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Analytic solutions for degenerate Raman-coupled model*

Zhang Zhi-Ming(张智明)† and Yu Ya-Fei(於亚飞)

Laboratory of Photonic Information Technology, School of Information and Photoelectronic Science & Engineering, South China Normal University, Guangzhou 510006, China

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The Raman-coupled interaction between an atom and a single mode of a cavity field is studied. For the cases in which a light field is initially in a coherent state and in a thermal state separately, we have derived the analytic expressions for the time evolutions of atomic population difference W, modulus B of the Bloch vector, and entropy E. We find that the time evolutions of these quantities are periodic with a period of π. The maxima of W and B appear at the scaled interaction time points $\tau = k\pi(k = 0, 1, 2, \ldots)$. At these time points, $E = 0$, which shows that the atom and the field are not entangled. Between these time points, $E \neq 0$, which means that the atom and the field are entangled. When the field is initially in a coherent state, near the maxima, the envelope of W is a Gaussian function with a variance of $1/(4\bar{n})$ ($\bar{n}$ is the mean number of photons). Under the envelope, W oscillates at a frequency of $\bar{n}/\pi$. When the field is initially in a thermal state, near the maxima, W is a Lorentz function with a width of $1/\bar{n}$.

Keywords: Raman-coupled model, atomic dynamics, entropy, entanglement
PACC: 4250

1. Introduction

Raman-coupled model (RCM) is one of the most important models in quantum optics and has been studied extensively.[1−5] Recently, many authors have used the RCM to study quantum information science.[6−10] Feng *et al*[6] proposed to use the two-channel RCM for realizing the unconventional geometric quantum computation. Nunn *et al*[7] analysed the quantum memory based on the RCM. Wasilewski and Raymer[8] proposed to use the RCM for realizing the entanglement between light and atomic ensemble. Alexanian[9] used the RCM for realizing the cloning of coherent state in a cavity. Yang *et al*[10] applied the RCM to the implementing of the Deutsch–Jozsa algorithm. For a general RCM, it is not easy to find an analytic solution, so some numerical calculations are needed. In this paper, we consider a simple RCM, i.e. the degenerate RCM, which can be realized separately in atomic systems, in trapped ion systems,[11] and in superconducting quantum interference devices (SQUIDs).[12−14] For the trapped ion systems, it is shown[11] that if the Lamb-Dicke parameter is not very large, then the system can be described by a degenerate RCM. In the situation of SQUIDs, the energy levels and their separations can be adjusted by changing the local bias field,[12−14] therefore, a degenerate RCM can be realized in this system. In this paper, for the degenerate RCM, we derive the analytic expressions for the time evolutions of the population difference, the modulus of the Bloch vector, and the entropy. By analysing these analytic expressions we obtain some interesting results.

This paper is organized as follows. In Section 2 we present the model and derive the time-dependent state vector. In Section 3 we derive the exact analytic expressions for the time evolutions of the atomic population difference for the situations in which the field is initially in a coherent state and in a thermal state separately. Section 4 is devoted to the study of the entropy, from which we discuss the entanglement between the atom and the field. Section 5 presents some conclusions drawn from the present study.

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†E-mail: zmzhang@scnu.edu.cn

2. Model and time-dependent state vector

We consider the interaction between a Λ-type three-level atom and a single mode of a cavity field. We assume that the two lower levels are degenerate, and the detuning between the field frequency and the atomic transition frequency from the upper level to the lower levels is very large. Under this condition, the effective Hamiltonian of the system has the following form ($h = 1$):

$$H = \omega a^{+}a + \lambda a^{+}a \left(|e\rangle \langle g| + |g\rangle \langle e|\right), \tag{1}$$

where a^{+} and a are the photon creation and annihilation operators, respectively. $|e\rangle$ and $|g\rangle$ are the two lower levels of the atom, ω is the field frequency, and λ is the effective coupling constant between the atom and the field. It is not difficult to find the eigenvalues and eigenstates of H:

$$E_n^{(\pm)} = n\left(\omega + \lambda\right), \tag{2}$$

$$|\pm, n\rangle = \frac{1}{\sqrt{2}}\left(|e, n\rangle \pm |g, n\rangle\right). \tag{3}$$

