varepsilon\left(u_{n,n+1}^{(1)}-u_{n,n}^{(1)}\right)+\cdots\right]^3\\
&\left.-K_4\left[\varepsilon\left(u_{n,n}^{(1)}-u_{n,n-1}^{(1)}\right)+\cdots\right]^3\right\}. \qquad (8)
\end{aligned}
$$

Comparing the powers of ε in Eq.(8) yields

$$\varepsilon^1:\omega^2\frac{\partial^2 u_{n,n}^{(1)}}{\partial\phi_n^2}+\omega_0^2 u_{n,n}^{(1)}-\chi K_2\left(u_{n,n+1}^{(1)}+u_{n,n-1}^{(1)}-2u_{n,n}^{(1)}\right)=0, \qquad (9a)$$

$$
\begin{aligned}
\varepsilon^2:&\omega^2\frac{\partial^2 u_{n,n}^{(2)}}{\partial\phi_n^2}+\omega_0^2 u_{n,n}^{(2)}-\chi K_2\left(u_{n,n+1}^{(2)}+u_{n,n-1}^{(2)}-2u_{n,n}^{(2)}\right)\\
&=-\alpha u_{n,n}^{(1)2}-2\omega\lambda\frac{\partial^2 u_{n,n}^{(1)}}{\partial\phi_n\partial\xi_n}\pm\chi K_2 a\frac{\partial u_{n,n\pm1}^{(1)}}{\partial\xi_n}+\chi K_3\left(u_{n,n+1}^{(1)}-u_{n,n-1}^{(1)}\right)\left(u_{n,n+1}^{(1)}+u_{n,n-1}^{(1)}-2u_{n,n}^{(1)}\right), \qquad (9b)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^3:&\omega^2\frac{\partial^2 u_{n,n}^{(3)}}{\partial\phi_n^2}+\omega_0^2 u_{n,n}^{(3)}-\chi K_2\left(u_{n,n+1}^{(3)}+u_{n,n-1}^{(3)}-2u_{n,n}^{(3)}\right)\\
=&-2\alpha u_{n,n}^{(1)}u_{n,n}^{(2)}-\beta u_{n,n}^{(1)3}-\lambda^2\frac{\partial^2 u_{n,n}^{(1)}}{\partial\xi_n^2}+2\omega\frac{\partial^2 u_{n,n}^{(1)}}{\partial\tau\partial\phi_n}-2\omega\lambda\frac{\partial^2 u_{n,n}^{(2)}}{\partial\phi_n\partial\xi_n}\pm\chi\frac{K_2 a^2}{2}\frac{\partial^2 u_{n,n\pm1}^{(1)}}{\partial\xi_n^2}\pm\chi K_2\frac{\partial u_{n,n\pm1}^{(2)}}{\partial\xi_n}\\
&+\chi K_3\left[a\left(u_{n,n+1}^{(1)}-u_{n,n-1}^{(1)}\right)\left(\frac{\partial u_{n,n+1}^{(1)}}{\partial\xi_n}-\frac{\partial u_{n,n-1}^{(1)}}{\partial\xi_n}\right)+\left(u_{n,n+1}^{(1)}-u_{n,n-1}^{(1)}\right)\left(u_{n,n+1}^{(2)}+u_{n,n-1}^{(2)}-2u_{n,n}^{(2)}\right)\right.\\
&+\left(u_{n,n+1}^{(2)}-u_{n,n-1}^{(2)}\right)\left(u_{n,n+1}^{(1)}+u_{n,n-1}^{(1)}-2u_{n,n}^{(1)}\right)\\
&\left.+a\left(\frac{\partial u_{n,n+1}^{(1)}}{\partial\xi_n}+\frac{\partial u_{n,n-1}^{(1)}}{\partial\xi_n}\right)\left(u_{n,n+1}^{(1)}+u_{n,n-1}^{(1)}-2u_{n,n}^{(1)}\right)\right]\\
&+\chi K_4\left[\left(u_{n,n+1}^{(1)}-u_{n,n}^{(1)}\right)^3-\left(u_{n,n}^{(1)}-u_{n,n-1}^{(1)}\right)^3\right]. \qquad (9c)
\end{aligned}
$$

Equation (9a) has the solution as follows:

$$u_{n,n}^{(1)}=A\left(\xi_n,\tau\right)\mathrm{e}^{\mathrm{i}\phi_n}+\text{c.c.}, \qquad (10)$$

where c.c. stands for complex conjugate, which is also used hereinafter.

