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Implementation of n-qubit Deutsch–Jozsa algorithm using resonant interaction in cavity QED*

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We propose a scheme to implement the n-qubit Deutsch–Jozsa algorithm based on resonant interaction between the atoms and a single-mode cavity. In the scheme, the resonant transitions between two ground states and one excited state of an atom are changed alternately by adjusting the cavity frequency appropriately, and the operations required to complete the algorithm can be significantly simplified following the increment of the number of qubits. The implementation of the scheme in experiment would show the full power of quantum algorithm and would be significative and important for more complicated quantum algorithm in cavity quantum electrodynamics.

Keywords: Deutsch–Jozsa algorithm, cavity quantum electrodynamics, resonant interaction, CNOT gate operation

PACC: 0365, 4250

1. Introduction

The principles of quantum mechanics have suggested many important and interesting applications in the field of quantum information in the last decade, such as quantum communication[1−8] and quantum computation.[9−12] Quantum computers based on the fundamental quantum mechanical superposition principle have attracted considerable attention for they are more powerful than their classical counterparts.[13] Shor[9] first discovered a quantum mechanical algorithm for factorization that was exponentially faster than any known classical algorithm in 1994. Subsequently, Grover proposed a quantum mechanical algorithm for searching a sequence that was only polynomially faster than any classical algorithm,[10] however, it does not depend on its impact on the unproven difficulty of the factorization problem. The Deutsch–Jozsa algorithm,[14] which can decide whether an unknown function is constant or balanced, is also a very important and significant quantum algorithm. The experimental implementation of the Deutsch–Jozsa algorithm has been reported in the ion trap system,[15] linear optical system,[16] and nuclear magnetic resonance system.[17,18] Up to now, many schemes[19−25] have been proposed for implementing the Deutsch–Jozsa algorithm. Recently, Zheng[23] has proposed a scheme to implement one-qubit Deutsch–Jozsa algorithm based on resonant interaction of atoms with a cavity mode. In the scheme, the state of the cavity mode was used as the auxiliary working qubit after the first atom interacted resonantly with the cavity mode, which makes the scheme sensitive to the cavity decay and the thermal field. Dasgupta *et al*[24] proposed an optical scheme for implementing the one-qubit Deutsch–Jozsa algorithm by using light shifts and atomic ensembles. However, the implementations of required one- and two-qubit operations are very complex. To date, most of the schemes on the Deutsch–Jozsa algorithm are for the case of a single qubit, while they cannot be implemented in the case of multi-qubit. In order to show the full power of the Deutsch–Jozsa algorithm, the implementation of the algorithm for larger number of qubits is more significant.

In this paper, we first propose a scheme to implement the two-qubit Deutsch–Jozsa algorithm, using resonant interaction of the atoms with a single-mode cavity, and then generalize the scheme to implement the n-qubit Deutsch–Jozsa algorithm in cavity quantum electrodynamics (QED). In the scheme, the two lowest levels of an atom are used to represent two

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logical states of a qubit, while a higher-energy intermediate level is used to facilitate the realization of the operations. By adjusting the cavity frequency appropriately, the resonant transitions between the two ground states and one excited state of an atom can be changed alternately. The time required to complete the operations is very short due to the resonant interaction. Furthermore, we discuss the experimental feasibility of the scheme with current cavity QED technology. As far as we know, this is the first scheme for implementing the n-qubit Deutsch–Jozsa algorithm in cavity QED. The experimental implementation of the scheme would show the full power of quantum algorithm for larger number of qubits compared with the classical algorithm and would be an important step to implement more complicated quantum algorithm in cavity QED, thus showing the power of cavity QED for quantum-information processing.

The paper is organized as follows. In Section 2, we describe the basic character of the Deutsch–Jozsa algorithm. In Section 3, we show the basic model for implementing the Deutsch–Jozsa algorithm. In Section 4, we illustrate how to implement two-qubit Deutsch–Jozsa algorithm in cavity QED, and then generalize the scheme to the n-qubit Deutsch–Jozsa algorithm. In Section 5, we give a brief discussion on experimental issues for the realization of the proposed scheme. A summary will be given in Section 6.

2. Basic character of the Deutsch–Jozsa algorithm

The main feature of the Deutsch–Jozsa algorithm is designed to identify whether a given function $f(x)$ is constant or balanced. The function $f(x)$, which can only take the values 0 or 1, is a bivalued function of a variable x which can take any of the 2^n values between 0 and 2^n-1. If for half of all the possible 2^n inputs, the function values are equal to 0, and 1 for the other half, then the function is called balanced function. Otherwise, it is called constant function if the function values are constant for all 2^n inputs. In classical sense, at worst, it will need $2^n/2+1$ evaluations of the function $f(x)$ for different values of x to determine whether the function is constant or balanced. In contrast, the Deutsch–Jozsa algorithm requires only a single evaluation of the function for the same determination.