Assume that initially the atom is in state $|e\rangle$, and the field is in a superposition of Fock states $\sum_n c_n |n\rangle$, therefore, the initial state of the system is

$$\begin{aligned}|\psi(0)\rangle =& |e\rangle \sum_n c_n |n\rangle = \sum_n c_n |e, n\rangle \\ =& \frac{1}{\sqrt{2}} \sum_n c_n \left(|+, n\rangle + |-, n\rangle\right).\end{aligned} \tag{4}$$

At an arbitrary time t, the state of the system is

$$\begin{aligned}|\psi(t)\rangle =& U(t)|\psi(0)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi(0)\rangle \\ =& \frac{1}{\sqrt{2}} \sum_n c_n (\mathrm{e}^{-\mathrm{i}E_n^{(+)}t}|+, n\rangle \\ &+ \mathrm{e}^{-\mathrm{i}E_n^{(-)}t}|-, n\rangle).\end{aligned} \tag{5}$$

Substituting expressions (2) and (3) into expression (5) we obtain

$$\begin{aligned}|\psi(t)\rangle =& \sum_n c_n \mathrm{e}^{-\mathrm{i}n\omega t}[\cos(n\lambda t)|e, n\rangle \\ &- \mathrm{i}\sin(n\lambda t)|g, n\rangle].\end{aligned} \tag{6}$$

This is the time-dependent state vector of the system.

3. Time evolution of the atomic population difference

From expression (6) we can derive the expressions for the probabilities of the atom in state $|e\rangle$ and in state $|g\rangle$:

$$P_e(\tau) = \sum_n |c_n|^2 \cos^2(n\tau) = \sum_n p_n \cos^2(n\tau), \tag{7}$$

$$P_g(\tau) = \sum_n |c_n|^2 \sin^2(n\tau) = \sum_n p_n \sin^2(n\tau), \tag{8}$$

where $p_n = |c_n|^2$ is the photon number distribution function of the initial field, and $\tau = \lambda t$ is the scaled interaction time (the dimensionless interaction time).

The difference in atomic population between state $|e\rangle$ and state $|g\rangle$ can be found to be

$$W(\tau) = P_e(\tau) - P_g(\tau) = \sum_n p_n \cos(2n\tau). \tag{9}$$

In the following, we consider two kinds of initial field states, i.e. the coherent state and the thermal state.

3.1. Coherent state

Suppose that the field is initially in a coherent state with a mean photon number $\bar{n}$. Substituting its photon number distribution function $p_n = \mathrm{e}^{-\bar{n}}\frac{\bar{n}^n}{n!}$[15] into expression (9), and after some simple algebra we find

$$\begin{aligned}W(\tau) =& \exp\{-\bar{n}[1 - \cos(2\tau)]\} \\ &\times \cos[\bar{n}\sin(2\tau)].\end{aligned} \tag{10}$$

This is one of the main results of this paper. It is worthy to note that here we have expressed the time evolution of the atomic population difference $W(\tau)$ as an exact analytic expression. While in other interaction models, for example, in the Jaynes–Cummings model,[15] $W(\tau)$ cannot be expressed as an exact analytic expression. In the following we will discuss expression (10) in some detail.

(a) $W(\tau)$ has a period of π and it has maxima 1 at $\tau = k\pi (k = 0, 1, 2, \ldots)$.

