Vol 17 No 4, April 2008
1674-1056/2008/17(04)/1184-05

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Influence of dark energy on time-like geodesic motion in Schwarzschild spacetime*

Chen Ju-Hua(陈菊华)† and Wang Yong-Jiu(王永久)

College of Physics and Information Science, Hunan Normal University, Changsha 410081, *China*

(Received 26 April 2007; revised manuscript received 22 October 2007)

In this paper we investigate the influence of the dark energy on the time-like geodesic motion of a particle in Schwarzschild spacetime by analysing the behaviour of the effective potential which appears in an equation of motion. For the non-radial time-like geodesics, we find a bound orbit when the particle energy is in an appropriate range, and also find another possible orbit, which is that the particle drops straightly into the singularity of a black hole or escapes to infinity. For the radial time-like geodesics, we find an unstable circular orbit when the particle energy is the critical value, in which case it is possible for the particle to escape to infinity.

Keywords: dark energy, effective potential, time-like geodesic motion
PACC: 0420, 0430, 1180

1. Introduction

Many test gravitational effects have been predicted by the general relativity. The deflection of light was investigated by Lake[1] in the Schwarzschild–de Sitter spacetime. Especially, the planetary orbit of the test particle in a given spacetime is very interesting. By using the method of an effective potential, Jaklitsch *et al*[2] investigated the time-like geodesic structure with a positive cosmological constant and Cruz *et al*[3] studied the geodesic structure of the Schwarzschild Anti–de Sitter black hole. The analysis of the effective potential for null geodesics in Reissner–Nordström–de Sitter and Kerr–de Sitter spacetime was carried out in Ref.[4]. Podolsky[5] investigated all possible geodesic motions in the extreme Schwarzschild–de Sitter spacetime. Chen and Wang[6−9] have investigated the orbital dynamics of a test particle in several gravitational fields with an electric dipole and a mass quadrupole, and in the extreme Reissner–Nordström black hole spacetime. The exact solutions in the closed analytic form for the geodesic motion in the Kottler spacetime were considered by Kraniotis *et al.*[10] The exact solutions of the time-like geodesics were used to investigate the perihelion precession of the planet Mercury. By solving the Hamilton–Jacobi partial differential equation, Kraniotis and Whitehouse[11] investigated the geodesic motion of the massive particle in the Kerr and Kerr–(anti)de Sitter gravitational field.

It is well known that the accelerating expansion of the universe is strongly supported by a large number of astronomical observations, such as type Ia supernovae,[12−16] cosmic microwave background (CMB)[17−19] and large scale structure.[20−23] The expansion implies the valuable contribution of matter with a negative pressure to the evolution of the universe. The accelerating expansion is believed to arise from the fact that the universe is dominated by an exotic component with a negative pressure called 'dark energy' which makes up 70 percent of the energy density of the universe. There are several candidates for dark energy: the first one is the cosmological constant,[24,25] and the second one is the so-called dynamic candidates such as: phantom,[26−29] quintessence,[30−32] K-essence[33,34] and quintom.[35−37] The difference between these candidates for dark energy lies in the magnitude of parameter $\omega_{\rm q}$, namely the ratio of the pressure to the energy density of the dark energy. For the quintessence, the state equation is given by the relation between the pressure $p_{\rm q}$ and the energy density $\rho_{\rm q}$, i.e. $p_{\rm q} = \omega_{\rm q}\rho_{\rm q}$ with $\omega_{\rm q}$ being in a range of $-1 \leq \omega_{\rm q} \leq -1/3$ in which the acceleration occurs. The borderline case

*Project supported by the State Key Development Program for Basic Research Program of China (Grant No 2003CB716300) and the National Natural Science Foundation of China (Grant No 10547114).
†E-mail: jhchen@hunnu.edu.cn

of $\omega_{\rm q} = -1$ of the extraordinary quintessence covers the cosmological constant term. As has been recognized,[38−40] the acceleration is a challenge to a consistent theory of quantum gravity because of the outer horizon, which does not allow one to introduce the observable S-matrix in terms of asymptotic past and future states. The outer horizon of de Sitter space differs significantly from the inner horizon of a black hole, which has an asymptotically flat space far from the black hole.