Now we will show how quantum algorithm can outdo the classical one in detail. The specific quantum circuit for implementing $n-$qubit Deutsch–Jozsa algorithm is depicted in Fig.1. The input state is

$$|\psi_0\rangle = |0\rangle_A^{\otimes n}|1\rangle_B, \tag{1}$$

where $|0\rangle_A^{\otimes n}$ and $|1\rangle_B$ denote the input query qubits (system A) and auxiliary working qubit (system B), respectively. After the Hadamard transform on the input query qubits and auxiliary working qubit we have

$$|\psi_1\rangle = \frac{1}{2^{n/2}}\sum_{x=0}^{2^n-1}|x\rangle_A\left[\frac{1}{\sqrt{2}}(|0\rangle_B-|1\rangle_B)\right]. \tag{2}$$

The input query qubits are now a superposition of all values, and the auxiliary working qubit is in an evenly weighted superposition of 0 and 1. Next, the function f is evaluated using U_f:$|x,y\rangle \longrightarrow |x,y\oplus f(x)\rangle$, giving

$$|\psi_2\rangle = \frac{1}{2^{n/2}}\sum_{x=0}^{2^n-1}(-1)^{f(x)}|x\rangle_A\left[\frac{1}{\sqrt{2}}(|0\rangle_B-|1\rangle_B)\right]. \tag{3}$$

After the U_f operation, the result of the function evaluation is stored in the amplitude of the query qubits' superposition state. Applying the Hadamard transform to the query qubits again, we obtain

$$\begin{aligned}|\psi_3\rangle &= \frac{1}{2^n}\sum_{x=0}^{2^n-1}(-1)^{f(x)}\sum_{y=0}^{2^n-1}(-1)^{x\cdot y}|y\rangle_A \\ &\quad\times\left[\frac{1}{\sqrt{2}}(|0\rangle_B-|1\rangle_B)\right] \\ &= \sum_{x=0}^{2^n-1}\sum_{y=0}^{2^n-1}\frac{(-1)^{x\cdot y+f(x)}|y\rangle_A}{2^n} \\ &\quad\times\left[\frac{1}{\sqrt{2}}(|0\rangle_B-|1\rangle_B)\right],\end{aligned} \tag{4}$$

where $x\cdot y = x_1y_1\oplus x_2y_2\oplus\cdots\oplus x_ny_n$, and $\oplus$ represents an addition mod 2. Now we perform a measurement on the query qubits of the system A, noting that the amplitude for the state $|0\rangle^{\otimes n}$ is $\sum_{x=0}^{2^n-1}(-1)^{f(x)}/2^n$. Now consider two possible cases $-$ $f(x)$ constant and $f(x)$ balanced $-$ to investigate what happens. If $f(x)$ is constant, the amplitude for $|0\rangle^{\otimes n}$ is $+1$ or -1, which depends on the constant value $f(x)$ takes. Since the length of $|\psi_3\rangle$ is unity, all the other amplitudes must be zero, and a measurement will yield 0 for all query qubits. If $f(x)$ is balanced then the positive and negative contributions to the amplitude for $|0\rangle^{\otimes n}$ cancel each other, leaving an amplitude of zero, and a measurement must yield a result other than 0 on at least one qubit for the query qubits. In summary, if the measurement result is $|0\rangle^{\otimes n}$ for the state of all the

query qubits, then the function is constant; otherwise, the function is balanced. In this way, the Deutsch's problem can be solved with one evaluation of the function $f(x)$ by using a quantum computer compared to the classical requirement for $2^n/2+1$.

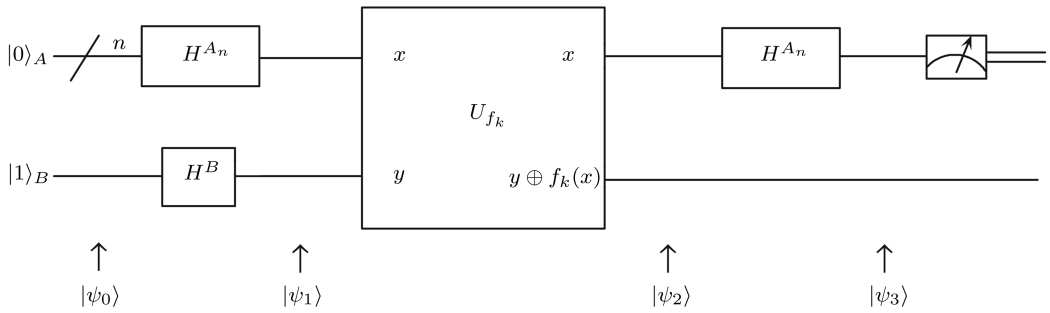


Fig.1. Basic quantum circuit for implementing the n-qubit Deutsch–Jozsa algorithm. All the n qubits in the system A are initialized to the state $|0\rangle$, while the qubit in the working system B is initialized to the state $|1\rangle$. H^A ($H^{A_n}=H^{\otimes n}$) and H^B represent the Hadamard transform on the qubits of the systems A and B, respectively, and U_{f_k} denotes the unitary operation which operates on the systems A and B simultaneously. A measurement on the system A gives the result.