(b) Expression (10) consists of two factors, one is the envelope function $\exp\{-\bar{n}[1 - \cos(2\tau)]\}$ and the other is the fast oscillation function $\cos[\bar{n}\sin(2\tau)]$. Now we analyse these two factors separately. Near $\tau = k\pi (k = 0, 1, 2, \ldots)$, we let $\tau = k\pi + \varepsilon$ (ε is small), then the envelope function changes as follows:

$$\begin{aligned}&\exp\{-\bar{n}[1 - \cos(2\tau)]\} \\ =& \exp\{-\bar{n}[1 - \cos(2(k\pi + \varepsilon))]\} \\ =& \exp\{-\bar{n}[1 - \cos(2\varepsilon)]\} \approx \mathrm{e}^{-2\bar{n}\varepsilon^2}.\end{aligned} \tag{11}$$

It can be seen that the envelope function near its maxima, is a Gaussian function with a variance of $1/(4\bar{n})$. This means that the variance decreases as $\bar{n}$ increases. On the other hand, the fast oscillation function is

$$\begin{aligned}&\cos\left[\bar{n}\sin\left(2\tau\right)\right]\\=&\cos\left[\bar{n}\sin\left(2\left(k\pi+\varepsilon\right)\right)\right]\\=&\cos\left[\bar{n}\sin\left(2\varepsilon\right)\right]\approx\cos\left(2\bar{n}\varepsilon\right).\end{aligned}\tag{12}$$

It can be seen that, near the maxima and under the envelope, $W(\tau)$ fast oscillates at a frequency of $\bar{n}/\pi$. This means that the stronger the field is, the faster the $W(\tau)$ oscillates. In order to have an intuitive picture about the behaviour of $W(\tau)$, we have drawn the curve of $W(\tau)$ versus τ by using expression (10) in Fig.1. It can be seen clearly that $W(\tau)$ has the features analysed above. However, we point out that the conclusions about the line-shape, the line-width, and the fast oscillation frequency can be drawn only from the analytic expressions above.

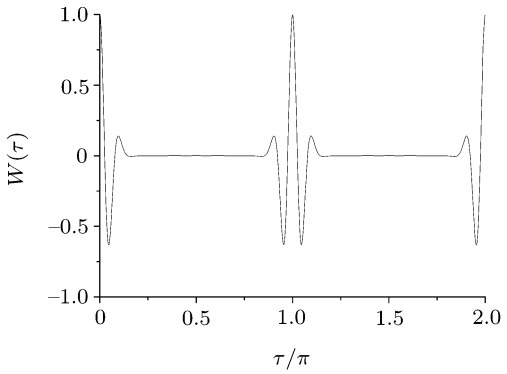


Fig.1. Time evolution of atomic population difference $W(\tau)$ for an initial coherent state with $\bar{n}=10$.

3.2. Thermal state

Suppose that the field is initially in a thermal state with a mean photon number of $\bar{n}$. Substituting its photon number distribution function $p_n=\frac{1}{(1+\bar{n})}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n$ [15] into expression (9), and after some simple algebra we find

$$W(\tau)=\frac{1+\bar{n}\left[1-\cos\left(2\tau\right)\right]}{\left\{1+\bar{n}\left[1-\cos\left(2\tau\right)\right]\right\}^2+\left[\bar{n}\sin\left(2\tau\right)\right]^2}.\tag{13}$$

This is another main result of this paper. The time evolution of the atomic population difference $W(\tau)$ is again described by an exact analytic expression. Now we come to discuss expression (13) in some detail.

(a) Similar to the situation of initial coherent state, $W(\tau)$ has also a period of π, and it has maxima 1 at $\tau=k\pi(k=0,1,2,\ldots)$.

(b) Near the maxima at $\tau=k\pi(k=0,1,2,\ldots)$, let $\tau=k\pi+\varepsilon$ (ε is small), then we will have

$$\begin{aligned}W(\tau)=&W(\varepsilon)\\\approx&\frac{1+2\bar{n}\varepsilon^2}{\left(1+2\bar{n}\varepsilon^2\right)^2+\left(2\bar{n}\varepsilon\right)^2}\\\approx&\frac{1}{1+\left(2\bar{n}\varepsilon\right)^2}=\frac{\left(\frac{1}{2\bar{n}}\right)^2}{\varepsilon^2+\left(\frac{1}{2\bar{n}}\right)^2}.\end{aligned}\tag{14}$$

It can be seen that, near the maxima, $W(\tau)$ has a Lorentz line-shape with a line-width of $1/\bar{n}$ (in the time-region). Different from the coherent state situation, there is no fast oscillation in the case of thermal state. In order to have an intuitive picture about the behaviour of $W(\tau)$, we have drawn the curve of $W(\tau)$ versus τ by using expression (13) in Fig.2. It can be seen clearly that $W(\tau)$ has the features analysed above. However, the conclusions about the line-shape and the line-width can be drawn only from the above analytic expressions.