In this paper we concentrate on the influence of dark energy on the time-like geodesic motion of a particle in the Schwarzschild spacetime. In Section 2 we construct the geodesic equation and its effective potential in the Schwarzschild spacetime surrounded by quintessence. In Section 3, we analyse the time-like bound geodesics of radial particle in detail. In Section 4 the time-like geodesics of a non-radial particle is investigated. The conclusions are given in the last section.

2. Geodesic equation and effective potential

Quintessence[30−32] is considered as one of the so-called dynamic candidates for dark energy. Kiselev[41−43] constructed a general solution to the spherically symmetric Einstein's equations for the quintessence surrounding the Schwarzschild black hole. It is very valuable to understand properties of dark matter and dark energy. The additivity and linearity condition under which fixes the free parameter of the energy–momentum tensor for the matter is fixed implies

$$T_t^t = T_r^r = \rho_{\rm q}, \tag{1}$$

$$T_\theta^\theta = T_\varphi^\varphi = \frac{1}{2}\rho_{\rm q}(3\omega_{\rm q}+1). \tag{2}$$

So the spherically symmetric metric of the Schwarzschild spacetime surrounded by quintessence is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{3}$$

where the lapse function is given by

$$f(r) = 1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}, \tag{4}$$

where M is the mass of the black hole, $\omega_{\rm q}$ is the quintessential state parameter, c is the normalization factor which depends on $\rho_{\rm q} = -\frac{c}{2}\frac{3\omega_{\rm q}}{r^{3(1+\omega_{\rm q})}}$, and $\rho_{\rm q}$ is the quintessence density. The above exact spherically symmetric solution for Einstein's equations describes the black hole surrounded the quintessential matter with the energy-momentum tensor, which satisfies the additivity and linearity condition in accordance with Eqs.(1) and (2).

In order to investigate the time-like geodesics of a particle in the Schwarzschild spacetime surrounded by quintessence, we solve the Euler–Lagrange equation for the variation problem associated with this metric (3). The corresponding Lagrangian is

$$2\mathrm{L} = -f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2), \tag{5}$$

where the dots represent the derivative with respect to the affine parameter τ, along the geodesics. The equations of motion are

$$\dot{\Pi}_q - \frac{\partial \mathrm{L}}{\partial q} = 0, \tag{6}$$

where $\Pi_q = \dfrac{\partial \mathrm{L}}{\partial \dot{q}}$ is the momentum conjugate to coordinate q. As the Lagrangian is independent of (t, φ), the corresponding conjugate momentum is conserved, thus

$$\Pi_t = -\left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right)\dot{t} = -E, \tag{7}$$

$$\Pi_\varphi = r^2\sin^2\theta\dot{\varphi} = L. \tag{8}$$

Choosing the conditions $\theta = \pi/2$ and $\dot{\theta} = 0$, then the above equation will be simplified into

$$\Pi_\varphi = r^2\dot{\varphi} = L. \tag{9}$$

Substituting Eqs.(7) and (8) into Eq.(5), we obtain

$$2\mathrm{L} = h = \frac{E^2}{1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}} - \frac{\dot{r}^2}{1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}} - \frac{L^2}{r^2}. \tag{10}$$

Now we solve the above equation for $\dot{r}^2$ in order to obtain the radial equation, which allows us to characterize possible moments of test particles without an explicit solution of the equation of motion in an invariant plane

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right)\left(h + \frac{L^2}{r^2}\right). \tag{11}$$

We can rewrite the above equation as a one-dimensional one

$$\dot{r}^2 = E^2 - V_{\rm eff}^2, \tag{12}$$

where $V_{\rm eff}^2$ is defined as an effective potential and expressed as

$$V_{\rm eff}^2 = \left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right)\left(h + \frac{L^2}{r^2}\right). \qquad (13)$$

For the time-like geodesics, $h = 1$, and thus the effective potential turns into

$$V_{\rm eff}^2 = \left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right)\left(1 + \frac{L^2}{r^2}\right). \qquad (14)$$

From this time-like geodesic effective potential, we investigate the radial equation (11) for two cases: non-radial and radial particles, respectively. Figure 1 shows the effective potential (14) for a non-radial particle at different values of $L^2 : 20, 15$ and $10M^2$ with $\omega_{\rm q} = -2/3$, $c = 0.0001$ and $M = 1$. Figure 2 shows the effective potential (14) of a non-radial particle at different values of the quintessential state parameter $\omega_{\rm q} : -1/2$, $-2/3$ and $-5/6$ with $L^2 = 15M^2$, $c = 0.0001$ and $M = 1$.