3. Basic model for implementing the Deutsch–Jozsa algorithm

3.1. Resonant interaction between an atom and a single-mode cavity

We consider a $\Lambda-$type three-level atom, which has one excited state $|2\rangle$ and two ground states $|0\rangle$ and $|1\rangle$. When the cavity mode is resonant with the transition $|0\rangle \longleftrightarrow |2\rangle$ while far off resonant with the transition $|1\rangle \longleftrightarrow |2\rangle$ and the transition $|0\rangle \longleftrightarrow |1\rangle$ of the atom (Fig.2(a)), in the rotating-wave approximation (RWA), the interaction Hamiltonian in the interaction picture is (assuming $\hbar=1$)

$$H_{\rm I}=g(a^+|0\rangle\langle 2|+a|2\rangle\langle 0|), \tag{5}$$

where a^+ and a are the creation and annihilation operators for the cavity mode, and g is the coupling strength of the cavity mode with the $|0\rangle \longleftrightarrow |2\rangle$ transition of the atom.

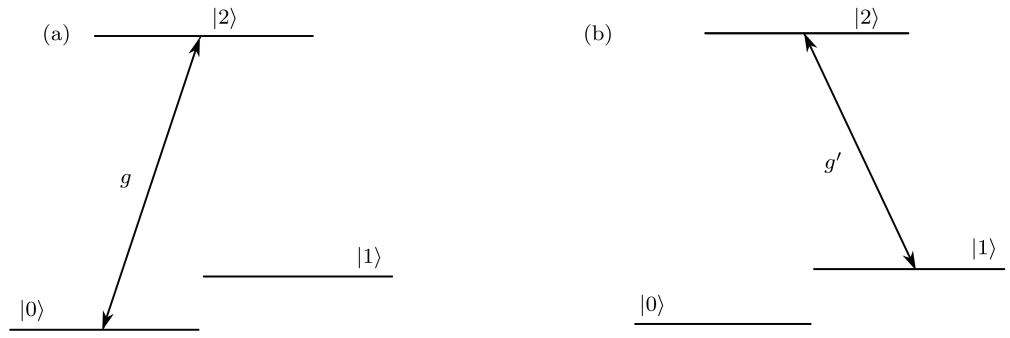


Fig.2. (a) Resonant interaction of the cavity mode with the $|0\rangle \longleftrightarrow |2\rangle$ transition while far off resonant with the $|1\rangle \longleftrightarrow |2\rangle$ transition and the $|0\rangle \longleftrightarrow |1\rangle$ transition of the atom, with the atom–cavity coupling strength g. (b) Resonant interaction of the cavity mode with the $|1\rangle \longleftrightarrow |2\rangle$ transition while far off resonant with the $|0\rangle \longleftrightarrow |2\rangle$ transition and the $|0\rangle \longleftrightarrow |1\rangle$ transition of the atom, the atom–cavity coupling strength being g'.

We assume the cavity mode is initially in the photon number state $|n\rangle$. After an interaction time t, the time evolution of the states of the atom–cavity system, governed by Hamiltonian (5), is given by

$$|0\rangle|n\rangle \longrightarrow \cos\sqrt{n}gt|0\rangle|n\rangle - \mathrm{i}\sin\sqrt{n}gt|2\rangle|n-1\rangle,$$

$$\begin{aligned}|2\rangle|n\rangle \longrightarrow & -\mathrm{i}\sin\sqrt{n+1}gt|0\rangle|n+1\rangle \\ & +\cos\sqrt{n+1}gt|2\rangle|n\rangle. \end{aligned} \quad (6)$$

Since the atoms may not be exactly placed at the same position in the cavity field, the coupling strength g may not be identical for different atoms.

Via adjusting the cavity frequency appropriately, we can obtain the resonant interaction between the cavity mode and the $|1\rangle \longleftrightarrow |2\rangle$ transition of the atom (Fig.2(b)). In this case, the interaction Hamiltonian in the interaction picture can be written as

$$H_{\mathrm{I}}' = g'(a^{+}|1\rangle\langle 2| + a|2\rangle\langle 1|), \quad (7)$$

where g' is the coupling strength of the cavity mode with the $|1\rangle \longleftrightarrow |2\rangle$ transition of the atom. The time evolution of the states of the atom–cavity system is similar to Eq.(6). One just needs to replace $|0\rangle$ in Eq.(6) by $|1\rangle$.

3.2. Arbitrary three-qubit controlled-NOT gate operations

The system of three atoms has 2^3 states: $|000\rangle_{123}$, $|001\rangle_{123}$, $|010\rangle_{123}$, $|011\rangle_{123}$, $|100\rangle_{123}$, $|101\rangle_{123}$, $|110\rangle_{123}$, and $|111\rangle_{123}$. When the first two are the control qubits and the last one is the target qubit, there are four different kinds of three-qubit controlled-NOT (CNOT) gate operations. The four three-qubit CNOT gate operations correspond respectively to the control conditions that atoms 1 and 2 is in the state $|00\rangle_{12}$, $|01\rangle_{12}$, $|10\rangle_{12}$, and $|11\rangle_{12}$. In Fig.3, we give the circuit representation of the four three-qubit controlled-CNOT gate operations. In the following we will show how to realize the CNOT gate operations corresponding to the different control conditions.