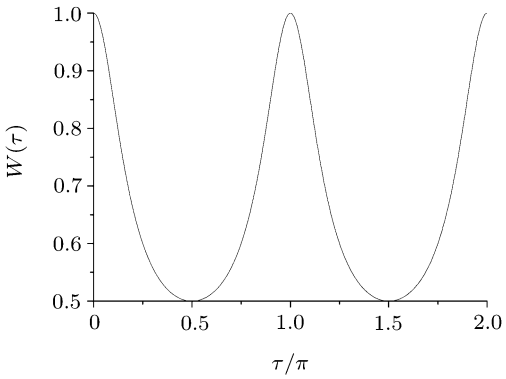


Fig.2. Time evolution of atomic population difference $W(\tau)$ for an initial thermal state with $\bar{n}=0.5$.

4. Time evolution of the entropy

The entropy is an important concept in physics and in information science as well. It can be used to measure the entanglement between subsystems of a composite system. Here we study the entropy for the degenerate RCM and discuss the entanglement between the field and the atom. From expression (6) we can construct the density operator of the system: $\rho_{\mathrm{A+F}}(t)=|\psi(t)\rangle\langle\psi(t)|$. By tracing over the field

variables we can obtain the reduced density operator of the atom: $\rho = \mathrm{Tr_F}\rho_{\mathrm{A+F}}$. After some simple calculations we find

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+W & U+\mathrm{i}V \\ U-\mathrm{i}V & 1-W \end{pmatrix}, \quad (15)$$

where

$$\rho_{ee}(\tau) = \sum_n |c_n|^2 \cos^2(n\tau), \quad (16a)$$

$$\rho_{eg}(\tau) = \mathrm{i}\sum_n |c_n|^2 \sin(n\tau)\cos(n\tau), \quad (16b)$$

$$\rho_{ge}(\tau) = -\mathrm{i}\sum_n |c_n|^2 \sin(n\tau)\cos(n\tau), \quad (16c)$$

$$\rho_{gg}(\tau) = \sum_n |c_n|^2 \sin^2(n\tau), \quad (16d)$$

and (W, U, V) are the three components of the Bloch vector:[15]

$$\begin{aligned} W(\tau) =& \rho_{ee}(\tau) - \rho_{gg}(\tau) \\ =& \sum_n |c_n|^2 \cos(2n\tau), \end{aligned} \quad (17a)$$

$$U(\tau) = \rho_{eg}(\tau) + \rho_{ge}(\tau) = 0, \quad (17b)$$

$$\begin{aligned} V(\tau) =& -\mathrm{i}\left[\rho_{eg}(\tau) - \rho_{ge}(\tau)\right] \\ =& \sum_n |c_n|^2 \sin(2n\tau). \end{aligned} \quad (17c)$$

The eigenvalues of ρ are

$$g_\pm = \frac{1}{2}(1 \pm B), \quad (18)$$

where B is the modulus of the Bloch vector and $B = \sqrt{W^2 + U^2 + V^2}$.

The von Neumann entropy[15] can be calculated from

$$E = -g_+ \ln g_+ - g_- \ln g_-. \quad (19)$$

In the following, we study the time evolution of the entropy for the situations in which the field is initially in a coherent state and in a thermal state separately. From expression (17b) we know that $U = 0$, therefore we only need to calculate W and V.