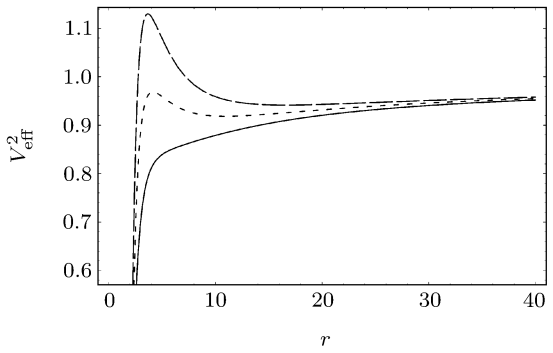


Fig.1. The effective potential $V_{\rm eff}^2$ of a non-radial particle vs r at different values of $L^2 : 20, 15$ and $10M^2$ (from top to bottom) with $\omega_{\rm q} = -2/3$, $c = 0.0001$ and $M = 1$.

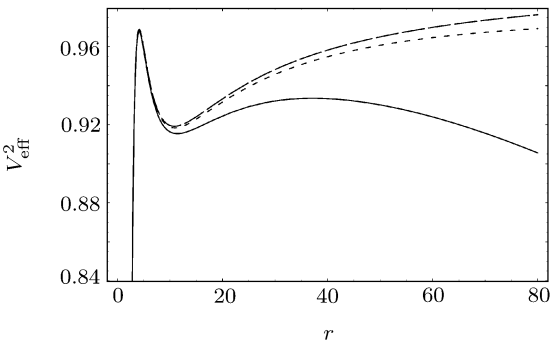


Fig.2. The effective potential $V_{\rm eff}^2$ of a non-radial particle vs r at different values of the quintessential state parameter $\omega_{\rm q} : -1/2$, $-2/3$ and $-5/6$ (from top to bottom) with $L^2 = 15M^2$, $c = 0.0001$ and $M = 1$.

3. Time-like bound geodesics

The effective potential $V_{\rm eff}^2$ of a particle with angular momentum $L \neq 0$ has the form

$$V_{\rm eff}^2 = \left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right)\left(1 + \frac{L^2}{r^2}\right), \qquad (15)$$

and the corresponding radial motion equation takes the following form:

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right)\left(1 + \frac{L^2}{r^2}\right). \qquad (16)$$

As an example we have plotted the effective potential of a non-radial particle with $\omega_{\rm q} = -1/2$, $c = 0.0001$, $L^2 = 15M^2$ and $M = 1$ in Fig.3. According to the equation of motion (16) and Fig.3, we can discuss the time-like geodesics in several cases according to different values of E of the particle as follows:

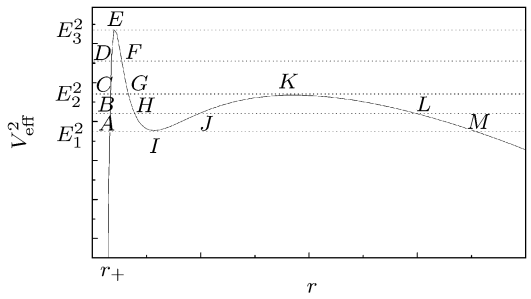


Fig.3. The effective potential $V_{\rm eff}^2$ of a non-radial particle vs r with the quintessential state parameter $\omega_{\rm q} = -5/6$, $c = 0.0001$, $L^2 = 15M^2$ and $M = 1$.

(I) when $E^2 > E_3^2$, the particle with enough energy will directly fall from rest(a finite distance) to the centre.

(II) when $E^2 = E_3^2$, the orbit of the particle is unstable at $r = r_E$. The particle in this kind of orbit escapes from $r = r_E$ to infinity, or to the singularity at $r = r_+$, which is determined by the initial condition of the particle.