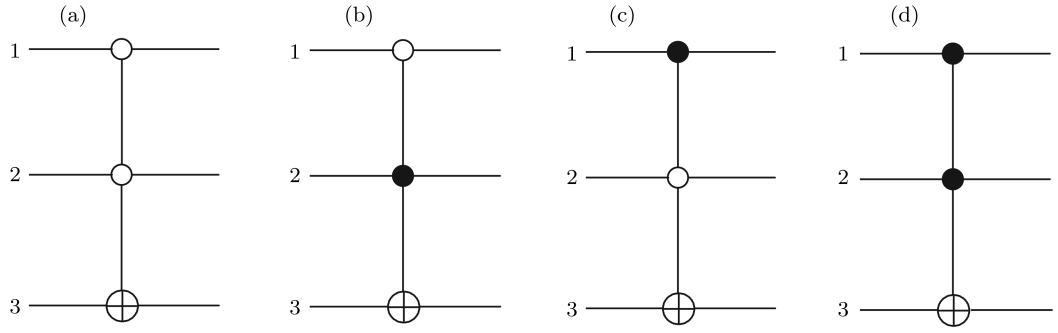


Fig.3. Schematic circuit of arbitrary three-qubit CNOT gate operations. (a) 00-CNOT gate operation. A NOT gate operation is performed on the target qubit 3 when the qubits 1 and 2 are both in the state $|0\rangle$. (b) 01-CNOT gate operation. A NOT gate operation is performed on the target qubit 3 when the qubit 1 is in the state $|0\rangle$ and the qubit 2 is in the state $|1\rangle$. (c) 10-CNOT gate operation. A NOT gate operation is performed on the target qubit 3 when the qubit 1 is in the state $|1\rangle$ and the qubit 2 is in the state $|0\rangle$. (d) 11-CNOT gate operation. A NOT gate operation is performed on the target qubit 3 when the qubits 1 and 2 are both in the state $|1\rangle$.

Realization of 00-CNOT gate operation. The action of the 00-CNOT gate operation is described by the evolution

$$\begin{aligned} |000\rangle &\longrightarrow |001\rangle, & |100\rangle &\longrightarrow |100\rangle, \\ |001\rangle &\longrightarrow |000\rangle, & |101\rangle &\longrightarrow |101\rangle, \\ |010\rangle &\longrightarrow |010\rangle, & |110\rangle &\longrightarrow |110\rangle, \\ |011\rangle &\longrightarrow |011\rangle, & |111\rangle &\longrightarrow |111\rangle. \end{aligned} \quad (8)$$

One can see that if and only if the first two qubits (control qubits) are in the state $|0\rangle$, the third qubit (target qubit) is flipped and nothing happens for the other cases. The explicit realization of the 00-CNOT gate operation is as follows.

Step 1 Apply a classical microwave pulse to atom 3. Choose the amplitudes and phases of the pulse appropriately so that atom 3 undergoes the transformation

$$\begin{aligned} |0\rangle_3 &\longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_3 + |1\rangle_3), \\ |1\rangle_3 &\longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_3 - |1\rangle_3). \end{aligned} \quad (9)$$

Step 2 Apply a classical microwave pulse to

atom 1. The pulse is resonant with the $|0\rangle \longleftrightarrow |2\rangle$ transition of atom 1. After that, we can obtain the transformation $|0\rangle \longrightarrow |2\rangle$.

Step 3 Send atom 1 into the cavity. The cavity mode is initially in the vacuum state $|0\rangle_{\rm c}$ and interacts resonantly with the $|1\rangle \longleftrightarrow |2\rangle$ transition of atom 1. After an interaction time $t_1 = \pi/(2g_1')$, atom 1 leaves the cavity.

Step 4 Send atom 2 into the cavity. Bring the $|1\rangle \longleftrightarrow |2\rangle$ transition to atom 2. After an interaction time $t_2 = \pi/(2g_2')$, atom 2 leaves the cavity.

Step 5 Adjust the cavity frequency so that the cavity mode interacts resonantly with the $|0\rangle \longleftrightarrow |2\rangle$ transition of atom 3. Then send atom 3 into the cavity. After an interaction time $t_3 = \pi/g_3$, atom 3 leaves the cavity.

Step 6 Adjust the cavity frequency so that the cavity mode interacts resonantly with the $|1\rangle \longleftrightarrow |2\rangle$ transition of atom 2. Then send atom 2 into the cavity. After an interaction time $t_4 = \pi/(2g_2')$, atom 2 leaves the cavity.

Step 7 Send atom 1 into the cavity. Bring the $|1\rangle \longleftrightarrow |2\rangle$ transition to atom 1. After an interaction time $t_5 = \pi/(2g_1')$, atom 1 leaves the cavity.

Step 8 Apply a classical microwave pulse to atom 1. The pulse is resonant with the $|0\rangle \longleftrightarrow |2\rangle$ transition of atom 1, leading to the transformation $|2\rangle \longrightarrow |0\rangle$.