4.1. Coherent state

For the initial coherent state, we have derived the expression of W above [expression (10)]. In a similar way we can find

$$\begin{aligned} V(\tau) =& \exp\{-\bar{n}[1-\cos(2\tau)]\} \\ & \times \sin[\bar{n}\sin(2\tau)]. \end{aligned} \quad (20)$$

Therefore,

$$B(\tau) = \exp\{-\bar{n}[1-\cos(2\tau)]\}. \quad (21)$$

By comparing expression (10) with expressions (21) and (20) we find respectively

$$W(\tau) = B(\tau)\cos[\bar{n}\sin(2\tau)], \quad (22a)$$

$$V(\tau) = B(\tau)\sin[\bar{n}\sin(2\tau)]. \quad (22b)$$

That is to say, $B(\tau)$ is the envelope function of $W(\tau)$ and $V(\tau)$. Substituting expression (21) into expression (18), and then into expression (19), we can obtain the analytic expression of entropy $E(\tau)$. In Fig.3 we draw the curves of $W(\tau)$, $B(\tau)$ and $E(\tau)$. In the following we discuss these curves by using the analytic expressions above.

(a) First we note that all the curves are periodic with a period of π. When $\tau = k\pi(k = 0, 1, 2, \ldots)$, $W(\tau)$ and $B(\tau)$ take maxima of 1, and $E(\tau)$ takes minima of 0 (these results can also be drawn from expression (21), (22a), (18), and (19)). In fact, when $\tau \equiv \lambda t = k\pi(k = 0, 1, 2, \ldots)$, i.e. $t = \dfrac{k\pi}{\lambda}$, from expression (6) we can obtain

$$\left|\psi\left(\frac{k\pi}{\lambda}\right)\right\rangle = \sum_n c_n \mathrm{e}^{-\mathrm{i}n\omega\frac{k\pi}{\lambda}}(-1)^{nk}|n\rangle \otimes |e\rangle. \quad (23)$$

This shows that at these time points, the atom and the field are in a product state, i.e. they are not entangled with each other, therefore, the entropy $E = 0$.

(b) During the time interval in which $W(\tau) \approx B(\tau) \approx 0$, we find that $E(\tau) \neq 0$. This shows that the atom and the field are in an entangled state.

(c) Near $\tau = k\pi(k = 0, 1, 2, \ldots)$, we know from expression (11) that $B(\tau)$ is a Gaussian function with a variance of $1/(4\bar{n})$.

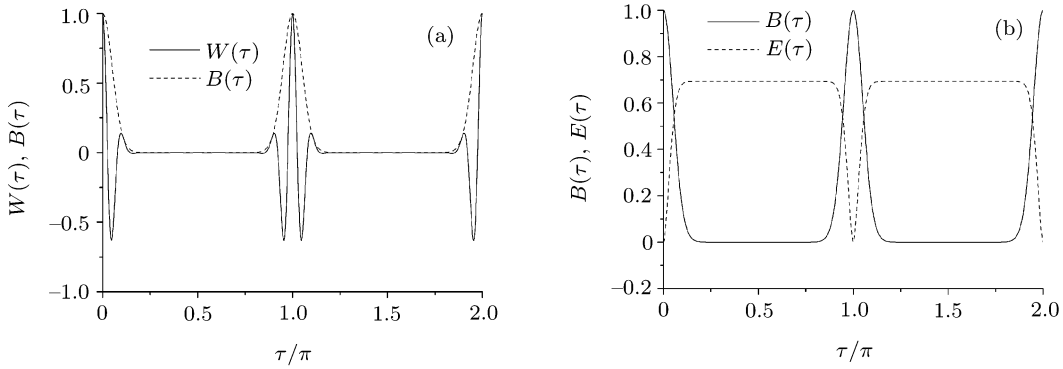


Fig.3. Time evolution of atomic population difference $W(\tau)$, modulus $B(\tau)$ of the Bloch vector, and entropy $E(\tau)$ for an initial coherent state with $\bar{n} = 10$. (a) Comparison between $W(\tau)$ and $B(\tau)$; (b) Comparison between $B(\tau)$and $E(\tau)$.