(III) when $E_2^2 < E^2 < E_3^2$, the particle in this kind of orbit rebounds from $r = r_F$ to infinity, or to the singularity at $r = r_+$ from $r = r_D$, which is also determined by the initial condition of the particle.

(IV) when $E^2 = E_2^2$, the particle has an unstable orbit. It goes from $r = r_K$ to infinity, or moves in a bound orbit in a range $r_J < r < r_K$, which is also determined by the initial condition of the particle. Another possible orbit is that the particle drops into the singularity at $r = r_+$ from $r = r_C$.

(V) when $E_1^2 < E^2 < E_2^2$, the particle moves in a bound orbit in a range $r_H < r < r_J$, where r_H and

r_J are the perihelion and aphelion distances, respectively. Another possible orbit of particle at point B is that the particle drops straightly into the singularity, or escapes to infinity from r_L.

(VI) when $E^2 = E_1^2$, the particle circles around the centre with a constant radius $r = r_I$, or plunges directly into the singularity from point A, or rebounds from $r = r_M$ to infinity.

(VII) when $E^2 < E_1^2$, the particle plunges directly into the singularity.

4. Time-like radial geodesics

An angular momentum $L = 0$ corresponds to the radial geodesics, which falls from a position (with a finite distance) to the centre. Under this condition the effective potential reduces to

$$V_{\rm eff}^2 = \left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right). \tag{17}$$

Figure 4 shows the effective potential of the radial particle at different values of the quintessential state parameter. In this case the corresponding radial motion equation reduces in the following form:

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r} - \frac{c}{r^{3\omega_{\rm q}+1}}\right). \tag{18}$$

Now we detail the orbit of the radial particle in several cases according to different values of E as follows:

(I) When $E^2 > E_{\rm cri}^2$, the particle drops directly into the singularity from infinity.

(II) When $E^2 = E_{\rm cri}^2$, the particle has an unstable circular orbit, i.e. the particle drops directly into the singularity from the critic radius $r_{\rm cri}$, or escapes to infinity.

(III) When $E^2 < E_{\rm cri}^2$, the particle plunges into the singularity from the radius $r < r_{\rm cri}$, or comes back to infinity from the radius $r > r_{\rm cri}$.

By comparing the above with Fig.3, we can find that there are no bound orbit for the radial particle.

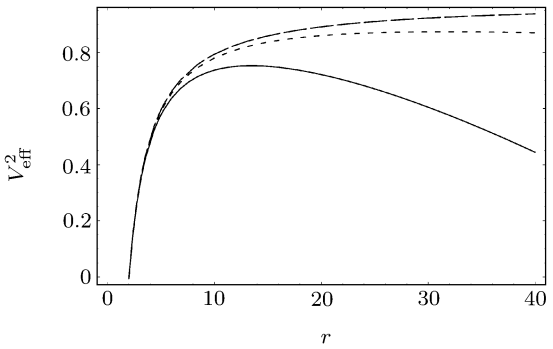


Fig.4. The effective potential $V_{\rm eff}^2$ of the radial particle vs r at different values of the quintessential state parameter $\omega_{\rm q}$: $-1/2$, $-2/3$ and $-5/6$ (from top to bottom) with $L^2 = 0$, $c = 0.002$ and $M = 1$.

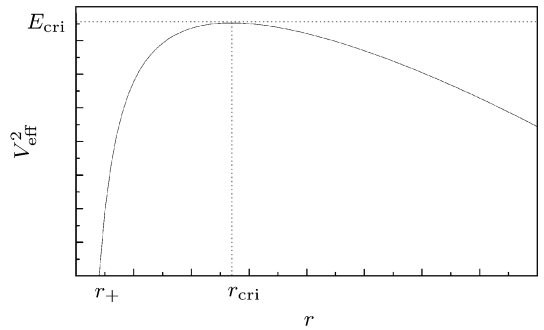


Fig.5. The effective potential $V_{\rm eff}^2$ of the radial particle vs r for the quintessential state parameter $\omega_{\rm q} = -5/6$ with $L^2 = 0$, $c = 0.002$ and $M = 1$.