The dispersion relation is

$$\omega^2=\omega_0^2-2\chi K_2\left(\cos ka-1\right). \qquad (11)$$

Inserting Eq.(10) into Eq.(9b), we find

$$
\begin{aligned}
&\omega^2\frac{\partial^2 u_{n,n}^{(2)}}{\partial\phi_n^2}+\omega_0^2 u_{n,n}^{(2)}-\chi K_2\left(u_{n,n+1}^{(2)}+u_{n,n-1}^{(2)}-2u_{n,n}^{(2)}\right)\\
=&-2\alpha\left|A\right|^2+2\mathrm{i}\left(\chi K_2 a\sin ka-\omega\lambda\right)\times\frac{\partial A}{\partial\xi_n}\mathrm{e}^{\mathrm{i}\phi_n}+\left[4\mathrm{i}\chi K_3\sin ka\left(\cos ka-1\right)-2\alpha\right]A^2\mathrm{e}^{2\mathrm{i}\phi_n}. \qquad (12)
\end{aligned}
$$

The terms proportional to $\mathrm{e}^{\mathrm{i}\phi_n}$ on the right-hand side of Eq.(12) are secular terms which must be eliminated for the theory to hold true. Thus we have

$$\omega^2\frac{\partial^2 u_{n,n}^{(2)}}{\partial\phi_n^2}+\omega_0^2u_{n,n}^{(2)}-\chi K_2\left(u_{n,n+1}^{(2)}+u_{n,n-1}^{(2)}-2u_{n,n}^{(2)}\right)=-2\alpha\left|A\right|^2+\left[4\mathrm{i}\chi K_3\sin ka\left(\cos ka-1\right)-2\alpha\right]A^2\mathrm{e}^{2\mathrm{i}\phi_n},\tag{13a}$$

$$\chi K_2a\sin ka-\omega\lambda=0.\tag{13b}$$

From Eq.(13a), we obtain

$$u_{n,n}^{(2)}=-2\frac{\alpha}{\omega_0^2}\left|A\right|^2+\delta A^2\mathrm{e}^{2\mathrm{i}\phi_n}+\text{c.c.},\tag{14}$$

where $\delta=\dfrac{8\mathrm{i}\chi K_3\sin ka\sin^2\left(ka/2\right)-2\alpha}{3\omega_0^2+16\chi K_2\sin^4\left(ka/2\right)}$. Substituting Eqs.(10) and (14) into Eq.(9c) yields

$$\omega^2\frac{\partial^2 u_{n,n}^{(3)}}{\partial\phi_n^2}+\omega_0^2u_{n,n}^{(3)}-\chi K_2\left(u_{n,n+1}^{(3)}+u_{n,n-1}^{(3)}-2u_{n,n}^{(3)}\right)=C_0+C_1\mathrm{e}^{\mathrm{i}\phi_n}+C_2A\frac{\partial A}{\partial\xi_n}\mathrm{e}^{2\mathrm{i}\phi_n}+C_3A^3\mathrm{e}^{3\mathrm{i}\phi_n},\tag{15}$$