Step 9 Apply a classical microwave pulse to atom 3. Choose the amplitudes and phases of the pulse appropriately so that atom 3 undergoes the transformation

$$\begin{aligned} |0\rangle_3 &\longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_3 + |1\rangle_3), \\ |1\rangle_3 &\longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_3 - |1\rangle_3). \end{aligned} \tag{10}$$

After each step of the above operations, the states $|100\rangle_{123}$, $|101\rangle_{123}$, $|110\rangle_{123}$, and $|111\rangle_{123}$ remain unchanged, while the time evolution of the states $|000\rangle_{123}$, $|001\rangle_{123}$, $|010\rangle_{123}$, and $|011\rangle_{123}$ are summarized as follows:

$$\begin{aligned}
&|000\rangle_{123}|0\rangle_{\rm c} \\
&|001\rangle_{123}|0\rangle_{\rm c} \\
&|010\rangle_{123}|0\rangle_{\rm c} \\
&|011\rangle_{123}|0\rangle_{\rm c}
\end{aligned}
\xrightarrow{\text{Step 1}}
\begin{aligned}
&|00\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|00\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c} \\
&|01\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|01\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c}
\end{aligned}
\xrightarrow{\text{Step 2}}
\begin{aligned}
&|20\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|20\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c} \\
&|21\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|21\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c}
\end{aligned}
\xrightarrow{\text{Step 3}}
\begin{aligned}
&-{\rm i}|10\rangle_{12}(|0\rangle_3 + |1\rangle_3)|1\rangle_{\rm c} \\
&-{\rm i}|10\rangle_{12}(|0\rangle_3 - |1\rangle_3)|1\rangle_{\rm c} \\
&-{\rm i}|11\rangle_{12}(|0\rangle_3 + |1\rangle_3)|1\rangle_{\rm c} \\
&-{\rm i}|11\rangle_{12}(|0\rangle_3 - |1\rangle_3)|1\rangle_{\rm c}
\end{aligned}$$

$$\begin{aligned}
&-{\rm i}|10\rangle_{12}(|0\rangle_3 + |1\rangle_3)|1\rangle_{\rm c} \\
&-{\rm i}|10\rangle_{12}(|0\rangle_3 - |1\rangle_3)|1\rangle_{\rm c} \\
&-|12\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&-|12\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c}
\end{aligned}
\xrightarrow{\text{Step 4}}
\begin{aligned}
&{\rm i}|10\rangle_{12}(|0\rangle_3 - |1\rangle_3)|1\rangle_{\rm c} \\
&{\rm i}|10\rangle_{12}(|0\rangle_3 + |1\rangle_3)|1\rangle_{\rm c} \\
&-|12\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&-|12\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c}
\end{aligned}
\xrightarrow{\text{Step 5}}
\begin{aligned}
&{\rm i}|10\rangle_{12}(|0\rangle_3 - |1\rangle_3)|1\rangle_{\rm c} \\
&{\rm i}|10\rangle_{12}(|0\rangle_3 + |1\rangle_3)|1\rangle_{\rm c} \\
&{\rm i}|11\rangle_{12}(|0\rangle_3 + |1\rangle_3)|1\rangle_{\rm c} \\
&{\rm i}|11\rangle_{12}(|0\rangle_3 - |1\rangle_3)|1\rangle_{\rm c}
\end{aligned}
\xrightarrow{\text{Step 6}}
\begin{aligned}
&|20\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c} \\
&|20\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|21\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|21\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c}
\end{aligned}$$

$$\xrightarrow{\text{Step 7}}
\begin{aligned}
&|00\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c} \\
&|00\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|01\rangle_{12}(|0\rangle_3 + |1\rangle_3)|0\rangle_{\rm c} \\
&|01\rangle_{12}(|0\rangle_3 - |1\rangle_3)|0\rangle_{\rm c}
\end{aligned}
\xrightarrow{\text{Step 8}}
\begin{aligned}
&|001\rangle_{123}|0\rangle_{\rm c} \\
&|000\rangle_{123}|0\rangle_{\rm c} \\
&|010\rangle_{123}|0\rangle_{\rm c} \\
&|011\rangle_{123}|0\rangle_{\rm c} .
\end{aligned}
\xrightarrow{\text{Step 9}} \tag{11}$$

So far a 00-CNOT gate operation is obtained. The essential character of the method is to produce a single photon through sending one of the control atoms (qubits) into the cavity and then exchanging the single photon between the control atoms and the cavity mode before and after one NOT gate operation is performed on the target atom. Similarly, the 01-CNOT gate operation, 10-CNOT gate operation and 11-CNOT gate operation can also be realized by using the same method with some modifications. Thus, all of the arbitrary three-qubit controlled-CNOT gate operations can be realized in cavity QED.

4. Implementing n-qubit Deutsch–Jozsa algorithm in cavity QED

In this section, for clarity, we will first give an explicit description of how to implement a two-qubit Deutsch–Jozsa algorithm with three atoms based on resonant interaction in cavity QED. Then we extend the method to implement an n-qutit Deutsch–Jozsa algorithm with a larger number n.