5. Conclusions

In this paper we have investigated the influence of dark energy on the time-like geodesic motion of particle in Schwarzschild spacetime by analysing the behaviour of the effective potential which appears in an equation of motion. For the non-radial time-like geodesics, the particle moves in a bound orbit between the perihelion and aphelion distances when the particle energy is in an appropriate range, and another possible orbit is that the particle may drop straightly into the singularity. For the radial time-like geodesics, there is an unstable circular orbit when the particle energy reaches a critical value.

References

[1] Lake K 2002 *Phys. Rev.* D **65** 087301

[2] Jaklitsch M J, Hellaby C and Matravers D R 1989 *Gen. Rel. Grav.* **21** 94

[3] Cruz N, Olivares M and Villanueva J R arXiv:gr-qc

0408016 1
[4] Stuchlik Z and Calvani M 1991 *Gen. Rel. Grav.* **23** 507
[5] Podolsky J 1999 *Gen. Rel. Grav.* **31** 1703
[6] Chen J H and Wang Y J 2003 *Class. Quant. Grav.* **20** 3897
[7] Chen J H and Wang Y J 2006 *Chin. Phys.* V **15** 1705
[8] Chen J H and Wang Y J 2007 *Chin. Phys.* V **16** 3212
[9] Chen J H and Wang Y J 2001 *Acta Phys. Sin.* V **50** 1833
[10] Kraniotis G V and Whitehouse S B 2003 *Class. Quant. Grav.* **20** 4817
[11] Kraniotis G V arXiv:gr-qc 0405095
[12] Perlmutter S 1999 *Astrophys. J.* **517** 565
[13] Knop R A 2003 *Astrophys. J.* **598** 102
[14] Riess A G 2004 *Astrophys. J.* **607** 665
[15] Riess A G 1998 *Astrophys. J.* **116** 1009 (Preprint Astro-ph/9805201)
[16] Zhang X and Wu F Q 2005 *Phys. Rev.* D **72** 043524
[17] De Bernardis P 2000 *Nature* **404** 955
[18] Halverson N W 2002 *Astrophys. J.* **568** 38
[19] Lamon R and Durrur R 2006 *Phys. Rev.* D **73** 023507
[20] Bacon D J 2000 *Mon. Not. R. Astron. Soc.* **318** 625
[21] Bacon D J 2003 *Mon. Not. R. Astron. Soc.* **344** 673
[22] Takeshi C 1998 *Astrophys. J.* **509** 74 (Preprint gr-qc/9903094)
[23] Tegmark M and Michael A S 2004 *Phys. Rev.* D **69** 103501
[24] Padmanabham T 2003 *Phys. Rep.* **380** 235
[25] Alcaniz J S 2004 *Phys. Rev.* D **69** 083501
[26] Caldwell R R and Rahul D 2002 *Phys. Lett.* B **545** 23
[27] Chimento L P and Lazkoz R 2003 *Phys. Rev. Lett.* **91** 211301
[28] Caldwell R R 1998 *Phys. Rev. Lett.* **80** 1582
[29] Sahniv and Wang L M 2000 *Phys. Rev.* D **62** 103517
[30] Capozziello S 2006 *Class. Quant. Grav.* **23** 1205
[31] Vikman A 2005 *Phys. Rev.* D **71** 023515
[32] Rogerio R and Joshua A F 2006 astro-ph/06112421V1
[33] Chiba T 2000 *Phys. Rev.* D **62** 023511
[34] Scherrer R J 2004 *Phys. Rev. Lett.* **93** 011301
[35] Wei H 2005 *Class. Quant. Grav.* **22** 3189
[36] Zhao G B 2005 *Phys. Rev.* D **72** 123515
[37] Feng B 2006 *Phys. Lett.* B **20** 2075
[38] Hellerman S and Takahiro O 2001 *J. High Energy Phys.* **JHEP06** 003 hep-ph/0104180
[39] Banks T and Fischler 2001 preprint hep-th/0102077
[40] Witten E 2001 preprint hep-th/0106109
[41] Kiselev V V 2003 *Class. Quant. Grav.* **20** 1187
[42] Landauand L D and Lifshitz E M 1978 *Field Theory* (Moscow: Nauka)
[43] Rinder W 2001 *Relativity* (Oxford: Oxford University Press)