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A potential integration method for Birkhoffian system*

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This paper is intended to apply the potential integration method to the differential equations of the Birkhoffian system. The method is that, for a given Birkhoffian system, its differential equations are first rewritten as $2n$ first-order differential equations. Secondly, the corresponding partial differential equations are obtained by potential integration method and the solution is expressed as a complete integral. Finally, the integral of the system is obtained.

Keywords: potential integration method, Birkhoffian system, differential equations, integral
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1. Introduction

The problem of integrating ordinary differential equations in analytical mechanics may be described by finding complete integrals of partial differential equations as do the Hamilton–Jacobi method,[1−3] field method,[4−8] Noether method[9−14] and so on. The Soviet author Arzhangh introduced a method called the potential integration method in 1965.[15] In Ref.[16] the present author applied the method to the holonomic and non-holonomic systems. The purpose of this paper is to show how the potential integration method is used for the Birkhoffian system.

2. Potential integration method

Consider a set of first-order differential equations

$$\dot{x}_i = F_i(t, x_j), \quad (i, j = 1, \ldots, n). \tag{1}$$

Using the potential integration method[15] one can express a solution to Eq.(1) as

$$x_i = \frac{\partial \psi}{\partial p_i}, \tag{2}$$

where $\psi(t, p_i, a_i)$ is a complete integral of the partial differential equation of first-order,

$$\frac{\partial \psi}{\partial t} = p_j F_j\left(t, \frac{\partial \psi}{\partial p_i}\right). \tag{3}$$

And furthermore, one must replace p_i with a value which is given by

$$\frac{\partial \psi}{\partial a_i} = b_i = \text{const}, \tag{4}$$

such that the solution (2) is rewritten as an explicit function of time t,

$$x_i = f_i(t, c_j), \tag{5}$$

where

$$c_i = \omega_i(a_j, b_j), \tag{6}$$

in which a_i, b_i, c_i are constants.

It is difficult to find the complete integral of Eq.(3) by applying the potential integration method. If only the complete integral is found, the next task will be to solve algebraic equations.

3. Differential equation of Birkhoffian system

The Birkhoff equations are written in the form[17]

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = 0, \tag{7}$$

where $B = B(t, \boldsymbol{a})$ is called the Birkhoffian, $R_\mu = R_\mu(t, \boldsymbol{a})$ are Birkhoffian functions. Assume

$$\det(\Omega_{\mu\nu}) = \det\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \neq 0, \tag{8}$$

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all $\dot{a}^\mu$ from Eq.(7) are therefore obtained as

$$\dot{a}^\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) \tag{9}$$

in which

$$\Omega^{\mu\nu}\Omega_{\nu\rho} = \delta^\nu_\rho.$$

In what follows, what we need to do is a very simple thing. Only taking

$$x_i = a^i, \quad F_i = \Omega^{ij}\left(\frac{\partial B}{\partial a^j} + \frac{\partial R_j}{\partial t}\right), \tag{10}$$

then Eq.(9) can be rewritten as

$$\dot{x}_i = F_i(t, x_j) \quad (i, j = 1, \ldots, 2n), \tag{11}$$

4. Illustrative example

Let us now consider a fourth-order Birkhoffian system[17]

$$\begin{aligned} &R_1 = a^2 + a^3, \quad R_2 = 0, \\ &R_3 = a^4, \quad R_4 = 0, \\ &B = \frac{1}{2}\{(a^3)^2 + 2a^2a^3 - (a^4)^2\}, \end{aligned} \tag{12}$$

which is the famous Hojman–Urrutia problem. It plays an important role in the development of Birkhoffian mechanics.

From Eq.(7), it is easy to establish

$$-\dot{a}^2 - \dot{a}^3 = 0, \quad \dot{a}^1 - a^3 = 0,$$

$$\dot{a}^1 - \dot{a}^4 - (a^3 + a^2) = 0, \quad \dot{a}^3 + a^4 = 0,$$

so that

$$\dot{a}^1 = a^3, \quad \dot{a}^2 = a^4, \quad \dot{a}^3 = -a^4, \quad \dot{a}^4 = -a^2.$$

Taking

$$x_i = a^i,$$

then

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_4, \quad \dot{x}_4 = -x_2.$$

Employing Eq.(3), we will have

$$\begin{aligned} \frac{\partial \psi}{\partial t} =& p_1\frac{\partial \psi}{\partial p_3} + p_2\frac{\partial \psi}{\partial p_4} + p_3\left(-\frac{\partial \psi}{\partial p_4}\right) \\ &+ p_4\left(-\frac{\partial \psi}{\partial p_2}\right). \end{aligned} \tag{13}$$

Its corresponding characteristic equation is

$$\frac{\mathrm{d}t}{-1} = \frac{\mathrm{d}p_1}{0} = \frac{\mathrm{d}p_2}{-p_4} = \frac{\mathrm{d}p_3}{p_1} = \frac{\mathrm{d}p_4}{p_2 - p_3},$$

from which we can solve

$$\begin{aligned} &p_1 = c_1, \quad \mathrm{d}p_2 = p_4\mathrm{d}t, \\ &\mathrm{d}p_3 = -p_1\mathrm{d}t, \quad \mathrm{d}p_4 = (-p_2 + p_3)\mathrm{d}t. \end{aligned}$$

Furthermore, we get

$$\begin{aligned} &p_3 + p_1 t = c_3, \\ &p_2 = c_2\cos t + c_4\sin t - p_1 t + c_3, \\ &p_4 = -c_2\sin t + c_4\cos t - p_1. \end{aligned}$$

Hence, the complete integral of Eq.(13) is expressed as

$$\begin{aligned} \psi =& b_1p_1 + b_3(p_3 + p_1t) \\ &+ b_2\{(p_2 - p_3)\cos t - (p_1 + p_4)\sin t\} \\ &+ b_4\{(p_2 + p_3)8t + (p_4 - p_1)\cos t\}. \end{aligned}$$

Finally, according to Formula (2), the solution of the system is

$$\begin{aligned} a^1 = x_1 &= \frac{\partial \psi}{\partial p_1} = b_1 + b_3 t - b_2\sin t - b_4\cos t, \\ a^2 = x_2 &= \frac{\partial \psi}{\partial p_2} = b_2\cos t + b_4\sin t, \\ a^3 = x_3 &= \frac{\partial \psi}{\partial p_3} = b_3 - b_2\cos t + b_4\sin t, \\ a^4 = x_4 &= \frac{\partial \psi}{\partial p_4} = -b_2\sin t + b_4\cos t. \end{aligned}$$

5. Conclusion

It is well known that the potential integration method is one of methods of integration in analytic mechanics. The aim of this paper is to apply the method to solve the equations of Birkhoffian system. A solution of Birkhoff's equations is equivalent to finding the complete integral of a partial differential equation. So long as the complete integral can be found, we can deal with the above problem by algebraic calculation.

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