where

$$\begin{aligned}
C_0 &= \left[4\chi K_3a\left(1-\cos ka\right)-2\frac{\alpha}{\omega_0^2}\chi K_2a\right]\frac{\partial\left|A\right|^2}{\partial\xi_n},\\
C_1 &= \mathrm{i}2\omega\frac{\partial A}{\partial\tau}+\left(\chi K_2a^2\cos ka-\lambda^2\right)\frac{\partial^2A}{\partial\xi_n^2}+\left[2\alpha\left(2\frac{\alpha}{\omega_0^2}-\delta\right)-3\beta-4\mathrm{i}\chi K_3a\sin ka\left(\cos 2ka-1\right)\delta\right.\\
&\quad\left.+4\mathrm{i}\chi K_3\sin 2ka\left(\cos ka-1\right)\delta+4\chi K_4\times\left(\cos ka-1\right)\left(3-3\cos ka-\cos 2ka\right)\right]A\left|A\right|^2,\\
C_2 &= 8\mathrm{i}\chi K_2a\sin ka\left(\cos ka-1\right)\delta-4\chi K_3a\cos ka\left(\cos ka-1\right),\\
C_3 &= -\left(2\alpha\delta+\beta\right)+4\mathrm{i}\chi K_3\sin ka\left(\cos 2ka-1\right)\alpha+4\mathrm{i}\chi K_3\sin 2ka\left(\cos ka-1\right)\delta+4\chi K_4\left(\cos ka-1\right)\\
&\quad\times\left(\cos 2ka-\cos ka\right).
\end{aligned}$$

Removing the secular terms $\mathrm{e}^{\mathrm{i}\phi_n}$ on the right-hand side of Eq.(15), we obtain

$$\mathrm{i}\frac{\partial A}{\partial\tau}+D_1\frac{\partial^2A}{\partial\xi_n^2}+D_2A\left|A\right|^2=0,\tag{16}$$

where

$$\begin{aligned}
D_1 &= \frac{\chi K_2a^2\cos ka-\lambda^2}{2\omega},\\
D_2 &= \frac{1}{2\omega}\left[16\alpha^2\left(\frac{\omega_0^2+4\chi K_2\sin^4\left(ka/2\right)}{\omega_0^2\left(3\omega_0^2+16\chi K_2\sin^4\left(ka/2\right)\right)}\right)-3\beta-\frac{64\chi^2K_2^2a\sin^4ka\sin^2\left(ka/2\right)}{3\omega_0^2+16\chi K_2\sin^4\left(ka/2\right)}\right.\\
&\quad\left.+\frac{128\chi^2K_3^2\sin^2ka\sin^4\left(ka/2\right)\cos ka}{3\omega_0^2+16\chi K_2\sin^4\left(ka/2\right)}+4\chi K_4\times\left(\cos ka-1\right)\left(3-3\cos ka-\cos 2ka\right)\right];
\end{aligned}$$

D_1 and D_2 must be real. Equation (16) is the nonlinear Schrödinger (NLS) equation, and its solutions have the following forms:[21]

When $D_1>0$ and $D_2>0$, we have

$$A=\pm\sqrt{\frac{2\left|D_1-\omega_1\right|}{D_2}}\operatorname{sech}\sqrt{\frac{\left|D_1-\omega_1\right|}{D_1}}\left(\xi_n-2D_1\tau\right)\cdot\mathrm{e}^{\mathrm{i}\left(\xi_n-\omega_1\tau\right)};\tag{17a}$$

When $D_1>0$ and $D_2<0$, we have

$$A=\pm\sqrt{-\frac{2\left|D_1-\omega_1\right|}{D_2}}\operatorname{csch}\sqrt{\frac{\left|D_1-\omega_1\right|}{D_1}}\left(\xi_n-2D_1\tau\right)\cdot\mathrm{e}^{\mathrm{i}\left(\xi_n-\omega_1\tau\right)};\tag{17b}$$

When $D_1 < 0$ and $D_2 > 0$, we have

$$A = \pm\sqrt{\frac{|D_1 - \omega_1|}{D_2}}\tanh\sqrt{-\frac{|D_1 - \omega_1|}{2D_1}}\,(\xi_n - 2D_1\tau)\cdot \mathrm{e}^{\mathrm{i}(\xi_n - \omega_1\tau)}. \tag{17c}$$

Let $\dfrac{|D_1 - \omega_1|}{D_2} = \mu$, $\dfrac{|D_1 - \omega_1|}{D_1} = \eta$, $\varepsilon\lambda + 2\varepsilon^2 D_1 = \lambda_0$, $\varepsilon + k = k_0$ and $\varepsilon\lambda + \varepsilon^2\omega_1 + \omega = \omega_b$, then Eq.(4) will have the solutions as follows:

When $D_1 >$ and $D_2 > 0$, we find

$$u_n = \varepsilon\sqrt{2\mu}\mathrm{sech}\sqrt{\eta}\,[\varepsilon na - \lambda_0 t]\cos\left[k_0 na - \omega_b t\right]; \tag{18a}$$

When $D_1 > 0$ and $D_2 < 0$, we find

$$u_n = \varepsilon\sqrt{-2\mu}\mathrm{csch}\sqrt{\eta}\,[\varepsilon na - \lambda_0 t]\cos\left[k_0 na - \omega_b t\right]; \tag{18b}$$

When $D_1 < 0$ and $D_2 > 0$, we find

$$u_n = \varepsilon\sqrt{\mu}\tanh\sqrt{-\eta/2}\,[\varepsilon na - \lambda_0 t]\cos\left[k_0 na - \omega_b t\right]. \tag{18c}$$

Figure 2 shows the forms of solutions (18a) and (18b), respectively. So they are the DBBs, i.e. the DBBs can exist in model (1).

Figure 3 shows the forms of solutions (18c), so they are the DDBs, i.e. the DDBs can exist in model (1). In a word, the DBBs and DDBs can be found in a one-dimensional monatomic chain with on-site and inter-site potentials indeed.

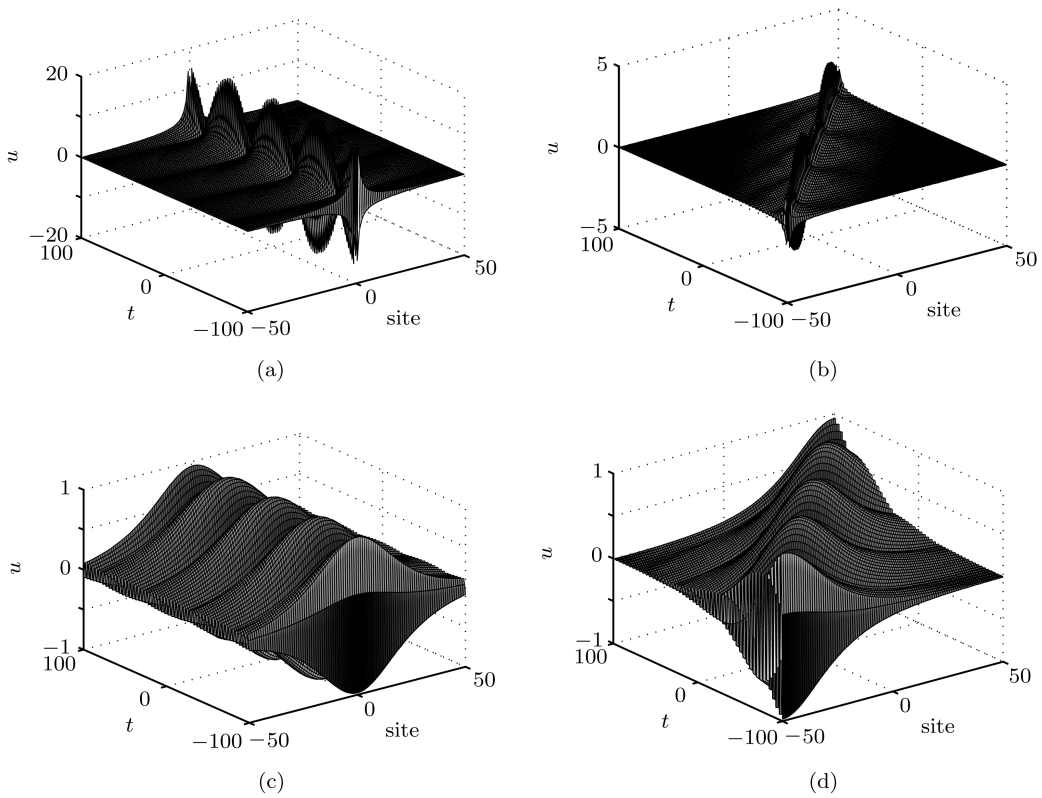


Fig.2. The DBBs in a 1D chain with the multi-potential. Panels (a) and (b) show the results from expression (18b). Panels (c) and (d) show the results from expression (18a). In Panels (a) and (c) there exists the relation $\varepsilon\lambda + 2\varepsilon^2 D_1 = 0$.