4.1. Two-qubit Deutsch–Jozsa algorithm

For the two-qubit Deutsch–Jozsa algorithm, we need to introduce three atoms 1, 2, and 3, each is initially in the state $|0\rangle_1$, $|0\rangle_2$, and $|1\rangle_3$. Firstly, apply a classical microwave pulse to atoms 1, 2, and 3, respectively. Choose the amplitude and phase of the pulse appropriately so that atom j ($j = 1, 2, 3$) undergoes the transformation

$$|0\rangle_j \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_j + |1\rangle_j),$$
$$|1\rangle_j \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_j - |1\rangle_j). \tag{12}$$

After that, the state of atoms 1, 2, and 3 becomes

$$\begin{aligned}|\Phi\rangle_0 &= \frac{1}{2\sqrt{2}}(|0\rangle_1 + |1\rangle_1)(|0\rangle_2 + |1\rangle_2)(|0\rangle_3 - |1\rangle_3)\\ &= \frac{1}{2\sqrt{2}}(|00\rangle_{12} + |01\rangle_{12} + |10\rangle_{12} + |11\rangle_{12})\\ &\quad\times(|0\rangle_3 - |1\rangle_3).\end{aligned} \tag{13}$$

Table 1. Different possible values of the function $f(x)$ that can hold for the two-qubit Deutsch–Jozsa algorithm.

Input	Constant		Balanced					
x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$
00	0	1	0	1	0	1	0	1
01	0	1	0	1	1	0	1	0
10	0	1	1	0	0	1	1	0
11	0	1	1	0	1	0	0	1

Next, the U_{f_k} operations which are the unitary operators corresponding to the function $f_k(x)$ ($k = 1, \ldots, 8$) are carried out. The implementation of U_{f_k} on $|\Phi\rangle$ yields

$$\begin{aligned}|\Phi\rangle_1 &= \frac{1}{2\sqrt{2}}[(-1)^{f_k(00)}|00\rangle_{12} + (-1)^{f_k(01)}|01\rangle_{12}\\ &\quad+(-1)^{f_k(10)}|10\rangle_{12} + (-1)^{f_k(11)}|11\rangle_{12}]\\ &\quad\otimes(|0\rangle_3 - |1\rangle_3).\end{aligned} \tag{14}$$

In Table 1 we show the different possible values of the function $f(x)$ ($x = 00, 01, 10, 11$), and the explicit implementation of the U_{f_k} operations is described as follows.

U_{f_1} **operation** One can see that it is an identity operation. In this case the atoms can be tuned far off resonance with the cavity mode, and thus the atom–cavity evolution is frozen. Hence the system remains in the state $|\Phi\rangle_1$.

U_{f_2} **operation** We perform successively the 00-CNOT operation the 01-CNOT operation the 10-CNOT and the 11-CNOT operation on atoms 1, 2, and 3. After the four CNOT operations, the U_{f_2} operation is obtained, and the state in Eq.(14) evolves to

$$\begin{aligned}|\Phi\rangle_2 &= \frac{1}{2\sqrt{2}}(-|00\rangle_{12} - |01\rangle_{12} - |10\rangle_{12} - |11\rangle_{12})\\ &\quad\times(|0\rangle_3 - |1\rangle_3).\end{aligned} \tag{15}$$

U_{f_3} **operation** We first perform the 10-CNOT operation on atoms 1, 2, and 3. Then we perform the 11-CNOT operation on them. The state in Eq.(14) evolves to

$$\begin{aligned}|\Phi\rangle_3 &= \frac{1}{2\sqrt{2}}(|00\rangle_{12} + |01\rangle_{12} - |10\rangle_{12} - |11\rangle_{12})\\ &\quad\times(|0\rangle_3 - |1\rangle_3).\end{aligned} \tag{16}$$

U_{f_4} **operation** We first perform the 00-CNOT operation on atoms 1, 2, and 3. Then we perform the 01-CNOT operation on them. The state in Eq.(14) evolves to

$$\begin{aligned}|\Phi\rangle_4 &= \frac{1}{2\sqrt{2}}(-|00\rangle_{12} - |01\rangle_{12} + |10\rangle_{12} + |11\rangle_{12})\\ &\quad\times(|0\rangle_3 - |1\rangle_3).\end{aligned} \tag{17}$$

U_{f_5} **operation** We first perform the 01-CNOT operation on atoms 1, 2, and 3. Then we perform the

11-CNOT operation on them. The state in Eq.(14) evolves to

$$|\Phi\rangle_5 = \frac{1}{2\sqrt{2}}(|00\rangle_{12} - |01\rangle_{12} + |10\rangle_{12} - |11\rangle_{12}) \times(|0\rangle_3 - |1\rangle_3). \quad (18)$$

U_{f_6} operation We first perform the 00-CNOT operation on atoms 1, 2, and 3. Then we perform the 10-CNOT operation on them. The state in Eq.(14) evolves to

$$|\Phi\rangle_6 = \frac{1}{2\sqrt{2}}(-|00\rangle_{12} + |01\rangle_{12} - |10\rangle_{12} + |11\rangle_{12}) \times(|0\rangle_3 - |1\rangle_3). \quad (19)$$