4.2. Thermal state

For an initial thermal field, W is written as expression (13). In a similar way we can find

$$V(\tau) = \frac{\bar{n}\sin(2\tau)}{\{1+\bar{n}[1-\cos(2\tau)]\}^2 + [\bar{n}\sin(2\tau)]^2}. \quad (24)$$

Therefore,

$$B(\tau) = \frac{1}{\sqrt{\{1+\bar{n}[1-\cos(2\tau)]\}^2 + [\bar{n}\sin(2\tau)]^2}}. \quad (25)$$

Substituting expression (25) into expression (18), and then into expression (19), we can obtain an analytic expression of entropy $E(\tau)$. In Fig.4 we draw the curves of $W(\tau)$, $B(\tau)$ and $E(\tau)$. In the following we discuss these curves in some detail by using the analytic expressions (13), (25), (18), and (19).

We note that all the curves are periodic with a period of π. When $\tau = k\pi(k = 0, 1, 2, \ldots)$, $W(\tau)$ and $B(\tau)$ take maxima of 1, and $E(\tau)$ takes the minima of 0. This means that at these time points, the atom and the field are not entangled. On the other hand, when $\tau = (2k+1)\frac{\pi}{2}(k = 0, 1, 2, \ldots)$, $W(\tau)$ and $B(\tau)$ take minima, while $E(\tau)$ takes maxima. This shows that at these time points, the atom and the field are in a maximum entangled state.

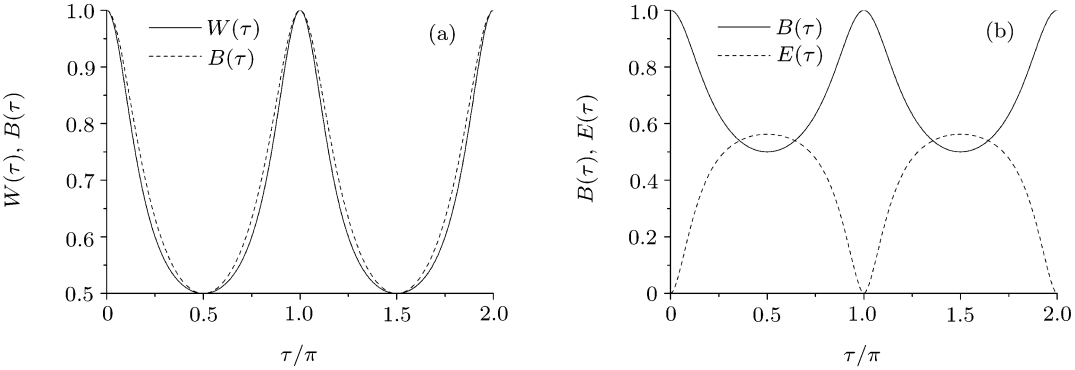


Fig.4. Time evolution of atomic population difference $W(\tau)$, modulus $B(\tau)$ of the Bloch vector, and entropy $E(\tau)$ for an initial thermal state with $\bar{n} = 0.5$. (a) Comparison between $W(\tau)$ and $B(\tau)$; (b) Comparison between $B(\tau)$ and $E(\tau)$.

5. Conclusions

We have studied the Raman-coupled interaction between an atom and a single mode of a cavity field. For the situations in which the field is initially in a coherent state and in a thermal state separately, we have derived the exact analytic expressions for the time evolution of atomic population difference $W(\tau)$, modulus $B(\tau)$ of the Bloch vector, and entropy $E(\tau)$. By analysing these analytic expressions we find that the time evolutions of these quantities are periodic with a period of π. The maxima of $W(\tau)$ and $B(\tau)$ appear at the scaled interaction time points $\tau = k\pi(k =$

$0, 1, 2, \ldots$). At these time points $E(\tau) = 0$, and this means that the atom and the field are not entangled. Between these time points, $E(\tau) \neq 0$ and the atom and field are entangled. When the field is initially in a coherent state, near its maxima, $W(\tau)$ has an envelope which is a Gaussian function with a variance of $1/(4\bar{n})$ ($\bar{n}$ is the mean number of photons). Under the envelope, $W(\tau)$ fast oscillates at a frequency of $\bar{n}/\pi$. When the field is initially in a thermal state, near its maxima, $W(\tau)$ can be expressed by a Lorentz function with a width of $1/\bar{n}$, and there is no fast oscillation, as appearing in the coherent state case.

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