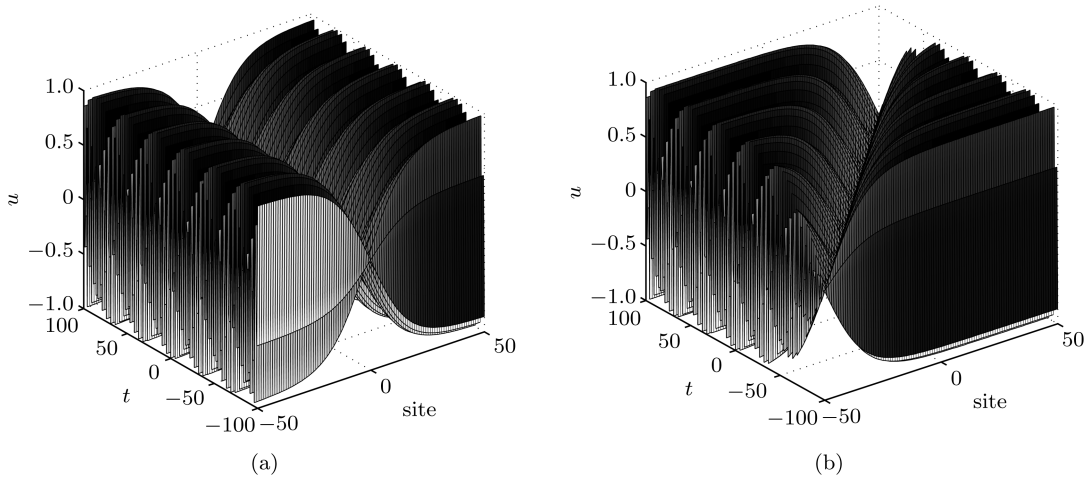


Fig.3. The DDBs in a 1D chain with the multi-potential. Panels (a) and (b) show the forms of solutions (18c). In Panels (a) there exists the relation $\varepsilon\lambda + 2\varepsilon^2 D_1 = 0$.

3.3. The several particular modes

From the above discussion, we know that only if $\chi K_2 a^2 \cos ka - \lambda^2 > 0 (D_1 > 0)$ and $D_2 \neq 0$, can the DBBs exist in model (1). On the contrary, only if $\chi K_2 a^2 \cos ka - \lambda^2 < 0 (D_1 < 0)$ and $D_2 > 0$, can the DDBs be found in model (1). These do not restrict the form of the coupling potential. Hence, only if $K_2 \neq 0$, can the DBBs and DDBs be found in a one-dimensional monatomic chain with on-site and inter-site potentials for any coupling potential. When $K_2 = 0$, the quasi-periodic and chaotic breathers have been found in a parametrically driven system by Maniadis and Bountis.[22] That is to say, we can expand the results of this model to several concrete models. For example, when $K_3 = K_4 = 0$, model (1) can be transformed into the KG model, and if $D_1 = (\chi K_2 a^2 \cos ka - \lambda^2)/2\omega$ and $D_2 = [2\alpha(2\alpha/\omega_0^2 - \delta) - 3\beta]/2\omega$, the DBBs and DDBs can also be found in the KG chain by discussing Eq.(16). The same question has been discussed by Alvarez *et al*[19] by using the method of anti-continuous limit. When $\omega_0^2 = \alpha = \beta = 0$, model (1) can be transformed into the FPU model, and if $D_1 = -\sqrt{\chi K_2}\sin(ka/2)/4$ and $D_2 = (\omega/\chi^2 K_2^2)\{3\chi K_4\omega^2/2 + \chi K_3^2[4(\omega^2/K_2)]\}$, the DBBs and DDBs can exist in the FPU chain by discussing Eq.(16). These results are the same as the conclusions discussed by Sanchez-Rey *et al*[20] by using numerical computations and approximate analytical expressions. When $\alpha = -1$, $\beta = 0$ and $\alpha = 0$, $\beta = -1$, the potentials expressed by Eq.(3) correspond to the cubic and the quartic soft on-site potentials, respectively. On the contrary, $\alpha = +1$, $\beta = 0$ and $\alpha = 0$, $\beta = +1$ the potentials expressed by Eq.(3) correspond to the cubic and the quartic hard on-site potentials, respectively. The DBBs and DDBs can also be found in these models by discussing Eq.(16). These questions have been discussed by Alvarez *et al*[19] and Morgante *et al.*[23] But they are different from model (1). Their main difference consists in the domain which contains the unperturbed orbit at the anti-continuous limit $\chi = 0$. The area containing the close orbits of model (1) is not restricted, but the domain with the close trajectories of these models is restricted by the separatrices which are indicated by thick lines in the phase space shown in Fig.4. That is to say, small amplitudes are restricted in these models except for the quartic hard on-site potentials, but such a restriction is not placed on model (1) and the models with the quartic hard on-site potentials. The former models only have the small-amplitude DBBs and DDBs, yet the large-amplitude DBBs and DDBs can exist in the latter models.