U_{f_7} operation We first perform the 01-CNOT operation on atoms 1, 2, and 3. Then we perform the 10-CNOT operation on them. The state in Eq.(14) evolves to

$$|\Phi\rangle_7 = \frac{1}{2\sqrt{2}}(|00\rangle_{12} - |01\rangle_{12} - |10\rangle_{12} + |11\rangle_{12}) \times(|0\rangle_3 - |1\rangle_3). \quad (20)$$

U_{f_8} operation We first perform the 00-CNOT operation on atoms 1, 2, and 3. Then we perform the 11-CNOT operation on them. The state in Eq.(14) evolves to

$$|\Phi\rangle_8 = \frac{1}{2\sqrt{2}}(-|00\rangle_{12} + |01\rangle_{12} + |10\rangle_{12} - |11\rangle_{12}) \times(|0\rangle_3 - |1\rangle_3). \quad (21)$$

4.2. N-qubit Deutsch–Jozsa algorithm

From the implementation of the two-qubit Deutsch–Jozsa algorithm, one can see that the key to the implementation of the n-qubit Deutsch–Jozsa algorithm is how to realize $(n+1)$-qubit CNOT gate operations under the different control conditions, namely, corresponding to the control conditions that atoms 1, 2, ..., n are respectively in the state $|00\ldots,0\rangle_{1,2,\ldots,n}$, $|00\ldots,1\rangle_{1,2,\ldots,n}$, ..., and $|11\ldots1\rangle_{1,2,\ldots,n}$. Reviewing the implementation of the arbitrary three-qubit CNOT gate operations carefully, we can obtain the $(n+1)$-qubit CNOT gate operations under the different control conditions with the cavity mode returning to the original vacuum state, by using the sequence of operators

$$S_{\rm CNOT} = S^{\dagger}_{(n+1)s} \otimes S^{\dagger}_1 \otimes \left(\prod_{k=n}^{1} S_{kr}\right) \otimes S^2_{(n+1)r} \otimes \left(\prod_{k=1}^{n} S_{kr}\right) \otimes S_1 \otimes S_{(n+1)s}, \quad (22)$$

where $\prod_{j=1}^{l} S_{jr} \equiv S_{lr}S_{(l-1)r}\ldots S_{2r}S_{1r}$; $S_{(n+1)s}$ and $S^{\dagger}_{(n+1)s}$ represent the Hadamard transformations on atom $n+1$ with the following matrices

$$S^{\dagger}_{n+1}S_{n+1} = I, \qquad S_{(n+1)s} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (23)$$

in the basis states $|0\rangle_1 = (1,0)^{\rm T}$ and $|1\rangle_1 = (0,1)^{\rm T}$. S_1 and $S^{\dagger}_1$ represent the unitary operations on atom 1 with the following matrices:

$$S^{\dagger}_1 S_1 = I, \qquad S_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (24)$$

in the basis states $|0\rangle_1 = (0,1)^{\rm T}$ and $|2\rangle_1 = (1,0)^{\rm T}$ (or in the basis states $|1\rangle_1 = (0,1)^{\rm T}$ and $|2\rangle_1 = (1,0)^{\rm T}$). S_{kr} is the joint operation on the atom k $(k = 1,2,\ldots,n)$ and the cavity mode, which is denoted by the matrix:

$$S_{kr} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \quad (25)$$

in the basis states $|0\rangle_k|1\rangle_{\rm c} = (0,1)^{\rm T}$ and $|2\rangle_k|0\rangle_{\rm c} = (1,0)^{\rm T}$ (or in the basis states $|1\rangle_k|1\rangle_{\rm c} = (0,1)^{\rm T}$ and $|2\rangle_k|0\rangle_{\rm c} = (1,0)^{\rm T}$); and $S_{(n+1)r}$ is the joint operation on the atom $n+1$ and the cavity mode with the same matrix as given by Eq.(25) (in the basis states $|0\rangle_{n+1}|1\rangle_{\rm c} = (0,1)^{\rm T}$ and $|2\rangle_{n+1}|0\rangle_{\rm c} = (1,0)^{\rm T}$).

Let us label the state of $n+1$ atoms as $|j_1 j_2\ldots,j_l\ldots j_n\rangle \otimes |j_{n+1}\rangle$, where the subscript l represents qubit l, $j_l \in \{0,1\}$, and j_1, j_2, ..., j_l, ..., j_n are control qubits and j_{n+1} is target qubit. From the above description of the $S_{\rm CNOT}$ operation, we give the explicit ways to implement the $S_{\rm CNOT}$ operations, as follows:

(i) $S_{(n+1)s}$ ($S^{\dagger}_{(n+1)s}$) can be realized with a $\pi/2$ classical microwave pulse resonant with the $|0\rangle \longleftrightarrow |1\rangle$ transition of the atom $n+1$.