From the above arguments, we know that the DBBs and DDBs can be found in a lot of nonlinear systems. In addition, because the sign of D_1 is closely associated with the sign of $\sin(ka/2)$, the DBBs and DDBs cannot be found at the same site in the Brillouin zone. However, the above discussion does not give the value of the coupling χ_C at which the DBBs and DDBs do not exist any more, and it is worth investigating whether they are stable or not, and up to which value the coupling parameter can reach if they are stable.

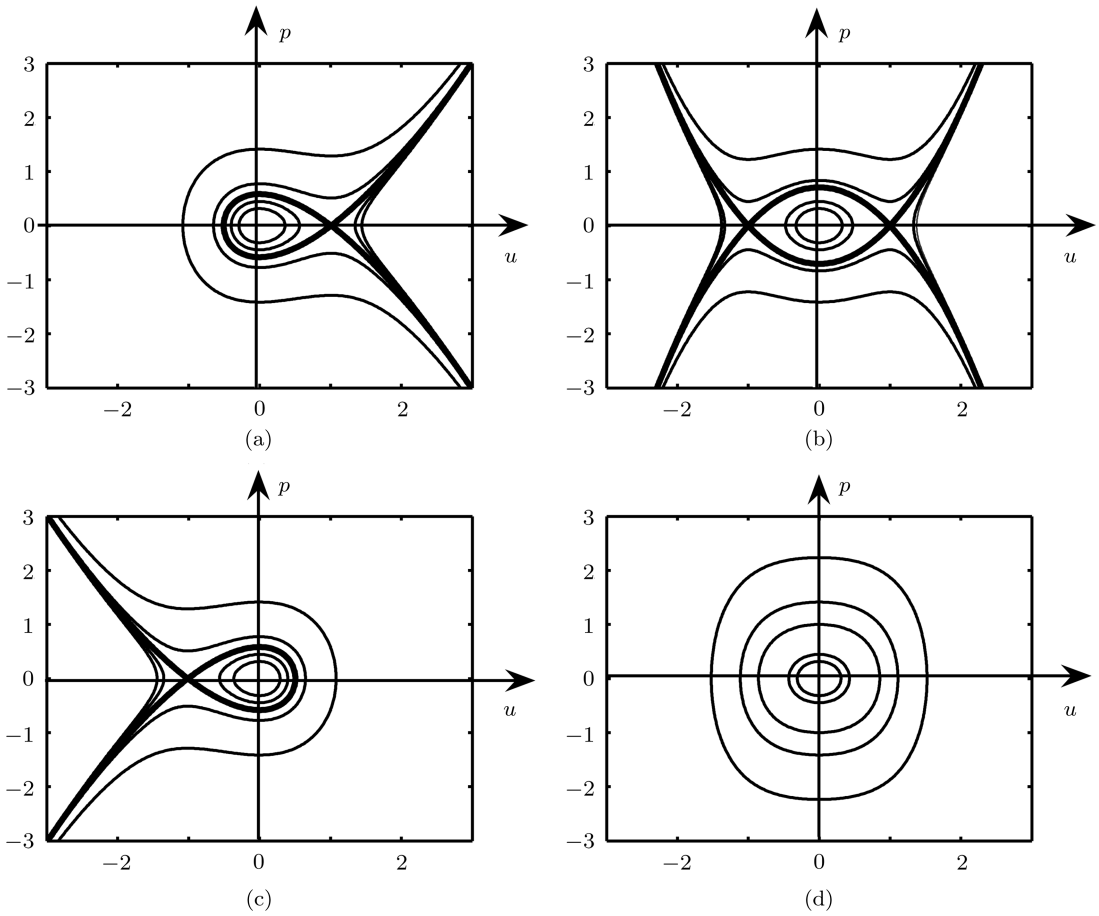


Fig.4. The soft and the hard on-site potentials, trajectories in the phase space. Panels (a) and (b) are for the cubic and the quartic soft on-site potentials, respectively, and panels (c) and (d) are for the cubic and the quartic hard on-site potentials, respectively.

4. The DBBs and DDBs stabilities

After Aubry,[4] a trajectory $u_n(t)$ of dynamical system (4) is said to be linearly stable if the time evolution of any infinitely small perturbation $\{\kappa_n(0)\}$ to its initial condition $\{u_n(0)\}$ remains unchanged in time. With the substitution of $u_n(t)$ for $u_n(t)+\kappa_n(t)$, the linearization of Eq.(4) yields a equation for κ_n