(ii) S_1 ($S^{\dagger}_1$) can be realized with a π classical microwave pulse resonant with the $|j_1\rangle \longleftrightarrow |2\rangle$ transition of the atom 1.

(iii) S_{kr} corresponds to the operation of the $|\overline{j_2}\rangle \longleftrightarrow |2\rangle$ transition of atom 1 and the $|\overline{j_m}\rangle \longleftrightarrow |2\rangle$ transition of atom m $(m = 2,3,\ldots,n)$ in resonance with the cavity mode for an interaction time $\tau_l = \pi/(2g_l)$ $(l = 1,2,\ldots,n)$.

(iv) $S_{(n+1)r}$ corresponds to the operation of the $|0\rangle \longleftrightarrow |2\rangle$ transition of atom $n+1$ in resonance with the cavity mode for an interaction time $\tau_{n+l} = \pi/g_{n+l}$.

The ways and means proposed here for implementing $(n+1)$-qubit CNOT gate operations under the different control conditions are the extension of the arbitrary three-qubit CNOT gate operations described in the previous subsection. Thus all the U_{f_k} $(k = 0, 1, 2, \ldots, 2^{n+1})$ operations in the n-qubit Deutsch–Jozsa algorithm can be implemented successfully by combining some of the $(n+1)$-qubit CNOT gate operations under the different control conditions, and the n-qubit Deutsch–Jozsa algorithm can be implemented successfully.

5. Discussion

In this section, we briefly discuss the feasibility of the proposed scheme to be used in experiment. For the sake of definiteness, we first consider the experimental feasibility of implementing the two-qubit Deutsch–Jozsa algorithm. For the Rydberg atoms with principal quantum numbers 50 and 51, the radiative time is $T_{\rm r} = 3.0 \times 10^{-2}$s, and the coupling strength is $g = 2\pi \times 25\,{\rm kHz}$.[26–30] Hence the interaction times of atoms 1, 2, and 3 with the cavity field are $\pi/2g = 1.0 \times 10^{-5}$s, and $\pi/g = 2.0 \times 10^{-5}$s, respectively. For simplicity, we have set $g_1 = g_1' = g_2 = g_2' = g_3 = g_3' = g$. Thus the time required to complete one CNOT operation is about 8.0×10^{-5}s, and the total time required to complete the U_{f_k} operation is $U_{f_1} = U_{f_3} = U_{f_4} = \cdots = U_{f_8}$=$1.6 \times 10^{-4}$s, and $U_{f_2} = 3.2 \times 10^{-4}$s, which are much shorter than $T_{\rm r}$. In recent experiments, the decay time of the cavity field was $T_{\rm c} \simeq 1.0 \times 10^{-3}$s,[26–30] which is longer than the required time. Therefore, based on the current cavity QED techniques, the proposed scheme for the two-qubit Deutsch–Jozsa algorithm may be experimentally realizable.

Without loss of generality, let us continue to consider the implementation of the n-qubit Deutsch–Jozsa algorithm. The time required to complete one CNOT operation in the n-qubit Deutsch–Jozsa algorithm is $\tau_{\rm CNOT} = 2n(\tau_{\rm r} + \tau_{\rm c}) + \tau_0 + 2\tau_\nu$ $(\tau_{\rm r} = \pi/(2g),\ \tau_0 = \pi/g)$. Here $\tau_{\rm c}$ is the typical time required to adjust the frequency of a cavity field, and τ_ν is the pulse duration. In theory, $\tau_{\rm CNOT}$ can be reduced, because (i) $\tau_{\rm r}$ and τ_0 can be decreased by enhancing the coupling strength g, (ii) $\tau_{\rm c}$ can be reduced by rapid adjustment of the frequency of the cavity field. On the other hand, we can utilize two cavities, one is adjusted to bring the $|0\rangle \longleftrightarrow |2\rangle$ transition of an atom and the other bring $|1\rangle \longleftrightarrow |2\rangle$ transition of the atom, thus the adjustment of the resonant transitions between two ground states and one excited state of an atom is unnecessary. Finally, we should point out that the one-qubit Deutsch–Jozsa algorithm can be implemented more simply and easily compared with the two-qubit Deutsch–Jozsa algorithm. Here we do not illustrate it again.

6. Summary

We have proposed a scheme for implementing an n-qubit Deutsch–Jozsa algorithm by using repetitious resonant interactions between the atoms and a single-mode cavity field. The scheme has the following merits. (i) The interaction among the atoms can be cancelled since only one atom interacts with the cavity field. (ii) All the unitary operations can be realized directly via resonant interactions between the atoms and a single-mode cavity, and no auxiliary atom or measurement is needed during the whole operation. (iii) The atom–cavity coupling strength may not be identical and thus all the atoms may not be sent through the cavity with precise tracks and velocities. (iv) With the increase of the number of qubits, the operations are significantly simplified as compared with the conventional operation-decomposition protocols. (v) In theory, the n-qubit Deutsch–Jozsa algorithm can be implemented with a large number n. Finally, the scheme can also be applied to the implementation of other multi-qubit quantum computations in cavity QED.

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