$$\begin{aligned}&\ddot{\kappa}_n(t)+\omega_0^2\kappa_n(t)+2\alpha u_n(t)\kappa_n(t)+3\beta u_n^2(t)\kappa_n(t)-\chi K_2(\kappa_{n+1}(t)+\kappa_{n-1}(t)-2\kappa_n(t))\\&-2\chi K_3[(u_{n+1}(t)-u_n(t))(\kappa_{n+1}(t)-\kappa_n(t))+(u_n(t)-u_{n-1}(t))(\kappa_n(t)-\kappa_{n-1}(t))]\\&-3\chi K_4\left[(u_{n+1}(t)-u_n(t))^2(\kappa_{n+1}(t)-\kappa_n(t))+(u_n(t)-u_{n-1}(t))^2(\kappa_n(t)-\kappa_{n-1}(t))\right]=0.\end{aligned}\tag{19}$$

In other words, a solution $\{u_n(t)\}$ of Eq.(4) is said to be linearly stable when there is no solution $\{\kappa_n(t)\}$ of Eq.(19) which diverges exponentially for a long time. In the case in which u_n is a real and time-independent solution of Eq.(4), the general solution of Eq.(19) can be searched as the following form:[23]

$$\kappa_n(t)=b_n\left(\mathrm{e}^{\mathrm{i}\omega t}\pm\mathrm{e}^{-\mathrm{i}\omega t}\right),\tag{20}$$

where the eigenfrequency $\omega=\omega_\lambda$ and the eigenmodes $\{b_n\}$ are determined by the eigenequation

$$\left[\omega_0^2+2\alpha u_n(t)+3\beta u_n^2(t)+2\chi K_3(u_{n+1}(t)+u_{n-1}(t)-2u_n(t))+3\chi K_4(u_{n+1}(t)-u_{n-1}(t))\right.$$

$$\left(u_{n+1}(t)+u_{n-1}(t)-2u_n(t)\right)]b_n-\chi\left[K_2+2K_3\left(u_{n+1}(t)-u_n(t)\right)+3K_4\left(u_{n+1}(t)-u_n(t)\right)^2\right]b_{n+1}$$
$$-\chi\left[K_2-2K_3\left(u_n(t)-u_{n-1}(t)\right)-3K_4\left(u_n(t)-u_{n-1}(t)\right)^2\right]b_{n-1}=\omega_\lambda^2 b_n. \tag{21}$$

Hence, linear stability is achieved if and only if all the eigenvalues of Eq.(21) are real. Thus for N sites in which N is large but finite, we should have N real eigenvalues ω_λ for a linearly stable solution. b_n in Eq.(21) has a solution as follows:

$$b_n=\mathrm{e}^{\mathrm{i}qn}. \tag{22}$$

Substituting expressions (18a) and (22) into Eq.(21) and eliminating high order terms of ε, we find

$$\omega_0^2+2\sqrt{2\mu}\varepsilon\alpha\mathrm{sech}\sqrt{\eta}\varepsilon na\cdot\cos k_0na-\chi\left[2K_2\cos q+4\sqrt{2\mu}\varepsilon K_3\mathrm{sech}\sqrt{\eta}\varepsilon na\cos k_0na\right]=\omega_\lambda^2. \tag{23}$$

For a real ω_λ, we have

$$\omega_0^2+2\sqrt{2\mu}\varepsilon\alpha\mathrm{sech}\sqrt{\eta}\varepsilon na\cdot\cos k_0na>\chi\left[2K_2\cos q+4\sqrt{2\mu}\varepsilon K_3\mathrm{sech}\sqrt{\eta}\varepsilon na\cos k_0na\right]. \tag{24a}$$

By taking the same procedure as the above, we also obtain

$$\omega_0^2+2\sqrt{-2\mu}\varepsilon\alpha\mathrm{csch}\sqrt{\eta}\varepsilon na\cdot\cos k_0na>\chi\left[2K_2\cos q+4\sqrt{-2\mu}\varepsilon K_3\mathrm{csch}\sqrt{\eta}\varepsilon na\cos k_0na\right], \tag{24b}$$
$$\omega_0^2+2\sqrt{\mu}\varepsilon\alpha\tanh\sqrt{-\eta/2}\varepsilon na\cdot\cos k_0na>\chi\left[2K_2\cos q+4\sqrt{\mu}\varepsilon K_3\tanh\sqrt{-\eta/2}\varepsilon na\cos k_0na\right]. \tag{24c}$$

Equation (24a)–(24c) suggest that the DBBs and DDBs are linearly stable only when χ is small, and its limited value is determined by Eqs.(24a)–(24c). Similarly, we can obtain the conditions of linear stabilities of the DBBs and the DDBs in all the above particular models.

5. Conclusions

In this paper we have investigated the existences and the linear stabilities of the DBBs and the DDBs in a one-dimensional discrete monatomic chain with no-site and inter-site potentials separately. The model covers a lot of one-dimensional discrete monatomic systems, so the results obtained from this model have general properties and can be applied to the KG and the FPU systems with the soft on-site and the hard on-site potentials and other systems. For this reason, we first prove the existences of the DBBs and DDBs in this general model by Aubry's theorem at the anti-continuous limit, then we discuss the discrete equations by using the method of the MSA, and the concrete localized modes of the DBBs and DDBs are obtained in this general model. As a result, the DBBs and DDBs can be found in one-dimensional discrete monatomic chain with on-site and inter-site potentials indeed, when $K_2\neq 0$. In addition, we also expand the general results to particular models, for example, KG model, FPU model, soft on-site potential model and hard on-site potential model and prove the existences of the DBBs and DDBs in these nonlinear systems. Because only the stable localized modes can be realized, we explore the linear stabilities of the DBBs and the DDBs in this general model by using the methods of linearization and analytical computation, and give the conditions of the linear stabilities of the DBBs and DDBs in this system, i.e. the coupling parameter χ must be very small, of which the limited value can be obtained. The same results can also be obtained from the KG and the FPU models, etc, by using the same method. That is to say, these models have stable DBBs and DDBs, when the coupling parameter χ is small, and their difference from system (1) appears as the difference in value of χ.

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